A.G. Mordkovich

Algebra and Elements of Mathematical Analysis

Mir Publishers · Moscow

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This is a translation of the second Russian edition published in 1987.

This book is written to cover the mathematics syllabus for preparatory departments of institutions for higher education. Its main purpose is to help those entering college to revise the algebra and mathematical analysis they covered at school and to prepare for college level of mathematics.

The text contains the basic theoretical material corresponding to the school course on algebra and analysis and is illustrated by a large number of worked examples. Each section ends in exercises for independent study.

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А. Г Мордкович

АЛГЕБРА И НАЧАЛА АНАЛИЗА

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PREFACE TO THE ENGLISH EDITION

This book is written to cover the mathematics syllabus for preparatory departments of institutions for higher education. Its main purpose is to help those entering college to revise the algebra and mathematical analysis they covered at school and to prepare for college level of mathematics.

The text contains the basic theoretical material corresponding to the school course in algebra and analysis and is illustrated by a large number of worked examples. Each section ends in exercises for independent study. However, it should be noted that these exercises are only the minimum necessary to go over the material in the section.

The first edition of this book in the original Russian was issued in 1979. This is a translation of the second Russian edition published in 1987. In the last decade many changes have been made to the education system due to the school reform. The reform was undertaken to raise the level of education, increase the formal rigour of teaching each subject, ensure students graduate with a sound knowledge of the fundamentals of science while removing overloading on the student and excessively complicated material. All this has affected the school syllabus for algebra and mathematical analysis. The new syllabus was carefully considered when preparing this second edition. In particular, the sections on topics related to continuity and limit of a function at a point have been revised, and a new chapter on complex numbers has been added. The whole system of exercises has been reviewed, the text has been made more concise, and a number of sections have been rearranged.

The following symbols are used in the book: the beginning and end of the proof of a theorem or assertion are marked by □ and ■, respectively. Similarly, the beginning and end of the solution of an example are marked by ○ and ●, respectively.

I should like to express my gratitude to Assistant Professor V.P. Modenov who read the manuscript and suggested a number of improvements. Special thanks to T.V. Terekhova who took an active part in preparing the new system of exercises.

A. Mordkovich

Chapter 1

REAL NUMBERS

§ 1.1. Natural Numbers

1. Properties of Natural Numbers. Positive integers 1, 2, 3, 4, ... used to count objects are called *natural* numbers. The set of all natural numbers is usually denoted by the capital letter N:

$$\mathbf{N} = \{1, 2, 3, \ldots, n, \ldots\}.$$

This is an infinite set. It has the least element 1, but has no greatest element. For every natural number we can indicate its successor: the number 7 is followed by the number 8, the number 124 by the number 125, and, in general, the number k is followed by the number $k + 1$.

For any natural numbers m, n, k the following equalities hold true:

1^0. $m + n = n + m$.

2^0. $m + (n + k) = (m + n) + k$.

3^0. $mn = nm$.

4^0. $m(nk) = (mn)k$.

5^0. $m(n + k) = mn + mk$.

6^0. $m \cdot 1 = m$.

Equalities 1^0 and 3^0 express the **commutative** law of addition and multiplication, equalities 2^0 and 4^0 the **associative** law of addition and multiplication, and equality 5^0 is known as the **distributive** law of multiplication over addition.

The above laws underlie any computations. For instance, when we have to find the value of the sum $37 + 25$, we first add (to 37) 20, and then add 5 to the obtained number. We actually use the associative law of addition: $37 + (20 + 5) = (37 + 20) + 5$.

Multiplying 18 by 13, we reason like this: multiplying 18 by 10, we get 180; multiplying 18 by 3, we get 54; adding together 180 and 54, we get 234. We actually use the distributive law: $18 \cdot (10 + 3) = 18 \cdot 10 + 18 \cdot 3$.

The result of addition or multiplication of two natural numbers is always a natural number. This is not so as far as subtraction and division are concerned: for instance, it is impossible to subtract 7 from 3 remaining within the set of natural numbers, to divide 15 by 4 exactly (with a zero remainder).

If a natural number m is divisible by a natural number n, then m is a *multiple* of the number n, and n, in turn, the *divisor* of the number

m. If m is a multiple of the number n, then there is a natural number k such that $m = kn$. For instance, 48 is divisible by 3, and we write: $48 = 16 \cdot 3$. Here, $m = 48$ (the dividend), $n = 3$ (the divisor), and $k = 16$ (the quotient).

If a natural number m is not exactly divisible by a natural number n, that is, if there is no natural number k such that $kn = m$, then we consider **division with a remainder.**

For example, 37 divided by 15 equals 2 (partial quotient) with a remainder of 7. Thus, $37 = 15 \cdot 2 + 7$.

Note that the equality $37 = 15 \cdot 1 + 22$ is also true, but, when carrying out division with a remainder, it is not taken into consideration since the remainder (22) exceeds the divisor (15). In general, if m is the dividend, n the divisor, p the partial quotient, and r the remainder $(r < n)$, then

$$m = np + r. \tag{1}$$

Here m, n, p, and r are natural numbers. If m is exactly divisible by n (that is, without a remainder), then $r = 0$.

Example. Find the partial quotient and the remainder of the division of the number 274 018 by the number 342.

○Let us perform a long division:

$$\begin{array}{r|l} & 801 \\ 342) & \overline{274018} \\ & 2736 \\ & \overline{418} \\ & 342 \\ & \overline{76} \end{array}$$

Thus, the partial quotient is 801, the remainder of the division being 76. By equality (1), we may write: $274\,018 = 342 \cdot 801 + 76$.●

2. Tests for Divisibility. In some cases, without performing the division of a natural number m by a natural number n, we can say whether the division of m by n leaves a remainder. We can do it with the aid of many special tests for divisibility of integers when written in decimal notation; some of them will be considered below.

Divisibility of a sum. *If each of the terms is divisible by a certain number, then their sum is also divisible by this number.*

□Let a and b be divisible by c; let us prove that $a + b$ is also divisible by c.

Since a is a multiple of c, there is a natural number n such that $a = nc$. Similarly, there is a natural number k such that $b = kc$. Then

$$a + b = nc + kc = cn + ck.$$

According to the distributive law, we have $cn + ck = c(n + k)$. Setting $n + k = m$, we get $a + b = cm$. But this means that the sum $a + b$ is divisible by the number c.■

For instance, without carrying out the division, we can assert that the sum $48 + 64 + 96$ is divisible by 16 since each term of this sum is divisible by 16.

However, one should not think that if each term of a sum is not divisible by a certain number, then the sum is not divisible by this number. For instance, the sum $37 + 19$ is divisible by 4 although neither 37 nor 19 is a multiple of 4. Thus, the formulated condition is sufficient but is not necessary for the sum $a + b$ to be divisible by the number c.

Divisibility of a product. *If at least one of the factors of a product is exactly divisible by a certain number, then the product is also divisible by this number.*

□ Let there be given a product ab of two natural numbers a and b, a being divisible by c; let us prove that ab is also a multiple of c.

Since a is a multiple of c, there is a natural number n such that $a = nc$. We then have

$$ab = (nc)\, b = (nb)\, c.$$

Setting $nb = m$, we get $ab = mc$. But this means that ab is a multiple of c. ■

For example, without performing the division, we may assert that the product $105 \cdot 48 \cdot 93 \cdot 54$ is divisible by 5 since 105 is divisible by 5.

As in the preceding case, the stated condition is sufficient for a product to be divisible by the number c but is not necessary. For example, the product $12 \cdot 18$ is divisible by 36, while neither 12 nor 18 is divisible by 36.

The above mentioned general tests for divisibility yield concrete tests for divisibility of a number by 2, 5, 10, 4, 3, and 9 which are listed below.

Divisibility by 2. *For a natural number to be divisible by* 2, *it is necessary and sufficient that its last digit be even.*

Divisibility by 5. *For a natural number to be divisible by* 5, *it is necessary and sufficient that its last digit be either* 0 *or* 5.

Divisibility by 10. *For a natural number to be divisible by* 10, *it is necessary and sufficient that it end in* 0.

Divisibility by 4. *For a natural number containing no less than three digits to be divisible by* 4, *it is necessary and sufficient that its last two digits be* 00, 04, 08 *or the number represented by the last two digits be divisible by* 4.

For example, the number 15 436 is divisible by 4 since the number 36 is divisible by 4, whereas the number 375 214 is not divisible by 4 since 14 is not divisible by 4.

Divisibility by 3 and by 9. *For a natural number to be divisible by* 3 (*or* 9), *it is necessary and sufficient that the sum of its digits be divisible by* 3 (*or* 9).

For example, the number 5 673 is divisible by 3 since the number $5 + 6 + 7 + 3 = 21$ is divisible by 3. The same number is not divisible by 9, since the sum of its digits, that is, the number 21 is not divisible by 9.

3. Factorizing a Natural Number into Prime Factors. If a natural number has only two divisors (a unity (one) and the number itself), then it is called a *prime*; if it has more than two divisors, then it is called a *composite number*.

For example, 19 is a prime number since it has only two divisors: 1 and 19, while 35 is a composite number since it has four divisors: 1, 5, 7, and 35.

The set of prime numbers is infinite.

Note that the number 1 is classed neither as a prime nor as a composite number. It could be classed as a prime, but it is better to put it in a separate class outside the prime and composite numbers (this convention stems from the fact that many of the rules which hold true for all other primes are invalid when applied to unity).

Every composite number can be written as a product of prime factors in only one way. The method used in the following example shows how to find the prime factors of a composite number.

Example. Factor 360 into primes.

First, trying factors of 2, we get

$$\begin{array}{r|l} 2 & 360 \\ \hline 2 & 180 \\ \hline 2 & 90 \\ \hline 3 & 45 \\ \hline & 15 \\ \hline & 5 \\ \hline & 1 \end{array}$$

The first factor of 2 | The second factor of 2 | The third factor of 2

Remainder of 1 indicates no fourth factor of 2

Next, trying factors of 3, we get

$$\begin{array}{r|l} 3 & 45 \\ \hline 3 & 15 \\ \hline 5 & 5 \\ \hline & 1 \end{array}$$

The first factor of 3 | The second factor of 3 | Remainder of 2 indicates no third factor of 3

Next, trying factors of 5, we get

$$\begin{array}{r|l} 5 & \\ \hline 5 & 1) \\ \hline 5 & \\ \hline 0 & \end{array}$$

The first factor of 5

Factored into primes, $360 = 2\cdot2\cdot2\cdot3\cdot3\cdot5 = 2^3\cdot3^2\cdot5$. Consequently, $360 = 2\cdot2\cdot2\cdot3\cdot3\cdot5$.

If in factorizing a number into prime factors one and the same factor a is encountered n times, then we write briefly a^n, that is,

$$\underbrace{a\cdot a\ldots a}_{n \text{ factors}} = a^n.$$

The expression a^n, called the *power*, where a is the *base*, and n the *exponent*, represents a product. The exponent n indicates that the base a is to be used as a factor n times.

For instance, $360 = 2\cdot2\cdot2\cdot3\cdot3\cdot5 = 2^3\cdot3^2\cdot5$.

Example. Factorize the number 911 250 into prime factors.

○ Using the tests for divisibility, we conclude that the given number is divisible by 2, 3, and 5:

$$\begin{array}{r|l} 911\,250 & 2 \\ 455\,625 & 5 \\ 91\,125 & 5 \\ 18\,225 & 5 \\ 3645 & 5 \\ 729 & 3 \\ 243 & 3 \\ 81 & 3 \\ 27 & 3 \\ 9 & 3 \\ 3 & 3 \\ 1 & \end{array}$$

Thus, $911\,250 = 2\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot5\cdot5\cdot5\cdot5 = 2\cdot3^6\cdot5^4$. ●

4. Greatest Common Divisor and Least Common Multiple of Several Natural Numbers. Let there be given two numbers 72 and 96. We write all the divisors of the number 72:

$$1,2,3,4,6,8,9,12,18,24,36,72,$$

then, of the number 96:

$$1,2,3,4,6,8,12,16,24,32,48,96.$$

These two number sets contain the following common numbers:

$$1,2,3,4,6,8,12,24.$$

All of them are called the *common divisors* of the numbers 72 and 96, the largest of all common divisors, the *greatest common divisor* being often abbreviated as G.C.D. Thus, the G.C.D. of 72 and 96 is 24.

For any two given natural numbers a and b we can find the greatest common divisor. If the numbers a and b are such that the G.C.D. $= 1$, then these numbers are called *relatively prime.*

For instance, 72 and 35 are relatively prime numbers (although each of them is a composite number). Indeed, the divisors of the number 72 are: 1,2,3,4,6,8,9,12,18,24, 36,72; the divisors of the number 35 are: 1,5,7,35. Hence, the G.C.D. $= 1$.

If numbers are factorized into prime factors, then it is easy to find their greatest common divisor.

Example 1. Find the G.C.D. of the numbers 3780 and 7056.

○We have

3780	2	7056	2
1890	2	3528	2
945	3	1764	2
315	3	882	2
105	3	441	3
35	5	147	3
7	7	49	7
1		7	7
		1	

Consequently, $3780 = 2\cdot2\cdot3\cdot3\cdot3\cdot5\cdot7 = 2^2\cdot3^3\cdot5\cdot7$, $7056 = 2\cdot2\cdot2\cdot2\cdot3\cdot3\cdot7\cdot7 = 2^4\cdot3^2\cdot7^2$.

Let us find the prime factors of the G.C.D. (3780, 7056). It must include the prime factors contained in both factorizations (of the numbers 3780 and 7056). If these prime factors enter the factorizations with different exponents, then we take the factor with a lesser exponent. For instance, the number 2 enters both factorizations: one with the exponent 2, the other with the exponent 4; therefore we take 2^2. Similarly, we take 3^2 and 7, while the factor 5 is rejected since it is absent from the factorization of the number 7056.

Thus, G.C.D. (3780, 7056) is $2^2\cdot3^2\cdot7 = 252$.●

Let us now introduce the notions of common multiple and least common multiple. Suppose we are given two numbers 12 and 18. We write the numbers multiple of 12:

$$12,24,36,48,60,72, \ldots .$$

We then write the numbers multiple of 18:

$$18,36,54,72, \ldots .$$

The above number sets contain common numbers:

$$36,72, \ldots .$$

All these numbers are *common multiples* of the numbers 12 and 18, the least of them being the *least common multiple* of the numbers 12 and 18; it is denoted by L.C.M.

Similarly, we determine the least common multiple of natural numbers a and b. Note that any common multiple of a and b is divisible by their L.C.M. In other words, if a number m is divisible by a and b, then it is also exactly divisible by the L.C.M. This note is often used for determining the divisibility. For instance, the number 2340 is divisible by 2,3,4,5,9,10. Hence, this number is also divisible by the least common multiple of the indicated numbers, that is, by the number 180.

If numbers are factorized into prime factors, then the least common multiple is readily found.

Example 2. Find the L.C.M. of the numbers 3780 and 7056.

○ In the preceding example, we determined that $3780 = 2^2 \times 3^3 \cdot 5 \cdot 7$, $7056 = 2^4 \cdot 3^2 \cdot 7^2$. Let us find the factorization of their L.C.M. It must contain all the prime factors which enter at least one of the given numbers. If some of the prime factors enters both factorizations, then it is taken with a greater exponent. We have

$$\text{the L.C.M. of 3780 and 7056 is } 2^4 \cdot 3^3 \cdot 5 \cdot 7^2 = 105\,840. \bullet$$

Here, we draw reader's attention to the following fact:

$$3780 \cdot 7056 = (2^2 \cdot 3^3 \cdot 5 \cdot 7) \cdot (2^4 \cdot 3^2 \cdot 7^2) = (2^2 \cdot 3^2 \cdot 7)\ (2^4 \cdot 3^3 \cdot 5 \cdot 7^2)$$

$$= \text{G.C.D. } (3780,7056) \cdot \text{L.C.M. } (3780,7056).$$

Thus, G.C.D. (3780, 7056)·L.C.M. (3780, 7056) = 3780·7056.

We can prove that a similar equality holds true for any natural numbers a and b:

$$\text{G.C.D. } (a, b) \cdot \text{L.C.M. } (a, b) = ab. \tag{2}$$

In particular, if a and b are relatively prime numbers, that is, G.C.D. $(a, b) = 1$, then L.C.M.$(a, b) = ab$. This means that the *least common multiple of two relatively prime numbers is equal to the product of these numbers.*

Equality (2) can be used for finding the least common multiple.

Example 3. Find L.C.M. of the numbers 276 and 282.

○ It is easy to notice that the given numbers have a common divisor 6. Since 282 — 276 = 6, the numbers 282 and 276 cannot have a common divisor larger than 6. Hence, 6 is the greatest common divisor. Thus, the G.C.D. (276, 282) = 6. Then from equality (2) we get

$$\text{L.C.M. } (276,282) = \frac{276 \cdot 282}{\text{G.C.D. } (276,282)} = (276 \div 6) \cdot 282 = 12\,972. \bullet$$

Exercises

In Exercises 1 to 5, perform the indicated operations:

1. $625 \div (1053 \div 9 - 29 \cdot 4) - 3750 \div 30$.
2. $(81\,108 \div 27 - 125 \cdot 12) \div 4$.
3. $(520 \cdot 43 \div 26 - 217 \div 7 \cdot 17) - (315 \div 63 \cdot 5 + 13 \cdot 16)$.
4. $(25 \cdot 9 - 15 \cdot 9) \cdot (25 \cdot 9 + 15 \cdot 9) \div 8100$.
5. $(18 \cdot 93 - (1927 - 1873) \cdot 31) + 56$.

6. Find the common divisors of the given numbers: (a) 39 and 42; (b) 28 and 36; (c) 42 and 56; (d) 14, 21, and 28; (e) 12, 18, and 24.

7. Find the greatest common divisor of the given pairs of numbers having factorized them into prime factors: (a) 28 and 42; (b) 504 and 540; (c) 616 and 396; (d) 1960 and 4725; (e) 2205 and 2475.

8. Among the numbers 4, 8, 3, 13, and 26, find all the pairs of relatively prime numbers.

9. Find three common multiples of each pair of numbers: (a) 2 and 5; (b) 7 and 3; (c) 6 and 15; (d) 14 and 21; the same for the triple of numbers 15, 20, and 30.

10. Find the least common multiple of the given pairs of numbers on having factorized them into prime factors: (a) 12 and 18; (b) 45 and 72; (c) 165 and 550; (d) 168 and 360; (e) 525 and 588.

11. Prove that the sum of: (a) two even numbers is even; (b) two odd numbers is even; (c) an even and an odd number are odd.

12. Prove that the product of: (a) two even numbers is even; (b) two odd numbers is odd; (c) an even and odd number are even.

13. Find the formula of the numbers: (a) divisible by 3 with 2 as a remainder; (b) divisible by 4 with 1 as a remainder; (c) divisible by 5 with 3 as a remainder.

14. Write the formula of numbers multiple of: (a) 2; (b) 3; (c) 5.

15. Write the formula of odd numbers.

16. The number a is divisible by 5. When dividing the number b by 5, we get 1 as a remainder and when dividing the number c by 5 we get 2 as a remainder. What remainder will be obtained after the division by 5 of the sum: (a) $a + b$, (b) $b + c$?

17. What digit must be inserted instead of the asterisk in the notation of a three-digit number 46 * to get a number divisible by: (a) 2; (b) 3; (c) 5; (d) 9; (e) 4; (f) 6; (g) 10?

§ 1.2. Rational Numbers

1. Common Fractions. Let us recall basic information on common fractions, that is, on numbers of the form m/n, where m and n are natural numbers.

Let there be given a common fraction m/n. The number m is called the numerator of the fraction, the number n the denominator of the fraction. In particular, n may be equal to 1. In this case we usually write m rather than $m/1$, i. e., any natural number can be represented in the form of a common fraction whose denominator is 1.

Two fractions a/b and c/d are regarded to be equal if $ad = bc$. Here is a couple of examples of equal fractions: 3/5 and 12/20 (since $3 \cdot 20 = 5 \cdot 12$), 7/1 and 28/4 (since $7 \cdot 4 = 28 \cdot 1$).

Obviously, the fractions a/b and am/bm are also equal. This means that *if the numerator and the denominator of a given fraction are mul-*

tiplied or divided by the same natural number, then an equivalent fraction is obtained: $\frac{a}{b}=\frac{am}{bm}$. It is usually called the **basic property of fractions.**

Taking advantage of this property, we sometimes may replace a given fraction by another equivalent fraction but with a smaller numerator and a smaller denominator by dividing all common factors out of the numerator and denominator. This operation is called **reduction of a fraction to its lowest terms or, simply, reduction of a fraction.**

For instance, let there be given the fraction 36/48. The numbers 36 and 48 have the G.C.D. equal to 12. Then $\frac{36}{48}=\frac{3\cdot 12}{4\cdot 12}=\frac{3}{4}$.

In the general case, reduction of a fraction is always possible if its numerator and denominator are not relatively prime numbers. If the numerator and denominator are relatively prime numbers, then the fraction is called *irreducible.*

Let now there be given two fractions 2/5 and 3/7. As we see, they have different denominators: 5 and 7. Using the basic rule of fractions, we can replace these fractions by equivalent like ones. We shall see later that such a process of **reducing fractions to a common denominator** often turns out to be useful. Multiplying the numerator and denominator of the fraction 2/5 by 7, we get

$$\frac{2}{5}=\frac{2\cdot 7}{5\cdot 7}=\frac{14}{35}.$$

Multiplying the numerator and denominator of the fraction 3/7 by 5, we get

$$\frac{3}{7}=\frac{3\cdot 5}{7\cdot 5}=\frac{15}{35}.$$

Thus, the fractions 2/5 and 3/7 have been reduced to a common denominator: $\frac{2}{5}=\frac{14}{35}$, $\frac{3}{7}=\frac{15}{35}$.

Note that this is not a unique solution of the given problem; for instance, the same fractions could be reduced to the common denominator 70:

$$\frac{2}{5}=\frac{2\cdot 14}{5\cdot 14}=\frac{28}{70},\quad \frac{3}{7}=\frac{3\cdot 10}{7\cdot 10}=\frac{30}{70}$$

and, in general, to any denominator which is simultaneously divisible by 5 and 7.

Consider one more example: reduce the fractions 7/24 and 11/30 to a common denominator. Reasoning as in the preceding example, we get

$$\frac{7}{24}=\frac{7\cdot 30}{24\cdot 30}=\frac{210}{720},\quad \frac{11}{30}=\frac{11\cdot 24}{30\cdot 24}=\frac{264}{720}.$$

But in this case it is possible to reduce the fractions to a common denominator smaller than the product of the denominators of these fractions. Let us find the least common multiple of the numbers 24 and 30: the L.C.M. of 24 and 30 = 120. We have $120 \div 24 = 5$; therefore, to write the fraction 7/24 with the denominator 120, it is then necessary to multiply both the numerator and denominator by 5; this number will be called the *multiplier for a fraction*. Consequently, $\frac{7}{24} = \frac{7 \cdot 5}{24 \cdot 5} = \frac{35}{120}$.

Further, we have $120 \div 30 = 4$. Multiplying the numerator and denominator of the fraction 11/30 by the multiplier for a fraction 4, we get

$$\frac{11}{30} = \frac{11 \cdot 4}{30 \cdot 4} = \frac{44}{120}.$$

Thus, the given fractions are reduced to a common denominator.

It is clear that the least common multiple of the denominators of the given fractions is the smallest common denominator. Later on, we shall often have to reduce fractions to a *least common denominator*.

2. Arithmetical Operations on Common Fractions. The **addition** of common fractions is defined in the following way:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

For example, $\frac{3}{5} + \frac{2}{7} = \frac{3 \cdot 7 + 2 \cdot 5}{5 \cdot 7} = \frac{31}{35}$.

If, in particular, $b = d$, then we have

$$\frac{a}{b} + \frac{c}{b} = \frac{ab + cb}{b \cdot b} = \frac{(a + c)\, b}{b \cdot b} = \frac{a + c}{b}.$$

This means that if the denominators of the fractions are the same, then the fractions may be added by adding their numerators. This sum will be the numerator of the resultant fraction, the denominator remaining unchanged. For example, $\frac{2}{7} + \frac{3}{7} = \frac{2+3}{7} = \frac{5}{7}$.

If the denominators differ, then we first reduce the fractions to a common denominator and then add the numerators together. For example,

$$\frac{7}{24} + \frac{11}{30} = \frac{35}{120} + \frac{44}{120} = \frac{35 + 44}{120} = \frac{79}{120}.$$

The **subtraction** of common fractions is defined similarly.

The **multiplication** of common fractions is defined as follows:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

For example, $\frac{3}{5} \cdot \frac{2}{7} = \frac{3 \cdot 2}{5 \cdot 7} = \frac{6}{35}$.

The **division** of common fractions is defined in the following way:

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

For example, $\frac{3}{5} \div \frac{2}{7} = \frac{3 \cdot 7}{5 \cdot 2} = \frac{21}{10}$.

3. Proper and Improper Fractions. We distinguish between proper and improper fractions. The fraction m/n is called a *proper fraction* if its numerator is less than the denominator; it is termed *improper* if its numerator is greater than the denominator (note that if the numerator of a fraction is equal to its denominator, that is, $m = n$, then $m/n = 1$; in this case the fraction m/n is neither proper nor improper).

Consider an improper fraction m/n and suppose that m is not multiple of n (if m is a multiple of n, then the fraction m/n may be replaced by a natural number). Since m is greater than n, as it was noted above (see § 1.1. Item 1), there are two numbers p and r ($r < n$) such that $m = pn + r$. Then

$$\frac{m}{n} = \frac{pn+r}{n} = \frac{pn}{n} + \frac{r}{n} = p + \frac{r}{n}.$$

Since the remainder is always less than the divisor, r/n is a proper fraction. This means that we have succeeded in representing the improper fraction m/n in the form of a sum of a natural number p and a proper fraction r/n. This operation is called the **reduction of an improper fraction to a whole or mixed number.** For example,

$$\frac{28}{5} = \frac{25+3}{5} = \frac{25}{5} + \frac{3}{5} = 5 + \frac{3}{5}.$$

The sum of a natural number and a proper fraction is customarily written without plus sign, that is, instead of $5 + \frac{3}{5}$ we usually write $5\frac{3}{5}$. A number consisting of an integer and a fraction is called a *mixed number*.

Thus, we have shown that any improper fraction can be written in the form of a mixed number. The converse is also true: any mixed number can be written in the form of an improper fraction. For example,

$$2\frac{3}{7} = 2 + \frac{3}{7} = \frac{2 \cdot 7}{1 \cdot 7} + \frac{3}{7} = \frac{14}{7} + \frac{3}{7} = \frac{14+3}{7} = \frac{17}{7},$$

or more briefly: $2\frac{3}{7} = \frac{2 \cdot 7 + 3}{7} = \frac{17}{7}$.

Example 1. Perform the indicated operations:

$$(1)\ \frac{77}{34} \cdot \frac{17}{33};\quad (2)\ \frac{12}{25} \div \frac{18}{35}.$$

○ (1) We have

$$\frac{77}{34}\cdot\frac{17}{33}=\frac{77\cdot17}{34\cdot33}=\frac{7\cdot11\cdot17}{2\cdot17\cdot11\cdot3}=\frac{7}{2\cdot3}=\frac{7}{6}.$$

(2) We find

$$\frac{12}{25}\div\frac{18}{35}=\frac{12\cdot35}{25\cdot18}=\frac{2\cdot6\cdot5\cdot7}{5\cdot5\cdot3\cdot6}=\frac{2\cdot7}{5\cdot3}=\frac{14}{15}.$$ ●

Example 2. Perform the indicated operations:

$$(1)\ 2\frac{1}{7}+3\frac{2}{3};\quad (2)\ 4\frac{1}{8}-\frac{7}{16};\quad (3)\ 1\frac{2}{5}\cdot2\frac{1}{7}.$$

○ (1) We have

$$2\frac{1}{7}+3\frac{2}{3}=\left(2+\frac{1}{7}\right)+\left(3+\frac{2}{3}\right)=(2+3)+\left(\frac{1}{7}+\frac{2}{3}\right)$$

$$=5+\left(\frac{1^{|3}}{7}+\frac{2^{|7}}{3}\right)=5+\frac{3+2\cdot7}{21}=5+\frac{17}{21}=5\ \frac{17}{21}.$$

(2) Note that in case of addition and subtraction of mixed numbers it is sometimes more convenient first to transform them into improper fractions and then carry out operations with improper fractions. Hence, we have in our example:

$$4\frac{1}{8}-\frac{7}{16}=\frac{4\cdot8+1}{8}-\frac{7}{16}=\frac{33}{8}-\frac{7}{16}$$

$$=\frac{33^{|2}}{8}-\frac{7^{|1}}{16}=\frac{33\cdot2-7\cdot1}{16}=\frac{66-7}{16}=\frac{59}{16}.$$

Now, it remains to change the obtained fraction to a mixed number (although not at all necessarily); we find $\frac{59}{16}=\frac{48+11}{16}=3\frac{11}{16}$.

Thus, $4\frac{1}{8}-\frac{7}{16}=3\frac{11}{16}$.

(3) When multiplying and dividing mixed numbers, we change them, as a rule, to improper fractions. We have $1\frac{2}{5}=\frac{7}{5}$, $2\frac{1}{7}=\frac{15}{7}$; then $1\frac{2}{5}\cdot2\frac{1}{7}=\frac{7}{5}\cdot\frac{15}{7}=\frac{7\cdot15}{5\cdot7}=3$. ●

4. Decimal Fractions. Any proper fraction whose denominator is equal to 10, 100, 1000, and so on can be written in the form of a decimal fraction or, simply, a decimal. For example,

$$\frac{3}{10}=0.3,\quad \frac{17}{100}=0.17,\quad \frac{847}{1000}=0.847,\quad \frac{3}{100}=0.03.$$

Mixed numbers can also be written in the same manner. For example, $3\frac{3}{10}=3.3$, $122\frac{17}{100}=122.17$. In these cases, the integer of a

mixed number is separated from the numerator of the fraction by a decimal point.

We can represent in the form of a decimal not only common fractions whose denominator is a power of 10, but also some other common fractions, say, $\frac{3}{4}$, $\frac{7}{25}$, $\frac{111}{40}$. Indeed, we have

$$\frac{3}{4}=\frac{3\cdot 25}{4\cdot 25}=\frac{75}{100}=0.75,\quad \frac{7}{25}=\frac{7\cdot 4}{25\cdot 4}=\frac{28}{100}=0.28,$$

$$\frac{111}{40}=\frac{111\cdot 25}{40\cdot 25}=\frac{2775}{1000}=2.775.$$

This is the general rule for changing a common fraction to a decimal fraction: *if the prime factorization of the denominator of a common fraction yields only* 2's *and* 5's, *then this fraction can be written in the form of a decimal. But if a common fraction is irreducible, and the factorization yields, besides* 2's *and* 5's, *other prime factors, then this fraction cannot be written as a decimal.*

Consider the decimal 7.234. We have

$$7.234=7\frac{234}{1000}=7+\frac{200+30+4}{1000}=7+\frac{200}{1000}+\frac{30}{1000}+\frac{4}{1000}$$

$$=7+\frac{2}{10}+\frac{3}{100}+\frac{4}{1000}$$

Hence, 7.234 signifies seven ones, two tenths, three hundredths, and four thousandths.

In general, a decimal fraction may have arbitrarily many decimal places (digits) after the decimal point: tenths, hundredths, thousandths, tens of thousandths, and so forth.

Again, let us take the decimal 7.234; we have

$$7.234=7\frac{234}{1000}=7\frac{2340}{10\,000}=7\frac{23\,400}{100\,000}.$$

But $7\frac{2340}{10\,000}=7.2340$, and $7\frac{23\,400}{100\,000}=7.23400$. Hence, $7.234=7.2340=7.23400$. Thus, annexing zeros to the right of a finite decimal does not change its value. Dropping zeros at the end of a decimal fraction does not change the value of the fraction.

Recall how arithmetical operations with decimal fractions are performed. Suppose we have to add the decimals 1.234 and 3.317; we find

$$1.234+3.317=1\frac{234}{1000}+3\frac{317}{1000}=4\frac{234+317}{1000}=4\frac{551}{1000}=4.551.$$

However, we can perform the addition without resorting to common fractions, just as in the addition of natural numbers. In doing so, we must arrange the numbers to be added so that all the decimal points are exactly lined up with one another (the decimal point under

the decimal point, tenths under tenths, hundredths under hundredths, and so on). We have

$$\begin{array}{r} 1.234 \\ +3.317 \\ \hline 4.551 \end{array}$$

Consider one more example: add 12.7 and 3.442. The first decimal has one digit after the decimal point, whereas the second three. To perform the addition, we convert the first decimal so that it has three digits after the decimal point: $12.7 = 12.700$. Then

$$\begin{array}{r} 12.700 \\ +3.442 \\ \hline 16.142 \end{array}$$

Subtraction of decimals is performed in a similar way. Let us, for example, subtract 0.37 from 13.1

$$\begin{array}{r} 13.10 \\ -0.37 \\ \hline 12.73 \end{array}$$

Let us now multiply two decimals: 1.12 and 2.3; we find

$$1.12 \cdot 2.3 = 1\frac{12}{100} \cdot 2\frac{3}{10} = \frac{112}{100} \cdot \frac{23}{10} = \frac{112 \cdot 23}{100 \cdot 10} = \frac{2576}{1000} = 2.576.$$

However, we could perform the multiplication without resorting to common fractions: it suffices to multiply the given numbers as if they were whole numbers. Count the total number of decimal places in both numbers being multiplied. Then insert the decimal point in the product so that the number of digits to the right of the decimal point is equal to the total number of decimal places in the two numbers being multiplied.

Let us multiply 1.12 by 2.3. We have $112 \cdot 23 = 2576$. Now point off three digits from the right since the factors have two or one digit after the decimal point, respectively. As a result, we get $1.12 \cdot 2.3 = 2.576$.

Let us multiply 2.317 by 5. We have $2317 \cdot 5 = 11\,585$. Now, we point off three decimal places since the first of the factors has three digits, while the second has no digits after the decimal point. Finally, we get $2.317 \cdot 5 = 11.585$.

If the obtained product has less digits than we are to point off, then annexing zeros to the left, we point off the required number of decimal places. For example,

$$\begin{array}{r} 2.12 \\ \times 0.13 \\ \hline 636 \\ 212 \\ \hline 0.2756 \end{array} \qquad \begin{array}{r} 3.43 \\ \times 0.02 \\ \hline 0.0686 \end{array}$$

Let us now consider the multiplication of a decimal fraction by 10, 100, 1000, etc. Suppose we have to multiply the decimal 12.733 by 10. We have: $12\,733 \cdot 10 = 127\,330$. Pointing off three digits from the right, we get $12.733 \cdot 10 = 127.330$. But $127.330 = 127.33$. Hence, $12.733 \cdot 10 = 127.33$ Thus, multiplying a decimal by 10 is reduced to moving the decimal point one place to the right.

In general, *to multiply a decimal fraction by* 10, 100, 1000, *etc., we have to move decimal point one, two, three, etc. places to the right, respectively.*

Let us, finally, dwell on the divison of decimal fractions. Let it be required to divide 22.1 by 13. In this case, the division is performed in the same way as the division of a natural number by a natural number. The decimal point in the quotient is placed after the division of the integral part is completed. We have

$$\begin{array}{r|l} & 1.7 \\ 13) & 22.1 \\ & 13 \\ & \hline & 91 \\ & 91 \\ & \hline & 0 \end{array}$$

If the integral part of the dividend is less than the divisor, write a zero in the integral part of the quotient. For example,

$$\begin{array}{r|l} & 0.0017 \\ 13) & 0.0221 \\ & 0 \\ & \hline & 00 \\ & 0 \\ & \hline & 02 \\ & 0 \\ & \hline & 22 \\ & 13 \\ & \hline & 91 \\ & 91 \\ & \hline & 0 \end{array}$$

Let us now consider the division of a decimal by a decimal. Let it be required to divide 2.576 by 1.12. We multiply both the dividend and the divisor by 100, thus leaving the quotient unchanged. Now the decimal 257.6 must be divided by the natural number 112, that is, the problem has been reduced to the case just considered:

$$\begin{array}{r|l} & 2.3 \\ 112) & 257.6 \\ & 224 \\ & \hline & 336 \\ & 336 \\ & \hline & 0 \end{array}$$

Thus, $2.576 \div 1.12 = 2.3$.

In the set of decimals (the same as in the set of natural numbers) division is not always possible. In such cases we usually change to common fractions. Let us, for example, divide 2.8 by 0.09:

$$\begin{array}{r|l} & 31.11\ldots \\ \hline 9) & 280 \\ & 27 \\ \hline & 10 \\ & 9 \\ \hline & 10 \\ & 9 \\ \hline & 1 \end{array}$$

As a result, we have obtained a so-called *nonterminating decimal.* Let us now perform the division by changing to common fractions:

$$2.8 \div 0.09 = \frac{28}{10} \div \frac{9}{100} = \frac{28 \cdot 100}{10 \cdot 9} = \frac{280}{9} = 31\frac{1}{9}.$$

When performing operations on numbers, it may turn out that some of the given numbers are written as common fractions, others as mixed numbers, or decimals. In such cases, we may proceed in different ways: (1) change the decimals to common fractions and apply the rules for performing operations on common fractions; (2) change the common fractions and mixed numbers to decimals (if possible) and use the rules for performing operations on decimals.

Example. Evaluate the expression

$$\frac{\left(1\frac{1}{5} \div \left(\frac{17}{40} + 0.6 - 0.005\right)\right) \cdot 1.7}{\frac{5}{6} + 1\frac{1}{3} - 1\frac{23}{30}} + \frac{4.75 + 7\frac{1}{2}}{33 \div 4\frac{5}{7}} \div 0.25.$$

○ (1) $\frac{17}{40} + 0.6 - 0.005$. Here, it is convenient to change 17/40 to a decimal: $\frac{17}{40} = \frac{17 \cdot 25}{40 \cdot 25} = \frac{425}{1000} = 0.425$. Then

$$\frac{17}{40} + 0.6 - 0.005 = 0.425 + 0.600 - 0.005 = 1.020 = 1.02.$$

(2) $1\frac{1}{5} \div 1.02$. Carrying out the operation of division we often change to common fractions; we have

$$1\frac{1}{5} \div 1.02 = \frac{6}{5} \div \frac{102}{100} = \frac{6 \cdot 100}{5 \cdot 102} = \frac{20}{17}.$$

(3) $\frac{20}{17} \cdot 1.7 = \frac{20}{17} \cdot \frac{17}{10} = 2.$

(4) $\frac{5}{6} + 1\frac{1}{3} - 1\frac{23}{30} \quad \frac{5\,|\,5}{6} + \frac{4\,|\,10}{3} - \frac{53}{30} = \frac{25 + 40 - 53}{30} = \frac{12}{30} = \frac{2}{5}.$

(5) $2 \div \frac{2}{5} = \frac{2 \cdot 5}{2} = 5.$

(6) $4.75 + 7\frac{1}{2} = 4.75 + 7.50 = 12.25.$

(7) $33 \div 4\frac{5}{7} = \frac{33}{1} \div \frac{33}{7} = \frac{33 \cdot 7}{1 \cdot 33} = 7.$

(8)
$$\begin{array}{r|l} & 1.75 \\ 7 & 12.25 \\ \hline & 7 \\ \hline & 52 \\ & 49 \\ \hline & 35 \\ & 35 \\ \hline & 0 \end{array}$$

(9) $1.75 \div 0.25 = 175 \div 25 = 7.$

(10) $5 + 7 = 12.$ ●

5. Per cent. In practice, among decimals especially frequent use is made of the decimal 0.01 which is called a *per cent* (from the Latin 'per centum' for per hundred). Thus, the expression 'per cent' means a hundredth part. The symbol for per cent is %: $0.01 = 1\%$. In various economic and statistic calculations, in many fields of science and engineering parts of quantities are customarily expressed in per cent. For example, to find 23% of 60 kg, it is necessary to multiply the decimal 0.23 by 60, that is, $0.23 \cdot 60 = 13.8$. Hence, 23% of 60 kg is 13.8 kg.

Example. For the "Lada" car the following petrol consumption norms are established: 8 litres for summer and 9.2 litres for winter per 100 km of travel. By what per cent does the winter norm exceed the summer norm?

○ We see that the winter consumption of petrol per 100 km of travel is increased by 1.2 litres. Hence, we have to find the so-called percentage ratio of the numbers 1.2 and 8. To this end, it is necessary to form the ratio of 1.2 to 8, to change this ratio to a decimal, and then to write it in per cent. We find

$$\frac{1.2}{8} = \frac{12}{80} = \frac{3}{20} = \frac{15}{100} = 0.15 = 15\%,$$

that is, the winter norm exceeds the summer norm by 15%. ●

6. Nonterminating Repeating (Periodic) Decimals. Let us take a common fraction such that its denominator is factored into prime factors other than 2 and 5. As it was noted above, such a fraction cannot be represented as a decimal. By asserting so, we implied a *terminating decimal*, that is, a decimal containing a finite number of digits after the decimal point. But if we assume that infinitely many

decimal digits may occur after the decimal point, that is, if we consider nonterminating decimals, then it turns out that any common fraction can be represented as a decimal.

Let us take, for example, the fraction 3/14 and divide its numerator by the denominator, thus obtaining the decimal digits one by one. Here we should like to note that any whole number (integer) can be represented as a nonterminating decimal whose all decimal places are equal to zero (for instance, $3 = 3.00000000...$). We have

$$
\begin{array}{r|l}
 & 0.214285714\ldots \\
\hline
14 & 3.00000000\ldots \\
 & 30 \\
 & 28 \\
\hline
 & 20 \\
 & 14 \\
\hline
 & 60 \\
 & 56 \\
\hline
 & 40 \\
 & 28 \\
\hline
 & 120 \\
 & 112 \\
\hline
 & 80 \\
 & 70 \\
\hline
 & 100 \\
 & 98 \\
\hline
 & 20 \\
 & 14 \\
\hline
 & 60 \\
 & 56 \\
\hline
 & 4 \\
 & \ldots
\end{array}
$$

Thus, $3/14 = 0.214285714...$.

Let us write out, in succession, the remainders obtained while performing the division:

$$3, 2, 6, 4, 12, 8, 10, 2, 6, \ldots .$$

It is clear that all these remainders are less than the divisor, that is, less than the number 14. This means that at a certain step of the division a remainder which has been obtained before must appear. Thus, at the eighth step the remainder 2, which was obtained at the second step, reappeares. Moreover, it is clear that as soon as a remainder which has been obtained once appears, it will be followed by the same sequence of remainders which followed it before. In our example, the remainder 2 is followed by the remainder 6; then follows 4, then 12, and so on. Hence, we get such a sequence of remainders:

$$3, 2, 6, 4, 12, 8, 10, 2, 6, 4, 12, 8, 10, \ldots .$$

Periodically repeating blocks (sets) of remainders lead to a periodically repeating block (set) of digits in the decimal notation of a number. Thus, in our example we get

$$\frac{3}{14} = 0.2142857\,142857\,142857\ldots.$$

Such a repeating block (set) of digits after the decimal point in decimal notation of a number is called the *period*, and a nonterminating decimal having a period in its notation is called *periodic* (or *repeating*). A shorthand notation for the period is to write a bar or two dots over the repeating set of digits:

$$0.2142857142857142857\ldots = 0.\overline{2142857}.$$

If the period begins immediately after the decimal point, then the decimal is called *purely periodic*, and if there are other decimal digits between the decimal point and the period, then the decimal is termed *mixed periodic*. For example, $2.\overline{23} = 2.23232323\ldots$ is a purely periodic (repeating) decimal, whereas $0.2\overline{142857}$ is a mixed periodic (repeating) decimal.

It was shown above that the common fraction 3/14 can be represented in the form of a nonterminating repeating decimal, the periodicity appearing due to the repetition of remainders. This is characteristic for the division of any natural numbers, and therefore the following statement holds true.

Theorem. *Any common fraction can be represented in the form of a nonterminating periodic decimal.*

For the sake of generality, a terminating decimal is regarded as a nonterminating periodic decimal with zero as the period. For example, $21.377 = 21.377000\ldots = 21.377\overline{0}$.

Thus, we have learnt how to change common fractions to periodic (repeating) decimals. Let us now consider the inverse problem: change a repeating decimal to a common fraction. To solve this problem, assume (this is permissible, as we shall see later) that in order to multiply a nonterminating decimal by 10, 100, 1000, it suffices, as was the case for a terminating decimal, to move the decimal point one, two, or three places to the right.

For example, $0.1\overline{23}\cdot 100 = 0.1232323\ldots\cdot 100 = 12.323232\ldots = 12.\overline{32}$.

Let us give a few examples to show some methods of changing a repeating decimal to a common fraction.

Example 1. Change the decimal $0.\overline{13}$ to a common fraction.

○ We set $x = 0.\overline{13} = 0.131313\ldots$. We then multiply the purely periodic decimal x by a number such that the decimal point moves exactly one period to the right. Since the period consists of two digits,

the decimal point must be moved two places rightwards, for which purpose it suffices to multiply the number x by 100. Then

$$100x = 0.131313\ldots\cdot 100 = 13.1313\ldots = 13.\overline{13}.$$

Subtracting x from $100x$, we get

$$100x - x = 13.\overline{13} - 0.\overline{13}.$$

Hence, $99x = 13$, whence we find $x = \frac{13}{99}$. ●

Example 2. Change (convert) the number $2.\overline{273}$ to a common fraction.

○ We set $x = 2.\overline{273}$. This is a purely periodic decimal whose period contains three digits. Multiplying x by 1000, we get $1000x = 2273.\overline{273}$. Further, we have

$$1000x - x = 2273.\overline{273} - 2.\overline{273},\quad 999x = 2271,$$
$$x = \frac{2271}{999} = 2\,\frac{91}{333}.$$ ●

Example 3. Change the number $0.2\overline{54}$ to a common fraction.

○ We set $x = 0.2\overline{54}$. This is a mixed periodic decimal. Let us move its decimal point to the right so as to obtain a purely periodic decimal. To this end, it suffices to multiply x by 10; we get: $10x = 2.\overline{54}$. We now set $y = 2.\overline{54}$ and change this purely periodic decimal to a common fraction, as it was done in the previous examples. We have

$$\begin{gathered} y = 2.\overline{54} \\ 100y = 254.\overline{54} \\ \hline 100y - y = 254.\overline{54} - 2.\overline{54}, \\ 99y = 252,\quad y = \frac{252}{99} = \frac{28}{11}. \end{gathered}$$

Hence, $10x = \frac{28}{11}$, whence we find $x = \frac{28}{11 \cdot 10} = \frac{14}{55}$. ●

Example 4. Change the number $3.254\overline{9}$ to a common fraction.

○ Setting $x = 3.254\overline{9}$, we get $1000x = 3254.\overline{9}$. Putting $y = 1000x$, we then have

$$\begin{gathered} y = 3254.\overline{9} \\ 10y = 32549.\overline{9} \\ \hline 10y - y = 32\,549.\overline{9} - 3254.\overline{9}, \end{gathered}$$
$$9y = 29\,295,\quad y = 3255;\quad 1000x = 3255,\quad x = \frac{3255}{1000} = 3\,\frac{51}{200}.$$ ●

Note that $\frac{3255}{1000} = 3.255 = 3.255\overline{0}$, that is, we have obtained a terminating decimal or a nonterminating decimal with zero as the period. Hence, $3.254\overline{9} = 3.255\overline{0}$. This occurs for any decimal with 9 as the period: such a decimal can be represented by a decimal with a zero as the period. To do so, it only suffices to increase by unity the last decimal digit standing before the period.

For example, $0.45\overline{9} = 0.46\overline{0}$; $14.\overline{9} = 15.\overline{0}$, and so on.

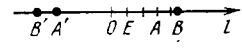


Fig. 1

7. Negative Numbers. Let us draw a straight line l, mark on it a point 0 which will be taken as the origin. We then choose a unit segment $0E$ and specify the direction (Fig. 1).

Let P be a set whose elements are all natural numbers and all common fractions. Let us establish a correspondence between the set P and the set of points of the straight line. Let, for instance, there be given the number 3. Laying off the line segment from the point 0 in the given direction three times, we obtain the point A which corresponds to the number 3. Let us now take the number $4\frac{1}{7}$. Laying off the line segment from the point 0 in the given direction four times and then $\frac{1}{7}$th part of this segment, we get the point B which exactly corresponds to the number $4\frac{1}{7}$. We can do the same for any number r. In general, to any number r from the set P there corresponds only one point of the straight line l.

If a point M of the line l corresponds to some number r, then this number is called the *coordinate of the point* M. In this case we write $M(r)$. Thus, we can show the coordinates of points E, A, and B in Fig. 1: $E(1)$, $A(3)$, and $B\left(4\frac{1}{7}\right)$. The coordinate of the point 0 is zero.

Let us now lay off three unit segments from the zero point in the direction opposite to the specified. We then get the point A' which is symmetric to the point A about the origin 0 (Fig. 1). The coordinate of the point A is the number 3, then the coordinate of the point A' is written as $A'(-3)$ and is read: "minus three". Analogously, the coordinate of the point B', which is symmetric to the point B (see Fig. 1), is the number $\left(-4\frac{1}{7}\right)$. The numbers 3 and -3, $4\frac{1}{7}$ and $-4\frac{1}{7}$ are said to be *opposite*. The numbers situated on the number line l in the specified direction (that is, being the coordinates of points located in the given direction from the origin) are said to

be *positive*; thus, $1, 3, 4\frac{1}{7}$, are positive numbers. At times, especially for emphasis, the plus sign is used: $+1$, $+3$, $+4\frac{1}{7}$. The numbers located on the number line in the direction opposite to the given one are said to be *negative*; thus, -3, -4, $-4\frac{1}{7}$, are negative numbers. The number zero is regarded to be neither positive nor negative, it separates positive numbers from negative ones.

The direction given for the straight line l is called the positive direction (usually from left to right), and the opposite direction is the negative direction. Such a line l is called the *number line.*

The *absolute value* of a number is defined as the distance from the origin to the point corresponding to that number. The absolute value of the number r is denoted by $|r|$.

Thus, to the number 3 there corresponds the point A (see Fig. 1), its distance from zero being equal to 3. Hence, the absolute value of the number 3 equals 3, that is, $|3| = 3$. Similarly, $|4\frac{1}{7}| = 4\frac{1}{7}$. To the number -3 there corresponds the point A' whose distance from zero is equal to 3. Hence, the absolute value of the number (-3) is equal to 3, that is, $|-3| = 3$. Similarly, $|-4\frac{1}{7}| = 4\frac{1}{7}$.

In general, *the absolute value of any positive number is the number itself; the absolute value of any negative number is equal to its opposite number; the absolute value of the number zero is zero.*

Let us recall the rules for performing operations on positive and negative numbers.

Addition. *The sum of two negative numbers is a negative number. To find the absolute value of a sum, it is necessary to add together the absolute values of the addends.*

Let us, for instance, evaluate the sum $(-12) + (-7)$. Since $|-12| = 12$, $|-7| = 7$, and $12 + 7 = 19$, we have $(-12) + (-7) = -19$.

The sum of two numbers having unlike signs is a number which has the same sign as the addend with greatest absolute value. To find the absolute value of the sum, it is necessary to subtract the smaller value from the larger one.

Let us, for instance, add together the numbers 12 and -7. We have $|12| = 12$, $|-7| = 7$, $12 - 7 = 5$; hence, the absolute value of the sum is equal to 5. Since 12 is greater than 7, the sum of the numbers 12 and -7 is positive: $12 + (-7) = 5$.

Let us also evaluate the sum $(-12) + 7$. Here, $|-12| = 12$, $|7| = 7$, $12 - 7 = 5$. We have $(-12) + 7 = -5$.

Subtraction. *To subtract one number from another, it is necessary to add to the minuend a number which is the opposite of the subtrahend.*

For example,

$$41 - 12 = 41 + (-12) = 29,\ 41 - (-12) = 41 + 12 = 53,$$
$$-41 - 12 = (-41) + (-12) = -53,$$
$$-41 - (-12) = (-41) + 12 = -29.$$

Multiplication and division. *The product (quotient) of two negative numbers is a positive number; the product (quotient) of two numbers having unlike signs is a negative number. To find the absolute value of a product (quotient), it is necessary to multiply (divide) the absolute values of these numbers.*

For example,

$$(-8)\cdot(-6) = 48,\ (-8)\cdot 6 = -48,\ 8 \div (-6) = -\frac{4}{3}.$$

8. Whole and Rational Numbers. Natural numbers, their opposites, and the number 0 form together the *set of whole numbers* (integers) which is denoted by **Z**.

Whole and fractional numbers (both positive and negative) form together the *set of rational numbers* denoted by **Q**.

The notation $m \in \mathbf{Z}$ means that m is a whole number, and the notation $r \in \mathbf{Q}$ implies that r is a rational number *.

Note that any rational number can be represented in the form of a ratio m/n, where m is a whole number, and n is a natural number, and the same number can be thus written in many ways. For example, $-2 = -4/2 = -6/3 = -100/50$; $0.3 = 3/10 = 6/20 = 300/1000$.

Among the fractions representing a given rational number there is one and only one irreducible fraction. For whole numbers this is a fraction with denominator 1.

For rational numbers, the operations of addition, subtraction, multiplication, and division (except division by zero) are defined, the following equalities being valid for any rational numbers a, b, c:

1°. $a + b = b + a$.
2°. $a + (b + c) = (a + b) + c$.
3°. $a + 0 = a$.
4°. $a + (-a) = 0$, where $(-a)$ is a number opposite to the number a.
5°. $ab = ba$.
6° $a\,(bc) = (ab)\,c$.
7° $a\,(b + c) = ab + ac$.
8°. $a \cdot 1 = 0$.

For any nonzero rational number a, there exists only one rational number x such that $ax = 1$. This number x is called the *multiplicative inverse* or *reciprocal* of the number a and is denoted by $1/a$.

* In general, the notation $x \in A$ means that the element x belongs to the set A.

For example, 1/3 is the reciprocal of the number 3, and -2 is the reciprocal of the number $-1/2$. The following equality holds true:

9^0. $a \cdot (1/a)$, where $a \neq 0$.

9. Variables. In mathematics, numbers are denoted by digits and letters. With the aid of digits, letters, algebraic signs, and parentheses we can compose various expressions. The letters entering an expression are referred to as *variables*, while the expression itself is called an *expression with variables*. Variables are usually denoted by small letters.

If the variables in an expression are assigned concrete numerical values, then the expression, as a whole, will take a concrete numerical value. For example, for $y = 1$ the expression $\frac{2y}{y+3}$ takes on the value $\frac{2 \cdot 1}{1+3} = \frac{1}{2}$, for $y = 0$ the value $\frac{2 \cdot 0}{0+3} = 0$, while for $y = -3$ this expression is meaningless (that is, has no numerical value).

Exercises

1. Change the given common fractions to decimals: (a) 12/5; (b) 3/625; (c) 11/3; (d) $-8/7$; (e) 41/24.

2. Change the given decimals to common fractions: (a) $0.\overline{3}$; (b) $2.\overline{18}$; (c) $0.\overline{03}$; (d) $5.2\overline{25}$.

In Exercises 3 to 18, perform the indicated operations:

3. $(+2.5) - (+4.9) + (-3.7) - (-5.8)$.

4. $\left(2 - \frac{1}{9} + \frac{4}{3}\right) - \left(\frac{3}{5} + \frac{2}{3} - \frac{1}{15}\right) + \left(-\frac{2}{9} + 1 - \frac{5}{3}\right)$.

5. $(-2) \cdot \left(+\frac{1}{4}\right) \cdot \left(-\frac{3}{5}\right) \cdot \left(-\frac{4}{3}\right)$.

6. $\left(\frac{1}{2} - \frac{1}{8}\right) \div (-2) - 16\frac{1}{4} \div (-4)$.

7. $5 \div \left(-\left(-\frac{1}{12}\right) \div \left(-\frac{1}{4}\right) + \frac{1}{6} \div (-2)\right) + 1\frac{1}{3}$.

8. $16.75 + \frac{10}{77}(70.84 \div 23) \div \left(2.025 - 1\frac{5}{6}\right) \div 4\frac{19}{24}$.

9. $\left(\left(8\frac{7}{55} - 6\frac{17}{100}\right) \cdot 1\frac{3}{217}\right) \div \left((0.4 - 0.15) \div \frac{1}{4}\right) + \left(\left(3\frac{5}{8} + 1.375\right) \div 0.5\right) \div \left(2\frac{3}{4} \div \left(3\frac{7}{20} - 2.8\right)\right)$.

10. $1.35 \div 2.7 + 6.02 - 5.9 + 0.4 \div 2.5 \cdot (4.2 - 1.075)$.

11. $\frac{(4.561 + 5.439) \cdot 0.1}{(7.01 - 5.01) \div 0.5} - \frac{(4.45 - 2.2) \div 0.3}{(0.823 + 0.177) \cdot 30}$.

12. $52 \div \left(\frac{6 \div (0.4 - 0.2)}{2.5\,(0.8 + 1.2)} + \frac{(34.06 - 33.81) \cdot 4}{6.84 \div (28.57 - 25.15)}\right) - 8$.

13. $(90.09 \div 91 + 3.774 \div 0.34) \div (232.31 \div 17.87 + 186.85 \div 5.05)$.

14. $24.57 \div 3.5 + \left(3.35 - 2\frac{13}{15} + \frac{5}{8}\right) \cdot \left(225 \div 12.5 - 3\frac{14}{19} \cdot 2\right)$.

15. $\left(17\frac{1}{18}\cdot 3.6-0.476\div 14\right)\div\left(0.009\cdot 8700-120\div 4\frac{2}{7}\right)+0.306\div 0.3.$

16. $\left(\left(4.625-\frac{13}{18}\cdot\frac{9}{26}\right)\div 2\frac{1}{4}+2\frac{1}{2}\div 1.25\div 6\frac{3}{4}\right)\div 1\frac{53}{68}.$

17. $\left(2\frac{3}{20}-0.645\div 0.3\right)\cdot\left(4\div 6\frac{1}{4}-0.2+\frac{1}{7}\cdot 1.96\right).$

18. $\dfrac{\left(0.666\ldots+\frac{1}{3}\right)\div 0.25}{0.1233\ldots\div 0.0925}+12.5\cdot 0.64.$

19. Which is greater: (a) $(-2.5)^{2n}$ or $(-1)^{2n}$, (b) $(-2.5)^{2n+1}$ or $(-1)^{2n+1}$ (n a natural number)?

20. Write in per cent: (a) 0.35, (b) 0.02, (c) 1.05, (d) 2.24.

21. Write in the form of decimals: (a) 5%, (b) 1.08%, (c) 120%, (d) 0.2%.

22. Find: (a) 5% of the number 120, (b) 70% of the number 320, (c) 1.02% of the number 150, (d) 120% of the number 3.

23. Find the percentage ratio of the given numbers: (a) 4 and 8, (b) 21 and 300, (c) 250 and 12.5.

24. 21 g of salt was added to 119 g of water. What is the percentage of salt in the obtained solution?

25. The original price of goods was 100 roubles. This price was reduced three times, each time by 10%. What was the price after the third reduction?

In Exercises 26 to 28, evaluate the given expressions:

26. $\dfrac{2a^2-4a-1}{a^2+a+1}$ for $a=-\frac{3}{4}$.

27. $\dfrac{(x-2y)\cdot x+z}{3y-z}$ for $x=2.5,\ y=0.72,\ z=1.04.$

28. $\dfrac{a^3-2ab\,(a-b)-b^3}{a^2+b\,(b-a)}$ for $a=-\frac{1}{2},\ b=-\frac{2}{3}$.

§ 1.3. Real Numbers

1. Irrational Numbers. In the preceding section we established that to every rational number r there corresponds a single point M (r) of the straight line l (the coordinate or number line) with the origin, direction, and scale specified; the number r was called the coordinate of the point M. Naturally, the following question arises: is the converse true? Does the only rational number, the coordinate of this point, correspond to any point of the straight line l? The answer to this question is negative; we are going to give an example of a point which has no rational coordinate.

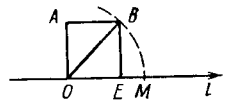


Fig. 2

Let us construct a square $OABE$ on the unit segment OE and lay off in the positive direction a line segment OM equal in length to the diagonal OB, that is, $OM = OB$ (Fig. 2).

Let us prove that the point M *does not correspond to any rational number.*

□ Let us assume the contrary, that is, that there is a rational number r which is the coordinate of the point M. Then $OM = r$;

but $OM = OB$, hence, $OB = r$. By the Pythagorean theorem, $OE^2 + EB^2 = OB^2$. Consequently, we have $1^2 + 1^2 = r^2$, that is, $r^2 = 2$. Since r is a positive rational number, it can be represented as an irreducible fraction $r = m/n$, where m and n are relatively prime natural numbers. We have

$$\left(\frac{m}{n}\right)^2 = 2, \quad \frac{m^2}{n^2} = 2, \quad m^2 = 2n^2.$$

The last equality means that m^2 is an even number. Then m is also an even number, that is, $m = 2k$. Substituting the expression $2k$ for m into the equality $m^2 = 2n^2$, we get

$$(2k)^2 = 2n^2, \quad 4k^2 = 2n^2, \quad 2k^2 = n^2.$$

The last equality means that n^2 is an even number; then n is also an even number.

Thus, m and n are even numbers which contradicts the assumption that m and n are relatively prime numbers. The contradiction thus obtained implies that there is no rational number r whose square is 2, and that the constructed point M has no rational coordinate. ■

But, nevertheless, it is natural to consider that the point M also has some coordinate. This coordinate, as we saw, is not a rational number, but a number of a new nature, that is, an *irrational number*; it is denoted by $\sqrt{2}$ (read: "a square root of two"). Analogously, we can prove that there is no rational number whose square is 5, 7, 10; the corresponding irrational numbers are denoted by $\sqrt{5}$, $\sqrt{7}$, $\sqrt{10}$. Their opposites are also irrational numbers; they are denoted as $-\sqrt{5}$, $-\sqrt{7}$, $-\sqrt{10}$, respectively.

It should be noted that irrational numbers appear not only as a result of seeking a number whose square is equal to a given positive number. For example, the number π expressing the ratio of the circumference of a circle to its diameter cannot be represented by a common fraction since it is irrational.

We know that the result of adding, subtracting, multiplying, and dividing (except dividing by zero) two rational numbers is a rational number. Therefore the sum of a rational and an irrational number is an irrational number. It is a similar thing for the cases of subtraction, multiplication, and division of two numbers one of which is rational, and the other irrational. And if an operation is performed on two irrational numbers, then the result may be a rational number. For example, $\sqrt{2} + (3 - \sqrt{2}) = 3$, $\sqrt{2} \cdot \sqrt{2} = 2$, $\pi - \pi = 0$, $\pi \div \pi = 1$.

2. **Real Numbers.** Rational and irrational numbers are united by the generic term *real numbers*. To every real number there corresponds a unique point on the coordinate line, and every point of the coordinate line corresponds to a unique real number. For the sake of brevity, we usually say "point a" instead of "point of the

number line corresponding to the real number a". Let us also agree, by the term "number a", to imply a real number a.

In practice, we do not distinguish between the set of real numbers and the set of points of the coordinate line applying to both sets the same term, vis. the *number line.*

The set of real numbers is customarily denoted by the capital letter **R**. The notation $x \in \mathbf{R}$ means: x is a member of the set of real numbers.

For any real numbers a, b, c the following equalities hold:

1^0. $a + b = b + a$.
2^0. $a + (b + c) = (a + b) + c$.
3^0. $a + 0 = a$.
4^0. $a + (-a) = 0$.
5^0. $ab = ba$.
6^0. $a(bc) = (ab)c$.
7^0. $a(b + c) = ab + ac$.
8^0. $a \cdot 1 = a$.
9^0. $a \cdot (1/a) = 1$; $a \neq 0$.

3. Numerical Inequalities and Their Properties. By comparing any real numbers a and b, we can say which of them is greater and which is less. The one lying further to the right on the number line is said to be greater than the other. If a is less than b, then we write $a < b$; if a is greater than b, then the notation is $a > b$. According to this definition, *any positive number is greater than zero, and any negative number is less than zero.*

On the basis of this definition, we may state the following: $a > b$ *if and only if the difference* $a - b$ *is a positive number*; $a < b$ *if and only if the difference* $a - b$ *is a negative number.*

For any given numbers a and b of the three relationships: $a < b$, $a > b$, $a = b$, one and only one is true.

The inequality signs $<$, $>$ are known as signs of *strict inequalities.* We sometimes use the signs $\geqslant$, $\leqslant$ which are the signs of *nonstrict inequalities.* The notation $a \leqslant b$ means that the number a is less than or equal to b, but only one of these two statements is true.

The notation $a < b < c$ means that $a < b$ and $b < c$ simultaneously and is said to be a *double inequality.*

Example. Compare the numbers 2/3 and 0.67.

○ We write the difference 2/3 — 0.67 and find its value:

$$\frac{2}{3} - 0.67 = \frac{2}{3} - \frac{67}{100} = \frac{200-201}{300} = -\frac{1}{300}.$$

Since the difference is negative, we have: $2/3 < 0.67$. ●

If two numbers a and b are connected by one of the following signs: $<, >, \leqslant, \geqslant$, then a numerical inequality is obtained. For example, $1 < 2$, $5 \geqslant 5$; $4 > 7$, $\sqrt{3} \leqslant 1$ are numerical inequalities. Every numerical inequality is actually the notation of a statement. For

instance, $1 < 2$ is the notation of the statement: "the number 1 is less than the number 2". Every statement may be true or false; accordingly, every numerical inequality may be correct or incorrect. Thus, of the four numerical inequalities written above the first two are correct, while the last two are incorrect.

It is hereafter assumed, that, when speaking of numerical inequalities, we imply only correct inequalities, that is, $a < b$ must be understood as the notation of the statement: the number a is less than the number b. The notations $a > b$, $a \leqslant b$, $a \geqslant b$ will be understood in a similar way.

Let us consider some properties of numerical inequalities:

1^0. *If* $a > b$, *then* $b < a$.

2^0. *If* $a > b$ *and* $b > c$, *then* $a > c$ **(transitivity).**

3^0. *If* $a > b$, *then* $a + c > b + c$, *where* c *is any real number.*

4^0. *If* $a > b$ *and* $c > 0$, *then* $ac > bc$.

□ Consider the difference $ac - bc$. We have $ac - bc = c(a - b)$. By hypothesis, $c > 0$. Further, since $a > b$, the difference $a - b > 0$. But the product of two positive numbers is a positive number and, hence, $c(a - b) > 0$. Therefore $ac - bc > 0$, that is, $ac > bc$. ■

The following two properties are proved in a similar way:

5^0. *If* $a > b$ *and* $c < 0$, *then* $ac < bc$.

6^0. *If* $a > b$ *and* $c \geqslant d$, *then* $a + c > b + d$.

Their proof is left to the reader.

Remarks. 1. Properties 1^0, 2^0, and 3^0 can be proved by the same method as Property 4^0 (the reader is recommended to carry out the appropriate proofs).

2. Properties 4^0 and 5^0 mean the following: *if both sides of a correct inequality are multiplied by the same positive number, and the sign of the original inequality remains unchanged, then a correct inequality is obtained; if both sides of a correct inequality are multiplied by the same negative number, and the sign of the original inequality is reversed, then a correct inequality is obtained.*

3. Property 6^0 means that *if two correct inequalities of the same sense are added termwise and the sense is retained, then a correct inequality is obtained.*

Let us dwell on one more property of inequalities.

7^0. *If* a, b, c, d *are positive numbers, where* $a > b$ *and* $c > d$, *then* $ac > bd$.

□ We have $a > b$, $c > 0$; then, by virtue of Property 4^0, $ac > bc$. Similarly, $c > d$, $b > 0$ and, hence, $bc > bd$. But, according to the property of transitivity, from the inequalities $ac > bc$, $bc > bd$ there follows the inequality $ac > bd$. ■

Thus, *termwise multiplication of correct inequalities of the same sense with positive terms yields a correct inequality having the same sign as the original inequalities.*

Remark. Properties 1^0-7^0 hold true not only for the sign $>$, but also for each of the remaining inequality signs $<, \geqslant, \leqslant$. For instance, the following properties are valid:

$2'$. If $a < b$ and $b < c$, then $a < c$.

$2''$. If $a \geqslant b$ and $b \geqslant c$, then $a \geqslant c$.

$4'$. If $a < b$ and $c > 0$, then $ac < bc$.

$5'$. If $a \geqslant b$ and $c < 0$, then $ac \leqslant bc$.

$6'$. If $a < b$ and $c < d$, then $a + c < b + d$.

4. The Integral and Fractional Parts of a Number. The *integral part of a number a* is the greatest whole number not exceeding the number a. The integral part of the number a is denoted by the symbol $[a]$. For any a we have $[a] \leqslant a < [a] + 1$.

For example: $[3.7] = 3$, $[5.128] = 5$, $[7] = 7$, $[-0{,}7] = -1$, since $-1 < -0.7 < 0$, $[-5.8] = -6$.

The *fractional part* of the number a is the difference between a and its integral part. The fractional part of the number a is denoted by the symbol $\{a\}$. Hence, $\{a\} = a - [a]$.

Since $[a] \leqslant a < [a] + 1$, it is possible to write that $0 \leqslant a - [a] < 1$ or $0 \leqslant \{a\} < 1$, that is, the *fractional part of a number is a nonnegative number less than unity.* For example:

$\{3.7\} = 3.7 - [3.7] = 0.7$,
$\{5.128\} = 5.128 - [5.128] = 0.128$
$\{7\} = 7 - [7] = 0$,
$\{-0.7\} = -0.7 - [-0.7] = -0.7 - (-1) = 0.3$,
$\{-5.8\} = -5.8 - [-5.8] = -5.8 - (-6) = 0.2$.

From the equality $\{a\} = a - [a]$ it follows that $a = [a] + \{a\}$, that is, *any number a can be written as the sum of its integral and fractional parts.* For example:

$3.7 = 3 + 0.7 = [3.7] + \{3.7\}$,
$5.128 = 5 + 0.128 = [5.128] + \{5.128\}$,
$7 = 7 + 0 = [7] + \{7\}$,
$-0.7 = -1 + 0.3 = [-0.7] + \{-0.7\}$,
$-5.8 = -6 + 0.2 = [-5.8] + \{-5.8\}$.

5. Infinite (or Nonterminating) Decimals. Let a be a positive real number and $[a] = n$. Consider the numbers $n + \frac{1}{10}$, $n + \frac{2}{10}$, $n + \frac{3}{10}$, . . ., $n + \frac{9}{10}$. There is n_1 such that

$$n + \frac{n_1}{10} \leqslant a < n + \frac{n_1 + 1}{10} \quad (0 \leqslant n_1 \leqslant 9)$$

or, which is the same,

$$n.n_1 \leqslant a < n.n_1 + \frac{1}{10}.$$

Let us now consider the numbers $n.n_1$, $n.n_1+\frac{1}{100}$, $n.n_1+\frac{2}{100}$, ..., $n.n_1+\frac{9}{100}$ and find n_2 such that

$$n.n_1n_2 \leqslant a < n.n_1n_2+\frac{1}{10^2}.$$

Continuing this process, we get a sequence of decimals

$$n,\ n.n_1,\ n.n_1n_2,\ \ldots,\ n.n_1n_2\ldots n_k,\ \ldots, \tag{1}$$

such that for any k we have

$$n.\ n_1n_2\ldots n_k \leqslant a < n.n_1n_2\ldots n_k+\frac{1}{10_k} \tag{2}$$

The decimals $n.n_1n_2\ldots n_k$ yield *rational approximations to a by deficiency*, while the fractions $n.\ n_1n_2\ldots n_k+\frac{1}{10^k}$ *rational approximations to a by excess*. These are called *decimal* approximations.

Instead of a sequence of decimals of form (1), we may write a nonterminating decimal $n.n_1n_2\ldots n_k\ldots$. If we truncate this notation at some decimal digit n_k, then we get an approximation to a by deficiency, adding the number $\frac{1}{10^k}$ to the truncated decimal, we find the approximation to a by excess.

If the obtained decimal is periodic, then, as we established in § 1.2, it is the notation of a rational number; if the obtained decimal is nonperiodic, then it is the notation of an irrational number.

We can represent negative numbers in a similar way. In this case, we write a bar over the integral part of a negative number. For example, $-3=\overline{3}.000\ldots$, $-7.23=-8+0.77=\overline{8}.77000\ldots$. Thus, it turns out that *any real number can be represented by a nonterminating decimal.*

Example. Using the above method, construct the sequence of rational approximations to the number $\sqrt{2}$ by deficiency and by excess.

○ We have

$$1^2<2<2^2,$$
$$1.4^2<2<1.5^2,$$
$$1.41^2<2<1.42^2,$$
$$1.414^2<2<1.415^2,$$
$$1.4142^2<2<1.4143^2,$$
$$1.41421^2<2<1.41422^2.$$

Hence, for the number $\sqrt{2}$ we get a sequence of rational approximations by deficiency:

$$1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ 1.41421,\ \ldots$$

and a sequence of rational approximations by excess:

$$2,\ 1.5,\ 1.42,\ 1.415,\ 1.4143,\ 1.41422\ \ldots\ .$$

We may write that $\sqrt{2} = 1.41421\ldots$.

The terms of the first sequence are the approximations to the number $\sqrt{2}$ by deficiency accurate to 1, 0.1, 0.01, 0.001, 0.0001, 0.00001 and so on, while the terms of the second sequence by excess with the corresponding accuracy. ●

Let us take a number line l with two points O and E marked on it. We shall assume that the distance between these points is equal to unity, that is, $OE = 1$ (Fig. 3). Then every point M of the ray OE

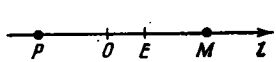


Fig. 3

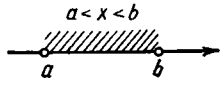


Fig. 4

Fig. 5

corresponds to the only nonnegative number $x_M = OM$, the coordinate of the point M. The notation of this number x_M in the form of a nonterminating decimal is obtained gradually in the process of measuring the line segment OM by deficiency accurate to 1, 0.1, 0.01, 0.001, ... The converse is also true: to every nonnegative number there corresponds (in the indicated sense) the only point of the ray OE.

The point O divides the straight line l into two rays: a positive ray OE and an oppositely directed negative ray. If the point P belongs to the negative ray (Fig. 3), then its coordinate is the number $x_P = -|OP|$.

6. Number Intervals. Let us take two numbers a and b (let $a < b$) and mark them by points on the number line (Fig. 4). Let us also take an arbitrary point x lying between a and b; then $x > a$ and $x < b$. Usually, instead of these two inequalities, we use the following notation: $a < x < b$ (so-called double inequality).

The set of all such real numbers x each of which satisfies the double inequality $a < x < b$ is denoted as (a, b) and is called an *open interval*. Figure 4 gives a geometrical representation of the open interval.

Consider now the set of numbers x such that $a \leqslant x \leqslant b$. It is denoted by $[a, b]$ and is called a *closed interval*. Geometrical representation of the closed interval $[a, b]$ is shown in Fig. 5.

Comparing Figs. 4 and 5, we see that the end points (a and b) of the closed interval are indicated by small closed circles (as in Fig. 5), whereas the end points of the open interval by open circles (see Fig. 4).

Figure 6 shows *half-open intervals* $[a, b)$ and $(a, b]$, that is, the sets of such numbers x which satisfy the inequalities $a \leqslant x < b$ and $a < x \leqslant b$, respectively.

The set of all numbers x such that $x > a$, denoted by $(a, +\infty)$, is called an *open ray*. The symbol $+\infty$ is read: "plus infinity". The geometrical representation of an open ray is given in Fig. 7. The set of all numbers x such that $x \geqslant a$, denoted by $[a, +\infty)$,

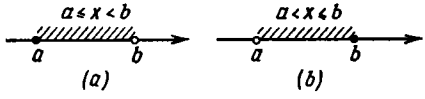


Fig. 6

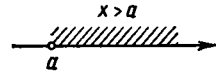


Fig. 7

is called a *ray*; the geometrical representation of a ray is given in Fig. 8. An open ray may also have the form $(-\infty, b)$, and a ray, the form $(-\infty, b]$ (see Fig. 9).

7. Absolute Value of a Real Number. In § 1.2 (Item 7) we introduced the notion of the absolute value of a rational number: the absolute value of the rational number r is understood as the distance

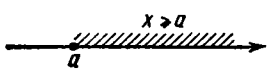


Fig. 8

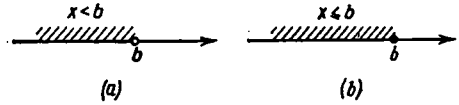


Fig. 9

of the point M (r) of the number line from the origin. It was shown that $|r| = r$ if $r \geqslant 0$, and $|r| = -r$, if $r < 0$. Just in the same way, it is possible to introduce the notion of the absolute value of any real number, but we usually proceed in a different way: we introduce a definition which is not connected with the number line.

Definition. The *absolute value of a real number* x is the number x itself if $x \geqslant 0$ and its opposite if $x < 0$. The absolute value of the number x is denoted by $|x|$. Thus,

$$x = \begin{cases} x \text{ for } x \geqslant 0, \\ -x \text{ for } x < 0. \end{cases}$$

For example, $|3.7| = 3.7$ since $3.7 > 0$, $|-\sqrt{2}| = -(-\sqrt{2}) = \sqrt{2}$ since $-\sqrt{2} < 0$.

Geometrically, $|x|$ means the distance from the point x to the zero point on the number line.

Example 1. Represent in the form of a number interval or in the form of a union of two number intervals the set M consisting of real numbers x such that: (1) $|x| < 3$, (2) $|x| \leqslant 1$, (3) $|x| > 2.7$, (4) $|x| \geqslant 4$.

○ (1) As is known, $|x|$ is the distance from the point x to the origin. Hence, the set M consists of all such points x whose distance from the point O is less than 3. Let us mark on the number line the points whose distance from the origin is equal to 3: these are -3 and 3. Then the set M is the interval from -3 to 3; $M = (-3, 3)$ (Fig. 10). Note that the notation $|x| < 3$ is equivalent to the notation $-3 < x < 3$.

(2) Reasoning just as in the preceding case, we get $M = [-1, 1]$ (Fig. 11). Note that the notation $|x| \leqslant 1$ is equivalent to the notation $-1 \leqslant x \leqslant 1$.

(3) The set M consists of all the numbers whose distance from the origin is greater than 2.7. Let us mark on the number line the points

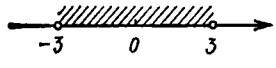


Fig. 10

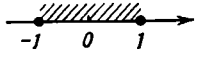


Fig. 11

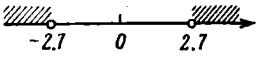


Fig. 12

whose distance from the origin is equal to 2.7; these are the points -2.7 and 2.7. Then, the set M consists of two open rays: from $-\infty$ to -2.7 and from 2.7 to $+\infty$ (Fig. 12). This can be written in the following way: $M = (-\infty, -2.7) \cup (2.7, +\infty)$, the symbol $\cup$ being the *union sign*.

(4) We have $M = (-\infty, -4] \cup [4, +\infty)$ (Fig. 13). ●

Theorem 1.3.1. *The distance* $\rho(x, a)$ *between the points* x *and* a *on the number line is equal to* $|x - a|$, *that is,* $\rho(x, a) = |x - a|$.

□ If $x > a$, then $\rho(x, a) = x - a$ (Fig. 14*a*); and if $x < a$, then $\rho(x, a) = a - x$ (Fig. 14*b*); if, finally, $x = a$, then $\rho(x, a) =$

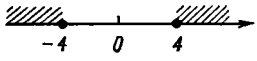


Fig. 13

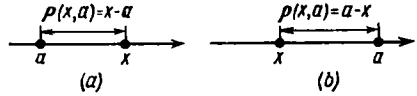


Fig. 14

$0 = x - a$. On the other hand, $|x - a| = x - a$ for $x > a$ or $x = a$; $|x - a| = a - x$ for $x < a$. Hence, for any arrangement of the points x and a we get $\rho(x, a) = |x - a|$. ■

Example 2. Find numbers x such that: (1) $|x - 2| = 3$; (2) $|x + 3| = 1$.

○ (1) Let us formulate the problem geometrically: $|x - 2|$ is the distance between the points x and 2 on the number line; hence, we have to find such points x whose distance from the point 2 is equal to 3. These points are $x_1 = -1$ and $x_2 = 5$ (Fig. 15).

(2) Since $|x + 3| = \rho(x, -3)$, it is necessary to find such points x whose distance from the point -3 is equal to 1. These points are $x_1 = -4$ and $x_2 = -2$ (Fig. 16). ●

Example 3. Represent in the form of a number interval or a union of number intervals the set M consisting of real numbers x such that: (1) $|x-1|<2$, (2) $|x+2|\leqslant 3$, (3) $|x-a|<r$, where $r>0$, (4) $|x-a|>r$, where $r>0$.

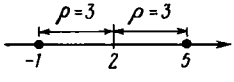


Fig. 15

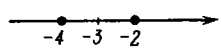


Fig. 16

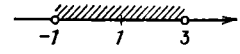


Fig. 17

○ (1) Let us represent the set of points x whose distance from the point 1 is less than 2. These are all the points contained in the open interval $(-1, 3)$ (Fig. 17).

(2) The points x whose distance from the point -2 is less than, or equal to, 3 fill in the closed interval $[-5, 1]$ (Fig. 18).

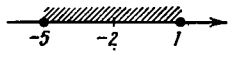


Fig. 18

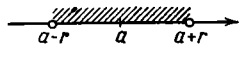


Fig. 19

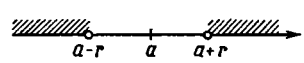


Fig. 20

Reasoning in a similar way, we get: (3) $(a-r, a+r)$ (Fig. 19); (4) $(-\infty, a-r)\cup(a+r, +\infty)$ (Fig. 20). ●

Note some properties of absolute values.

1°. *For any number x the equality $|x|=|-x|$ is valid.*

2°. *For any numbers x, y the following equalities hold true:*

$$(1)\ |xy|=|x|\cdot|y|,\quad (2)\ \left|\frac{x}{y}\right|=\frac{|x|}{|y|}\ (y\neq 0),\quad (3)\ |x|^2=x^2.$$

3°. *For any x the inequality $|x|\geqslant 0$ is valid.*

4°. *The inequality $|x|<r$, where $r>0$, defines the same set of points on the number line as the double inequality $-r<x<r$ does; the inequality $|x-a|<r$, where $r>0$, specifies the same set of points on the number line as the double inequality $a-r<x<a+r$ does.*

5°. *The union of two open rays $(-\infty, -r)\cup(r, +\infty)$ on the number line corresponds to the inequality $|x|>r$, where $r>0$; to the inequality $|x-a|>r$, where $r>0$, the union of two open rays $(-\infty, a-r)\cup(a+r, +\infty)$.*

6°. *For any numbers x, a the inequality $|x+a|\leqslant|x|+|a|$ is valid.*

□ We have $|x+a|=\rho(x,-a)$, $|x|=\rho(x,0)$, $|a|=|-a|=\rho(-a,0)$. Let us take advantage of the fact that if three points are taken on the number line, then the distance between two of them does not exceed the sum of the distances from each of them to the third one. In particular, $\rho(x,-a)\leqslant\rho(x,0)+\rho(-a,0)$. This just means that $|x+a|\leqslant|x|+|a|$. ■

The obtained inequality is sometimes called the **triangle inequality.**

8. Proportions. Let a, b, c, and d be real numbers other than 0. They are called *proportional* if

$$a/b = c/d.$$

This equality is called the *proportion*; the numbers a and d are termed the *extremes* of the proportion, and the numbers b and c the *means* of the proportion.

For example, the numbers 2, 6, 4, 12 are proportional since $2/6 = 4/12$. It is also possible to form a proportion from the numbers 2.5, -4, -5, and 8: $2.5/(-4) = (-5)/8$.

The following statement holds true.

Theorem 1.3.2. *The product of the extremes of a proportion is equal to the product of its means.*

□ Let us take advantage of the fact that the equality $a/b = m$ is equivalent to the equality $a = bm$. Let $a/b = c/d$. Then $a = b\frac{c}{d}$. But $\frac{c}{d} = c\cdot\frac{1}{d}$. Hence, $a = b\cdot\frac{c}{d} = bc\cdot\frac{1}{d} = (bc)\cdot\frac{1}{d} = \frac{bc}{d}$. From the equality $\frac{bc}{d} = a$ it follows that $bc = ad$. ■

The above statement is called the *fundamental rule of proportion*.

We have proved that from the equality $a/b = c/d$ there follows the equality $ad = bc$. The converse is also true.

Theorem 1.3.3. *If $ad = bc$, then the numbers a, b, c, d are proportional, that is, $a/b = c/d$.*

□ Indeed, if $ad = bc$, then $a\frac{bc}{d} = b\frac{c}{d}$, and then $a/b = c/d$. ■

Corollary 1. *The extremes of the proportion can be interchanged, that is, if $a/b = c/d$, then $d/b = c/a$.*

□ Indeed, if $a/b = c/d$, then $ad = bc$ or, which is the same, $da = bc$. But from the equality $da = bc$ it follows that $d/b = c/a$ (by Theorem 1.3.3). ■

The following assertion is proved in a similar way.

Corollary 2. *The means of the proportion can be interchanged, that is, if $a/b = c/d$, then $a/c = b/d$.*

9. Axiomatics of Real Numbers. The development of the concept of a real number lasted for a long time. The geometric form of this concept was described by R. Descartes* at the beginning of the seventeenth century in his only mathematical book *La geometrie*, but rigorous theories of real numbers were created only in the second half of the nineteenth century. These theories yield concrete realizations of such numbers in the form of infinite decimals or using some other methods.

* Descartes, René (1597-1650). Philosopher and mathematician; born in France, lived in several countries of Western Europe, settled in Holland.

However, it is not the method of introducing real numbers that is of essential importance for mathematics, but just the properties of the set **R** of real numbers. We can mention the principal properties from which the rest of the properties of these numbers follow. These basic properties define the set **R** of real numbers axiomatically, that is, they are **axioms** specifying this set.

Addition and multiplication are the fundamental operations in the set **R**: any two real numbers a and b are associated with the numbers $a + b$ (the sum of a and b) and ab (the product of a and b). In addition, the relationship "a is less than b ($a < b$)" or, which is the same, "b is greater than a ($b > a$)" is introduced into the set of real numbers. A special part is played by the numbers 0 and 1.

Numbers greater than zero are called *positive*, and numbers less than zero are termed *negative*.

Axioms of addition. For any a, b, c the following properties are valid:

I. $a + b = b + a$.

II. $(a + b) + c = a + (b + c)$.

III. $a + 0 = a$.

IV. *For any number a there is a number b such that $a + b = 0$.* This number b is said to be opposite to the number a and is denoted by $-a$.

Axiom IV enables us to introduce the operation of **subtraction** into the set of real numbers: by the difference $a - b$ the sum $a + (-b)$ is understood.

Axioms of multiplication. For any a, b, c the following properties are valid:

V. $ab = ba$.

VI. $(ab)\,c = a\,(bc)$.

VII. $a \cdot 1 = a$.

VIII. *For any nonzero number a there is a number b such that $ab = 1$.* This number is said to be *inverse* (or reciprocal) of the number a and is denoted by $\frac{1}{a}$.

IX. $(a + b)\,c = ac + bc$.

Axiom VIII enables us to introduce the operation of **division** into the set of real numbers: by $\frac{a}{b}$ the product $a \cdot \frac{1}{b}$ (where $b \neq 0$) is understood.

Axioms of order. For any a, b, c the following properties are valid:

X. *$a < a$ is a false proposition.*

XI. *If $a \neq b$, then either $a < b$ or $b < a$.*

XII. *If $a > b$ and $b > c$, then $a > c$* **(transitivity).**

XIII. *If $a > 0$ and $b > 0$, then $ab > 0$.*

XIV. *If $a > b$, then $a + c > b + c$.*

XV. *If $a > 0$ and $b > 0$, then there is a natural number n such that $na > b$* **(axiom of Archimedes).**

The above fifteen axioms are fulfilled in the set **Q** of rational numbers. The principal difference between the sets **Q** and **R** consists in that there are "holes" in the set **Q** (for instance, a "hole" for the number $\sqrt{2}$, a "hole" for the number π; in other words, the interval from the rational number a to the rational number b ($a < b$) is not filled completely with rational numbers, that is, there are some gaps), whereas the set **R** has no "holes", it is said to be continuous. This continuity of the set **R** is expressed by a special axiom. Prior to formulating this axiom (called the axiom of continuity), let us introduce a number of new notions.

Any set consisting of real numbers will be called a *number set*. A number set M is said to be *bounded above* if there is a number p such that for all $x \in M$ the inequality $x \leqslant p$ is valid. The number p is called the *upper bound* of the set M. A number set M is said to be *bounded below* if there is a number m such that for all x from M the inequality $x \geqslant m$ holds. The number m is called the *lower bound* of the set M.

For example, the set of all negative numbers is bounded above; the upper bound can be represented by any positive number or by the number 0. The set

of all positive numbers is bounded below. The lower bound can be represented by any negative number or by the number 0.

Let us now state the **axiom of continuity** of the set of real numbers.

XVI. *If a number set is bounded above, then there is the least bound among its upper bounds.*

Let us clarify this axiom. Suppose we discard some number a from the set $\mathbf{R}$; then the entire set $\mathbf{R}$ would fall into two parts: a set A consisting of numbers less than a and a set B containing numbers greater than a. The set A is bounded above, but among its upper bounds (these bounds form the set B) there is no least bound (this least bound must be represented just by the discarded number a). Thus, in a "punctured" set $\mathbf{R}$ the axiom of continuity is not valid. In other words, the axiom of continuity means that the set $\mathbf{R}$ has neither "punctures", nor "holes".

The above axioms of algebra, the same as the axioms of geometry, are accepted without proof. These self-evident principles are a generalized century-old experience of practical activity of man. The axioms of a mathematical system are the basic propositions from which all other propositions can be derived: in geometry theorems are deduced from the axioms of geometry; similarly, algebraic statements are derived from the axioms of real numbers.

As an example, let us show how, based on axioms, the rule for removing parentheses in multiplication is derived:

$$(a + b)(c + d) = ac + ad + bc + bd.$$

□ Denoting $c + d$ by m, we get $(a + b)(c + d) = (a + b)m$. According to axiom IX, $(a + b)m = am + bm$, and, by axiom V, $am = ma$, $bm = mb$. Hence

$$(a + b)(c + d) = ma + mb = (c + d)a + (c + d)b.$$

Applying axiom IX to the numbers $(c + d)a$ and $(c + d)b$, we get, respectively, $ca + da$ and $cb + db$ or, which is the same (by axiom V), $ac + ad$ and $bc + bd$. Thus, $(a + b)(c + d) = ac + ad + bc + bd$. ■

Let us also show that if $a > b$ and c is a positive number ($c > 0$), then $ac > bc$.

□ Consider the difference $ac - bc$. We have $ac - bc = (a - b) \cdot c$ (by axiom IX). By hypothesis, c is a positive number, and $a - b$ is also a positive number. According to axiom XIII, the product of two positive numbers is a positive number, and, hence, $(a - b) \cdot c > 0$. Thus, $ac - bc > 0$, and therefore $ac > bc$. ■

10. The Principle of a Separating Number. Let A and B be two nonempty sets of numbers, and for any a from A and any b from B the inequality $a \leqslant b$ holds true. In this case, we shall say that the *set A lies to the left of the set B.*

For example, the set $\mathbf{R}_-$ of negative real numbers lies on the left of the set $\mathbf{R}_+$ of positive real numbers. The set X consisting of the numbers lying between 3 and 5 is to the left of the set Y consisting of the numbers lying between 6 and 9.

Let the set A be to the left of the set B. Then the set A is bounded above (any element of the set B is the upper bound for A), and, hence, by the axiom of continuity, it has the least upper bound c. This number c possesses the following property: if $a \in A$ and $b \in B$, then $a \leqslant c \leqslant b$. Hence, the number c lies as if between the sets A and B, thus separating these sets. Therefore it is called the *separating number.*

Thus, we have proved the following assertion.

If the set A is to the left of the set B, then there is a number separating these sets (the **principle of a separating number**).

Thus, in the examples considered above we have: the sets $\mathbf{R}_-$ and $\mathbf{R}_+$ are separated by the number 0; the sets X and Y are separated by any number lying between 5 and 6. As is seen from these examples, both a unique separating

number and infinitely many separating numbers are possible. This depends on how close the sets under consideration are to each other: if they contain arbitrarily close elements (that is, such $a \in A$ and $b \in B$ that the difference $b - a$ is arbitrarily small), then the separating number is unique. This result is described more precisely in the following way.

The test for uniqueness of a separating number. *If a set A lies to the left of a set B, then for the separating number to be unique, it is necessary and sufficient that the following condition be satisfied: for any given positive number ε there are a from A and b from B such that $b - a < \varepsilon$.*

We shall not prove this assertion here.

Exercises

1. Compare the numbers a and b if: (a) $a = \frac{44}{71}$, $b = \frac{44}{69}$, (b) $a = 2\frac{7}{12}$, $b = 2\frac{23}{42}$, (c) $a = 1.41$, $b = 1\frac{3}{7}$, (d) $a = -8$, $b = -7.9$, (e) $a = -\frac{4}{7}$, $b = -\frac{13}{22}$, (f) $a = -1\frac{101}{102}$, $b = 0.0001$.

2. Prove that any proper common fraction is less than unity, and any improper common fraction is greater than unity.

3. Arrange the given numbers in increasing order: $-2\frac{1}{8}$, -3, $\frac{2}{7}$, 0, $-2\frac{1}{7}$, $\frac{3}{14}$.

4. Represent on the number line the number interval consisting of all real numbers x such that (a) $-1 < x < 2$, (b) $0 \leqslant x \leqslant 5$, (c) $-2 \leqslant x < 0$, (d) $-2/5 < x \leqslant 2.5$, (e) $x > 4.5$, (f) $x \geqslant -2.7$, (g) $x < 0$, (h) $x \leqslant -5$, (i) $|x| < 8.3$, (j) $|x| > 1.5$, (k) $|x| \leqslant 7/3$, (l) $|x| \geqslant 1/5$.

5. Represent on the number line the number interval M and describe the set of numbers belonging to this interval with the aid of the inequality signs: (a) $M = (2, 8)$, (b) $M = [-1, +\infty)$, (c) $M = (0, +\infty)$, (d) $M = [-10, 0]$, (e) $M = (-\infty, -4]$, (f) $M = (-\infty, 2.3)$, (g) $M = [-4, 1)$, (h) $M = (1, 4]$.

6. Find the whole numbers belonging to the given interval A: (a) $A = (-3.1, 8.3]$, (b) $A = [-3, 9)$, (c) $A = [-2, +\infty)$, (d) $A = (-\infty, 1]$, (e) $A = [-7, 0.3]$, (f) $A = (1, 9/7)$.

7. Show on the number line that if $a > b$, then $-a < -b$. Consider the following cases: a and b are positive; a and b are negative; a is positive and b is negative; $a = 0$ and b is negative.

8. What sign has the number a if: (a) $3a < 2a$, (b) $5a > 2a$, (c) $-2a < 5a$, (d) $-6a > -5a$?

9. It is known that $a > b$. On the basis of what properties of inequalities is it possible to assert that the following inequalities are correct: (a) $-5a < -5b$, (b) $\frac{a}{3} > \frac{b}{3}$, (c) $3a + 5 > 3b + 5$, (d) $0.1a - 0.4 > 0.1b - 0.4$, (e) $2 - a < 2 - b$, (f) $2 - \frac{a}{3} < 3 - \frac{b}{3}$?

10. Find the unknown term of the proportion: (a) $12/x = (-48)/96$, (b) $x/0.02 = 27.4/548$.

11. Give a few positive values of the variable for which the value of $\sqrt{a}$ is: (a) an irrational number; (b) a rational number.

12. Find the first three decimal approximations (both by deficiency and by (excess) and the first three decimal digits of the following numbers: (a) 3/14, b) $-10/3$, (c) $\sqrt{3}$; (d) $-\sqrt{5}$.

13. It is known that $x = 0.5638413...$, $y = 1.3414825...$. Find the first five decimal digits of the sum $x + y$.

14. It is known that $x = 2.1468...$, $y = 1.5431...$. Find the first two decimal digits of the product xy.

15. Prove that the numbers 2.2 and 2.3 are decimal approximations to the number $\sqrt{5}$ (by deficiency and by excess, respectively) accurate to 0.1.

16. What can be said about the numbers x and y if: (a) $x^2 = y^2$, (b) $|x| = |y|$, (c) $|x| + |y| = 0$, (d) $x^2 + y^2 = 0$, (e) $|x + y| = 0$?

17. Find the numerical values of the following expressions:

(a) $\dfrac{|a| - |b|^3}{c}$ for $a = -3$, $b = -2$, $c = 3$; (b) $\dfrac{xy^2}{|z|}$ for $|x| = 3$, $|y| = 1$, $z = -3$.

18. Indicate on the number line the set of such points x for which the given relationship holds true: (a) $|x - 4| = 2.5$, (b) $|x + 1.5| = 3$, (c) $|x - 3| < 0.5$, (d) $|x + 1.5| \leqslant 1.5$, (e) $|x - 4| > 4$, (f) $|x + 2| \geqslant 1.7$, (g) $|x + 1| > 0$.

19. Find the integral and the fractional part of the number x if: (a) $x = 12.4$, (b) $x = -4$, (c) $x = \sqrt{3}$, (d) $x = 17/5$, (e) $x = -0.374$.

§ 1.4. Simplest Transformations of Integral Rational Expressions

An expression composed of numbers and variables using a finite number of arithmetic operations (addition, subtraction, multiplication, and division) is said to be a *rational expression*. A rational expression is called *integral* if it involves no division by an expression containing variables. The present section is devoted to integral rational expressions.

1. Power with a Natural Exponent. Let a be a real number, and $n > 1$ a natural number. The product of n factors, each being equal to a, is called the nth *power of the number* a and is denoted by a^n; if $n = 1$, then we set $a^1 = a$. The number a repeated as a factor is called the *base of the power*, and the number n the *exponent of the power*. For example, $2^3 = 2 \cdot 2 \cdot 2 = 8$, $\left(\frac{1}{3}\right)^4 = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{81}$.

Recall the properties of the power with a natural exponent.

1^0. $a^m \cdot a^n = a^{m+n}$.

2^0. $a^m \div a^n = a^{m-n}$ if $m > n$.

3^0. $(a^m)^n = a^{mn}$.

4^0. $(ab)^n = a^n b^n$.

5^0. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

Let us prove, for instance, Property 1^0. We have

$$\underbrace{a \cdot a \ldots a}_{m \text{ factors}} \cdot \underbrace{a \cdot a \ldots a}_{n \text{ factors}} = \underbrace{a \cdot a \ldots a \cdot a \cdot a \ldots a}_{m+n \text{ factors}} = a^{m+n}.$$

2. Operations with Monomials. A *monomial* is such an expression which contains numbers, variables, powers of numbers and variables and their products.

For example, the expressions $2a \cdot 3.5a^2$, $7.5a^2b \cdot 0.4ba$, and $x^2 \times 2y \cdot 3 \cdot (-z) \cdot (-11)$ are monomials, whereas the expressions $a + b$, $\frac{2ab}{3c}$ are not monomials.

Using the properties of multiplication of real numbers and the properties of a power with natural exponent, we can always reduce a monomial to **standard form**, that is, to such a form when a monomial has only one numerical factor occupying the first place (*coefficient*), and each product of like variables is represented in the form of a power.

Let us, for example, reduce the above monomials to standard form. Thus, for the first monomial we have $2a \cdot 3.5a^2 = (2 \cdot 3.5) \times (a \cdot a^2)$. Since $a \cdot a^2 = a^1 \cdot a^2 = a^{1+2} = a^3$, we finally get: $2a \times 3.5a^2 = 7a^3$. Similarly, we find $7.5a^2b \cdot 0.4ba = 3a^3b^2$, $x^2 \cdot 2y \cdot 3\,(-z)\,(-11) = 66x^2yz$.

Now, let there be given two monomials. If we put the multiplication sign between them, then we get a monomial called the product of the given monomials. When raising a monomial to a natural power, we obtain again a monomial. The result is usually reduced to standard form.

Monomials reduced to standard form are said to be *similar* if they differ only in the coefficients or do not differ at all. Similar monomials can be added together and subtracted, the resulting monomial being similar to the original monomials. Adding and subtracting similar monomials is called **collecting like terms.**

Example 1. Multiply the monomials $22abc^2d^3$ and $\frac{1}{2}a^2bcx^3$.

○ We have

$$(22abc^2d^3) \cdot \left(\frac{1}{2}a^2bcx^3\right) = \left(22 \cdot \frac{1}{2}\right) \cdot (a \cdot a^2)\,(b \cdot b)\,(c^2 \cdot c)\,d^3x^3$$
$$= 11a^3b^2c^3d^3x^3. \quad \bullet$$

Example 2. Raise the monomial $-2a^2b^3c$ to the fourth power.

○ We find

$$(-2a^2b^3c)^4 = (-2)^4\,(a^2)^4\,(b^3)^4\,c^4 = 16a^8y^{12}c^4. \quad \bullet$$

Example 3. Add together the monomials $15a^2b$ and $-7a^2b$.

○ Applying the distributive law, we get

$$15a^2b + (-7a^2b) = 15 \cdot a^2b + (-7) \cdot a^2b$$
$$= a^2b\,(15 + (-7)) = 8a^2b. \quad \bullet$$

3. Identical Transformation of an Expression. Consider the expressions $x^2 - 2x$ and $4x - 5$. For $x = 2$ we have $x^2 - 2x = 2^2 - 2 \times$

$2 = 0$ and $4x - 5 = 4 \cdot 2 - 5 = 3$. The numbers 0 and 3 are called the *corresponding values* of the expressions $x^2 - 2x$ and $4x - 5$ for $x = 2$. Let us find the corresponding values of the same expressions for $x = 0$ and $x = 1$, and tabulate the obtained results:

x	$x^2 - 2x$	$4x - 5$
0	0	-5
1	-1	-1

As is seen from the table, the corresponding values of two expressions may sometimes coincide. If they coincide for all values of the variables, then the expressions are said to be *identically equal.*

An equality whose left and right sides are indentically equal expressions is called the *identity.*

Thus, the following pairs of expressions are identically equal: x^5 and $x^2 \cdot x^3$, $a + b + c$ and $c + a + b$, $(2ab)^2$ and $4a^2b^2$.

All the equalities expressing the fundamental laws of algebra are identities. For example,

$$a + 0 = a,\ a + b = b + a,\ a + (b + c) = (a + b) + c,$$

$$ab = ba,\ a(bc) = (ab)c,\ (a + b)c = ac + bc,\ a \cdot 1 = a.$$

Correct numerical equalities are also called identities.

The replacement of one expression by another identically equal expression is known as an **identical transformation** of the expression. Some examples of identical transformations were already given. Thus, the reduction of a monomial to standard form is an identical transformation carried out on the basis of the definition of the power or the properties of a power with natural exponent, and the commutative and associative laws of multiplication. Multiplying monomials and raising them to a natural power, collecting like terms are also identical transformations.

4. Polynomials. Reducing a Polynomial to Standard Form. The *polynomial* is defined as the sum of monomials. Writing all terms of a polynomial in standard form and collecting like terms, we get a *polynomial in standard form.*

For example, the expression $3a \cdot 5b + 3ab + 2a \cdot (-4b) + b \cdot b$ is a polynomial. To reduce it to standard form, it is necessary first to reduce its terms to standard form and then to collect like terms; finally, we get: $15ab + 3ab - 8ab + b^2 = 10ab + b^2$.

The main task of identical transformations of integral expressions consists in reducing them to standard form of a polynomial (or monomial). Such transformations are always feasible.

Example. Simplify (i.e., reduce a polynomial or a mononomial to standard form) the given expression:

(1) $(3a + 5b - c) + (2a - 7b - 3c)$,

(2) $(5a^2b + ab^2) - (2.5a^2b - ab^2)$, (3) $2x^2\left(x - \frac{1}{2}x^2 - 4\right)$.

○ (1) If the plus sign is put before the brackets, then the brackets may be removed, the signs of the terms in brackets being retained. Using this rule, we get

$$(3a + 5b - c) + (2a - 7b - 3c) = 3a + 5b - c + 2a - 7b - 3c$$

and, further,

$$(3a + 2a) + (5b - 7b) + (-c - 3c) = 5a - 2b - 4c.$$

(2) If the minus sign is put before the brackets, then the brackets may be removed, and the signs of all terms in brackets are reversed. Using this rule, we get

$$(5a^2b + ab^2) - (2.5a^2b - ab^2) = 5a^2b + ab^2 - 2.5a^2b + ab^2$$

and, further,

$$(5a^2b - 2.5a^2b) + (ab^2 + ab^2) = 2.5a^2b + 2ab^2.$$

(3) According to the distributive law, the product of a monomial and a polynomial is equal to the sum of products of this monomial by each term of the polynomial. Therefore we get

$$2x^2\left(x - \frac{1}{2}x^2 - 4\right) = 2x^2 \cdot x - 2x^2 \cdot \frac{1}{2}x^2 - 2x^2 \cdot 4 = 2x^3 - x^4 - 8x^2. \quad ●$$

Exercises

1. Represent each of the given numbers in the form of a power with an exponent other than unity: (a) 49, (b) −8, (c) 4/81, (d) 0.09, (e) 49/9, (f) −27/64.
2. Solve the following inequalities: (a) $14 < 2^n < 34$, (b) $25 < 3^n < 250$ ($n \in \mathbb{N}$).
3. Prove that: (a) $(+m^2)\cdot(+m^3) = m^5$, (b) $(+p^2)\cdot(-p^4) = -p^6$.
4. Compare the values of the given pairs of expressions: (a) $2^2 + 2^3$ and 2^{2+3}, (b) $(-2)^2 + (-2)^3$ and $(-2)^{2+3}$, (c) $2^2 \cdot 2^3$ and 2^{2+3}, (d) $(-2)^2 \cdot (-2)^3$ and $(-2)^{2+3}$.
5. Determine for what values of the variables the following inequalities are true: (a) $(-2.5m^3)^2 > 0$, (b) $(-2.5m^2)^3 > 0$, (c) $(2.5m^2)^3 > 0$, (d) $(0.75ab^2)^2 > 0$.
6. Replace the asterisks by numbers such that the given expressions turn into identities: (a) $(2a^*)^2 = 4a^4$, (b) $(0.1x^2y^3)^* = 0.001x^6y^9$.
7. Evaluate the given expressions: (a) $3x^4$ for $x = -2$, (b) $8x^3$ for $x = -0.5$, (c) xy^4 for $x = 0.125$, $y = -2$, (d) $16x^4y$ for $x = -0.5$, $y = 15$.
8. Perform the indicated operations:

(a) $\frac{5^3 \cdot 25^2}{5^8}$, (b) $\frac{2^5 \cdot 8^3}{4^6}$, (c) $\frac{5^5 \cdot 8^3}{4^5 \cdot 25^3}$, (d) $\frac{81^{11} \cdot 3^{17}}{27^{10} \cdot 9^{15}}$.

9. Compare the values of the given pairs of expressions: (a) $3^{10} \cdot 5^{10}$ and $(3 \cdot 5)^{10}$, (b) 21^4 and 28^3, (c) 8^{40} and 72^{20}.

10. Find the numerical value of the product of two monomials $15ax$ and $0.5a^3x$ for $a = 2$, $x = -1$.

11. Reduce the given polynomials to standard form:

(a) $-5x^2y + 3x^2y - 6x^2y + 4x^2y - 7x^2y + 1$,

(b) $-8p^2 + 2.2p^4 - 4p^4 + 8p^2 - 2.2p^4 - 3.7p^4$.

12. Prove that the sum of three successive natural numbers is multiple of three.

§ 1.5. Equations in One Variable

1. Basic Notions Pertaining to Equations. An equality with variables is called an *equation* if the problem is set to find all those values of the variables for which a true proposition is obtained (a correct numerical equality).

Let there be given an equation in one variable. The value of the variable for which the equation is true is called the *root* (or *solution*) of the equation. To solve an equation means to find all of its roots.

Example 1. The equation $3 + x = 7$ has the only root $x = 4$ (for this and only for this value of x the equality with a variable turns into a true proposition $3 + 4 = 7$).

Example 2. The equation $(x - 1)(x - 2) = 0$ has two roots 1 and 2. This can be written in the following way: $x_1 = 1$, $x_2 = 2$.

Example 3. The equation $|x| = x$ has infinitely many roots (any nonnegative number is a root of this equation).

Example 4. The equation $x^2 + 1 = 0$ has no roots. In this case, the answer is symbolized by $\varnothing$.

Equations having the same roots are said to be *equivalent*.

For example, the equations $x + 2 = 5$ and $x + 5 = 8$ are equivalent since either of them has the only root $x = 3$.

In the process of solving an equation, we try to replace it by a simpler one, but necessarily equivalent to the given equation. Therefore it is important to know what transformations yield an equivalent equation.

There is a number of equivalence theorems. Most of them will be studied later. Here, we are going to consider the following two theorems.

Theorem 1.5.1. *If some term is transposed with a reversed sign, then a new equation is obtained which is equivalent to the original one.*

□ Let there be given an equation $f(x) = g(x)$, where the expression $g(x)$ is the sum of two terms $h(x)$ and $p(x)$. Then the given equation can be written in the form

$$f(x) = h(x) + p(x). \tag{1}$$

We have to prove that equation (1) is equivalent to the equation

$$f(x) - p(x) = h(x). \tag{2}$$

Let b be a root of equation (1). This means that the numerical equality $f(b) = h(b) + p(b)$ holds true. Then $f(b) - p(b) =$

$h(b)$ is a true numerical equality, and this means that b is a root of equation (2).

Thus, we have proved that any root of equation (1) is at the same time a root of equation (2). Let us now prove that the converse is also true, that is, any root of equation (2) is a root of equation (1).

Let c be a root of equation (2), that is, the numerical equality $f(c) - p(c) = h(c)$ is true. Then $f(c) = h(c) + p(c)$ is a true numerical equality, and this just means that c is a root of equation (1).

Thus, equations (1) and (2) are equivalent. ■

Theorem 1.5.2. *If both sides of an equation are multiplied or divided by the same nonzero number, then the equation thus obtained is equivalent to the original equation.*

The proof of this theorem is similar to that of Theorem 1.5.1.

2. Systems and Collections of Equations. Consider the equation

$$(x^2 - 1)^2 + ((x - 1)(x - 2))^2 = 0.$$

It is clear that $(x^2 - 1)^2 \geqslant 0$ and $((x - 1)(x - 2))^2 \geqslant 0$, and the sum of two nonnegative numbers is equal to zero if and only if **each** term is equal to zero. Therefore, we have first to solve the equations $(x^2 - 1)^2 = 0$ and $((x - 1)(x - 2))^2 = 0$ and then to find their **common** roots. The equation $(x^2 - 1)^2 = 0$ has two roots 1 and -1, the equation $((x - 1)(x - 2))^2 = 0$ also has two roots 1 and 2. Hence, the number 1 is the root of the original equation.

When we seek the values of a variable which satisfy **both** given equations, we say that a *system of equations* or simultaneous equations are given. To denote the system of equations we use a brace:

$$\begin{cases} (x^2 - 1)^2 = 0, \\ ((x - 1)(x - 2))^2 = 0. \end{cases}$$

Let us now consider the equation $(x^2 - 1)(x^2 - 4) = 0$. The product of two numbers is equal to zero if and only if **at least one** of the numbers is zero. Therefore first we have to solve the equations $x^2 - 1 = 0$ and $x^2 - 4 = 0$ and then to unite their roots. The roots of the first equation are the numbers 1 and -1, the roots of the second the numbers 2 and -2. Hence, 1, -1, 2, -2 are roots of the original equation.

If we seek the values of the variable satisfying **at least one** of the given equations, we say that a *collection of equations* is given. A bracket is sometimes used to denote a collection of equations:

$$\begin{cases} x^2 - 1 = 0, \\ x^2 - 4 = 0. \end{cases}$$

3. Linear Equations. An equation of the form $ax + b = 0$, where x is a variable and a and b are real numbers ($a \neq 0$), is a *linear* equa-

tion. The numbers a and b are termed the coefficients of the equation.

The basic techniques involved in solving linear equations and equations reduced to linear ones are based on Theorems 1.5.1 and 1.5.2.

Example 1. Solve the equation $(1/5)\,x + 2/15 = 0$.

○ This is a linear equation, here $a = 1/5$ and $b = 2/15$. The solution will be obtained by two methods.

First method. According to Theorem 1.5.1, the given equation is equivalent to the equation $(1/5)\,x = -2/15$. Dividing both sides of this equation by the coefficient x, we get (by Theorem 1.5.2) an equivalent equation $x = (-2/15) \div (1/5)$, that is, $x = -2/3$. Thus, $-2/3$ is the root of the equation.

Second method. Multiplying both sides of the given equation by 15 (such a transformation is known as the **clearing of fractions**), we get the equation $3x + 2 = 0$. By Theorem 1.5.2, the obtained equation is equivalent to the original one. Further, we have $3x = -2$, that is, $x = -2/3$. ●

Example 2. Solve the equation $7 - 2x - \frac{1-3x}{7} = 2 - \frac{2x-1}{3}$.

○ First, we multiply both sides of the equation by 21 to clear the fractions:

$$147 - 42x - 3\,(1 - 3x) = 42 - 7\,(2x - 1)$$

and, further,

$$147 - 42x - 3 + 9x = 42 - 14x + 7, \quad 144 - 33x = 49 - 14x,$$
$$-33x + 14x = 49 - 144, \quad -19x = -95, \quad x = 5. \;\bullet$$

Example 3. Find the number whose 15% amount to 60.

○ Denoting the required number by x, we get the linear equation $(15/100)\,x = 60$, whence we find $x = 400$. ●

4. Equations Containing a Variable Under the Modulus Sign. Let us recall that the modulus of a number a is defined in the following way (see 1.3, Item 7):

$$|a| = \begin{cases} a \text{ for } a \geqslant 0, \\ -a \text{ for } a < 0. \end{cases}$$

Example 1. Solve the equation $|3x - 5| = 2$.

○ If $|a| = 2$, then either $a = 2$ or $a = -2$. This means that the given equation is equivalent to the collection of two equations: $3x - 5 = 2$; $-(3x - 5) = 2$. From the equation $3x - 5 = 2$ we find $x_1 = 7/3$; the equation $-(3x - 5) = 2$ yields $x_2 = 1$. ●

Example 2. Solve the equation $|2x - 8| = 3x + 1$.

○ If $2x - 8 \geqslant 0$, then $|2x - 8| = 2x - 8$, and the given equation takes the form $2x - 8 = 3x + 1$. This can be written

as follows:

$$\begin{cases} 2x-8 \geqslant 0, \\ 2x-8=3x+1. \end{cases}$$

From the equation $2x - 8 = 3x + 1$ we find $x = -9$. But for this value of the variable the proposition $2x-8 \geqslant 0$ is false; hence, the found value cannot be a root of the given equation.

If $2x - 8 < 0$, then $|2x - 8| = -(2x - 8)$, and the given equation takes the form $-(2x - 8) = 3x + 1$. This can be written in the following manner:

$$\begin{cases} 2x-8<0, \\ -(2x-8)=3x+1. \end{cases}$$

From the equation $-(2x - 8) = 3x + 1$ we find $x = 7/5$. The proposition $2 \cdot (7/5) - 8 < 0$ is true; hence, the root of the given equation is $x = 7/5$. ●

Equations of the form $|x - a| = b$ can be solved geometrically as well (see § 1.3, Item 7).

5. Problems on Setting Up Linear Equations. In the process of solution, many practical problems are reduced to linear equations.

Example 1. A piece of alloy of copper and tin weighs 12 kg; the alloy contains 45% of copper. How much pure tin must be added to this alloy in order to obtain a new alloy containing 40% of copper?

○ Let the mass of the tin added to the alloy be x kg. Then we obtain a new alloy of mass $(12 + x)$ kg containing 40% of copper. Hence, the new alloy contains $0.4\ (12 + x)$ kg of copper. The original piece of alloy (with 12 kg mass) contained $0.45 \cdot 12$ kg of copper. Since the original and the new alloy contained the same mass of copper, we have the following equation:

$$0.4\ (12 + x) = 0.45 \cdot 12.$$

Solving this equation, we get $x = 1.5$. Thus, 1.5 kg of tin must be added to the original alloy. ●

Example 2. There are two grades of steel with nickel content of 5% and 40%. How much steel of one and the other grade must be taken to obtain (after remelting) 140 t of steel containing 30% of nickel?

○ Let the mass of steel of the first grade be equal to x t; then it is necessary to take $(140 - x)$ t of steel of the second grade. The nickel content in the steel of the first grade is 5%; hence, x t of steel of the first grade contain $0.05x$ t of nickel. The nickel content in the steel of the second grade amounts to 40%; hence, $(140 - x)$t of steel of the second grade contain $0.4\ (140 - x)$ t of nickel. By the hypothesis, after fusing together the two grades of steel 140 t of steel with 30% nickel content must be obtained, that is, after re-

melting the obtained steel must contain $0.3 \cdot 140$ t of nickel. And this amount of nickel is just the sum of $0.05x$ t of steel of the first grade and $0.4\ (140 - x)$ t of steel of the second grade. Thus, we get the equation

$$0.05x + 0.4\ (140 - x) = 0.3 \cdot 140,$$

whence we find $x = 40$. Consequently, it must be taken 40 t of steel with 5% nickel content and 100 t of steel with 40% nickel content. ●

Exercises

In Exercises 1 to 10, solve the given equations:

1. $\frac{7x+4}{5} - x = \frac{3x-5}{2}$.

2. $x - \frac{3x-2}{5} = 3 - \frac{2x-5}{3}$.

3. $\frac{x-3}{8} + \frac{x+9}{12} = \frac{3x+7}{20} + 3$.

4. $\frac{x - \frac{1}{2}}{3} - \frac{1}{3}\left(\frac{2}{5} - \frac{x}{3}\right) = 0$.

5. $\frac{9x+7}{2} - \left(x - \frac{x-2}{7}\right) = 36$.

6. $\frac{x-1}{4} - \frac{1}{8}\left(\frac{x-5}{4} - \frac{14-12x}{2}\right) = \frac{x-9}{2} - \frac{7}{8}$.

7. $\frac{2x-1}{3} - \frac{5x+2}{12} = \frac{x-3}{4} + 1$.

8. $\frac{x+6}{2} + \frac{2(x+17)}{3} + \frac{5(x-10)}{6} = 2x + 6$.

9. (a) $|x| = 2$, (b) $|x| = 0$, (c) $|x| = -1$.

10. (a) $|x - 5| = 2$, (b) $|x - 4| = 0$, (c) $|x + 1| - 2 = 0$, (d) $|2x - 7| = 3$, (e) $|2x + 5| = 0$, (f) $|2x + 3| + 1 = 0$.

11. Solve the given equations using the definition of the modulus: (a) $|2x - 3| = 3x + 2$, (b) $|3x - 1| = 5 - 2x$, (c) $|5x - 2| = 3x - 8$.

12. Find the number if: (a) 13% of this number is 91, (b) 2.05% of it is 41, (c) 110% of it is 660.

13. Are the given equations equivalent:

(a) $6x + 4 = 8$ and $3x + 2 = 4$, (b) $x - 2 = 2 - x$ and $x - 2 + \frac{1}{x^2+1} = 2 - x + \frac{1}{x^2+1}$, (c) $5 - 3x - \frac{1}{x-1} = 2x - \frac{1}{x-1}$ and $5 - 3x = 2x$, (d) $(x+3) \times (x-3) = 0$ and $x + 3 = 0$, (e) $(2x-1)(x-2) = 2(x-2)$ and $2x - 1 = 2$?

14. Prove that the equations $x^2 = 2x - 1$ and $x^2(x^4 + 1) = (2x - 1) \times (x^4 + 1)$ are equivalent.

15. Solve the given equations: (a) $(6x - 1)^2 + (x - 5)^2 = 0$, (b) $(x - 2)^4 + (x^2 - 4)^2 = 0$.

16. Solve the given systems of equations:

(a) $\begin{cases} |x| = 2, \\ (x-2)(x+3) = 0, \end{cases}$ (b) $\begin{cases} -2x - 1 = 0. \\ -7x = 0. \end{cases}$

17. Solve the given collections of equations:

(a) $\begin{cases} 6x-5=9-x, \\ x^2+4=0, \end{cases}$ (b) $\begin{cases} (x-2)(x^2-9)=0, \\ x^2-4=0, \\ \dfrac{x+7}{3}=0. \end{cases}$

18. Solve the given equations with respect to x:

(a) $4+mx=3x+1$, (b) $ax-a=x-1$, (c) $\dfrac{2x-1-0.5(2x-10)}{ax+4a}=0$,

(d) $\dfrac{mx-3}{x-1}=0$.

19. For what values of k does the equation $2k+3(x+1)=\dfrac{3(kx+15)}{5}$ have: (a) a unique and, in particular, zero solution, (b) no solution?

20. Solve the given equations: (a) $S=ab$ with respect to a, (b) $Q=0.5bh$ with respect to b, (c) $U=Qh/3$ with respect to Q.

21. One tank contained 114 l of water, the other 60 l. 12 l of water is taken daily from the first tank, and 6 l from the second. In how many days will the second tank contain 2/3 of the amount of water in the first tank?

22. If we write the digit 2 in front of the thought-of two-digit number, then the three-digit number thus obtained turns out to be nine times greater than the thought-of number. What number was thought of?

23. A six-digit number ends in 5. If this digit opens the given number, then a new number is obtained which is four times the original number. Find this number.

24. 12 per cent of those entering a college failed to pass a written examination in mathematics, 32% of them got satisfactory and good marks, and the remaining 140 students got an excellent mark. What was the total number of those taking the entrance examinations and how many of them failed to pass the examination in mathematics?

25. A greengrocery received a supply of vegetables: carrot 36% of the total supply, red beet 185 kg, and cabbage whose amount constituted 75% of the amount of carrot. How many kilograms of vegetables were supplied?

26. A motor-cyclist has left the town A for the town B. If he moves with a speed of 35 km/h, then he will reach B two hours later than the planned time, and if he moves with a speed of 50 km/h, then he will come to B an hour before the schedule. What is the distance from A to B?

§ 1.6. Linear Equations in Two Variables

1. Equations in Two Variables. Up to this point we have considered equations in one variable. Let us now dwell on equations in two variables.

Suppose we are given an equation in two variables: $x-3y=10$. If we set $x=10$, $y=0$, then the equation turns into a correct numerical equality: $10-3\cdot 0=10$; if $x=16$, $y=2$, then we also get a correct equality: $16-2\cdot 3=10$; if, further, $x=-2$, $y=-4$, then we get a correct equality once again: $-2-3(-4)=10$; but if $x=1$ and $y=5$, then we have an incorrect equality: $1-3\cdot 5\neq 10$.

A *solution to an equation in two variables* is defined as a pair of values of the variables for which the equation is a correct equality.

If we are given an equation in two variables x and y, then in writing its solution we customarily put x first and y second.

Thus, according to the foregoing, the pairs of numbers (10, 0), (16, 2), and (—2, —4) are solutions to the equation $x - 3y = 10$, while the pair of numbers (1, 5) is not. Obviously, this equation has also other solutions; moreover, it has infinitely many solutions. To find them, it is suitable to express one variable in terms of the other, for instance, x in terms of y: $x = 10 + 3y$. Taking an arbitrary value of y, we compute the corresponding value of x. For example, if $y = 7$, then $x = 10 + 3 \cdot 7 = 31$; hence the pair (31, 7) is also a solution of the given equation.

The notion of equivalence of equations in two variables is formulated in the same way (word for word) as for equations in one variable (see § 1.5). The two equivalence theorems given in § 1.5 for equations in one variable are also valid for equations in two variables.

In the present section, we shall confine ourselves to considering *linear equations in two variables*, that is, equations of the form

$$ax + by + c = 0,$$

where at least one of the coefficients a or b is nonzero, and equations reducible to this form.

2. Systems of Two Equations in Two Variables. In the preceding subsection we considered the linear equation in two variables

$$x - 3y = 10. \tag{1}$$

Let us consider one more linear equation in two variables

$$3x - 2y = 2. \tag{2}$$

Expressing y in terms of x, we get $y = (3x - 2)/2$. Let us write out a number of solutions, assigning concrete values to x: (0, —1), (—1, —5/2), (—2, —4), (2, 2). We notice that, among the solutions of equations (1) and (2), there is a common solution (—2, —4).

If a problem is posed to find all common solutions for two equations in two variables, then this means that we have to solve a *system of equations*. Any pair of values of the variables turning each equation of a system into a correct equality is called a *solution of the system of equations*. To solve a system means to find **all** its solutions.

Thus, the system

$$\begin{cases} x - 3y = 10, \\ 3x - 2y = 2 \end{cases}$$

consisting of equations (1) and (2) has a solution (—2, —4). True, for the time being we have no right to assert that the indicated system has been solved, since it may have other solutions (we shall see later that, in fact, this system has no other solutions).

Two systems of equations are said to be *equivalent* if all of their solutions coincide. If, in particular, both systems have no solutions, then they are also assumed to be equivalent.

It was already mentioned that, when solving equations in one variable, one equation is replaced by another which is simpler and equivalent to the given one. When solving systems of equations, we usually proceed in a similar way: we replace one system by another, a simpler one or more "suitable" for us for some reason or other, but necessarily equivalent to the original system. The possibility of such a replacement is stipulated by the following two theorems which are given without proof.

Theorem 1.6.1. *If one equation in a system of two equations in two variables is left unchanged and the other is replaced by an equivalent equation, then the system obtained turns out to be equivalent to the original system.*

Thus, the systems

$$\begin{cases} x-3y=10, \\ 3x-2y=2, \end{cases} \text{ and } \begin{cases} x=3y+10, \\ 3x-2y=2 \end{cases}$$

are equivalent: we have replaced the equation $x - 3y = 10$ by an equivalent equation $x = 3y + 10$, leaving the equation $3x - 2y = 2$ unchanged.

Analogously, the following two systems are also equivalent:

$$\begin{cases} x-3y=10, \\ 3x-2y=2, \end{cases} \text{ and } \begin{cases} x-3y=10, \\ x-\frac{2}{3}y=\frac{2}{3}. \end{cases}$$

Here, the first equation of the given system is left unchanged, and the second is replaced by an equivalent equation (we have divided both sides of the second equation by 3).

Corollary. *If each equation of a system is replaced by an equivalent equation, then a new system is obtained which is equivalent to the original one.*

Thus, the systems

$$\begin{cases} x-3y=10, \\ 3x-2y=2, \end{cases} \text{ and } \begin{cases} x=3y+10, \\ x=\frac{2}{3}y+\frac{2}{3} \end{cases}$$

are equivalent.

Theorem 1.6.2. *If one equation of a system of two equations in two variables is left unchanged, and the other is replaced by the sum or the difference of both equations, then the system thus obtained turns out to be equivalent to the given one.*

Thus, the systems

$$\begin{cases} x-3y=10, \\ 3x-2y=2, \end{cases} \text{ and } \begin{cases} (x-3y)+(3x-2y)=10+2, \\ 3x-2y=2 \end{cases}$$

are equivalent: we have replaced the equation $x - 3y = 10$ by the sum of two equations of the given system, leaving the equation $3x - 2y = 2$ unchanged.

Similarly, the following systems are also equivalent:

$$\begin{cases} x - 3y = 10, \\ 3x - 2y = 2, \end{cases} \text{ and } \begin{cases} x - 3y = 10, \\ (3x - 2y) - (x - 3y) = 2 - 10. \end{cases}$$

Here, the first equation of the given system is left unchanged, while the second is replaced by the difference between the second and the first equation.

3. Methods for Solving Systems of Two Linear Equations in Two Variables. Consider two basic methods of solving systems of equations.

Substitution method. Let there be given the system

$$\begin{cases} x = 4, \\ 2x + 3y = 17. \end{cases} \tag{3}$$

The first equation of this system may be regarded as an equation in two variables x and y, since we may rewrite this equation in the form $x + 0 \cdot y = 4$. Any pair of the form $(4, y)$, where y is any real number, is a solution to this equation. It is clear that not any pair $(4, y)$ is a solution to the second equation of the given system: if $x = 4$, then the equation $2x + 3y = 17$ turns into the equation $8 + 3y = 17$, whence we find $y = 3$. Thus, if $x = 4$, then only the pair $(4, 3)$ is a solution to the equation $2x + 3y = 17$. Since this pair, as any pair of the form $(4, y)$, is at the same time the solution to the first equation of system (3), $(4, 3)$ is the only solution to system (3).

In practice, system (3) is solved in the following way: the value of the variable x determined by the first equation is substituted for x into the second equation. From the equation with variable y thus obtained, the corresponding value of y is found.

This method of solving a system of equations is referred to as the substitution method and is widely used, but, as a rule, in a somewhat more general way: one of the equations of a system is transformed to the form solved with respect to one variable, for instance, y is expressed in terms of x. Further, the obtained expression is substituted for y into the second equation. As a result, an equation with one variable x is obtained. The roots of this equation are found and then, expressing y in terms of x, the corresponding values of y are found.

Example 1. Solve the system of equations

$$\begin{cases} x - 3y = 10, \\ 3x - 2y = 2. \end{cases}$$

○ From the first equation we find $x = 3y + 10$. Substituting the expression $3y + 10$ for x into the second equation we get $3(3y + 10) - 2y = 2$, whence $y = -4$. The corresponding value of x is found from the equation $x = 3y + 10$; for $y = -4$ we have $x = -2$. Thus, the given system has the only solution $(-2, -4)$. ●

Example 2. Solve the system of equations

$$\begin{cases} 20x - 4y = 1, \\ 5x - y = 7. \end{cases}$$

○ From the second equation of the system we find $y = 5x - 7$. Substituting the expression $5x - 7$ for y into the first equation of the system, we get $20x - 4(5x - 7) = 1$ or $28 = 1$. The obtained proposition is false for any x. This means that the given system has no solution. ●

Addition method. We are going to explain this method by a couple of examples.

Example 3. Solve the system of equations

$$\begin{cases} 2x + 3y = 7, \\ 3x - y = 16. \end{cases} \tag{4}$$

○ Multiplying both sides of the second equation by 3, we get the system

$$\begin{cases} 2x + 3y = 7, \\ 9x - 3y = 48 \end{cases} \tag{5}$$

which is equivalent to the original system (on the basis of Theorem 1.6.1).

Now, adding together both equations of the newly obtained system, we get the system

$$\begin{cases} 2x + 3y = 7, \\ (9x - 3y) + (2x + 3y) = 48 + 7 \end{cases} \tag{6}$$

which is equivalent to system (5). System (6) is, in turn, transformed to the form

$$\begin{cases} 2x + 3y = 7, \\ 11x = 55. \end{cases}$$

From the equation $11x = 55$ we find $x = 5$. Substituting this value into the equation $2x + 3y = 7$, we get $y = -1$.

Thus, the solution of system (6) is $(5, -1)$, and, hence, this is also the solution of system (4) which is equivalent to it.

In practice many problems are reduced to systems of linear equations.

Example 4. Find the two-digit number about which the following is known: the sum of this number and the number written with the same digits but in the reverse order is 110; if the desired number is divided by the difference of its digits, then the quotient will be 18 with -1 as a remainder. It is also known that the digit representing tens is greater than the digit representing units.

○ Let x denote the tens digit and y the units digit of the desired number. Then the sought-for number has the form $10x + y$, while the number written with the same digits, but in the reverse order, has the form $10y + x$. As is known from the hypothesis,

$$(10x + y) + (10y + x) = 110.$$

Further, dividing the desired number $10x + y$ by the difference of digits $x - y$, we get 18 as a quotient and 1 as a remainder. Uzing the relationship among the dividend, divisor, quotient, and remainder (see 1.1, Item 1), we may write

$$10x + y = (x - y) \cdot 18 + 1.$$

Thus, the problem has been reduced to solving the following system of equations:

$$\begin{cases} (10x + y) + (10y + x) = 110, \\ 10x + y = (x - y) \cdot 18 + 1. \end{cases}$$

This system has the only solution (7, 3), that, is, the desired number is 73. ●

Exercises

1. Given the equation $3x - 2y = 5$. Write some of its solutions.
2. Set up the equation in two variables whose solution is the given pair of numbers: (a) (2, 5), (b) (−2, 0), (c) $(2 \div 5, -0.3)$.
3. Given the equation $x + y = 13$. Find all the pairs of natural numbers satisfying the given equation.
4. Solve the following systems of equations by two methods: by substitution and by addition:

(a) $\begin{cases} 3x + 4y = 29, \\ 9x - 2y = 17, \end{cases}$ (b) $\begin{cases} 3x + 5y + 9 = 0, \\ 2x - y = 7, \end{cases}$ (c) $\begin{cases} 9x + 8y - 21 = 0, \\ 6x + 4y - 13 = 0. \end{cases}$

5. Show that the following systems of equations have no solutions:

(a) $\begin{cases} 5x + 3y = 10, \\ 10x + 6y = -5, \end{cases}$ (b) $\begin{cases} 2x - 3y = 4, \\ 6x - 9y = 2. \end{cases}$

6. Show that the following systems of equations have infinitely many solutions:

(a) $\begin{cases} 2x - y = 3, \\ 6x - 3y = 9, \end{cases}$ (b) $\begin{cases} 2x + y = 0, \\ 4x + 2y = 0. \end{cases}$

7. Solve the given systems of linear equations:

(a) $\begin{cases} \dfrac{7x-3y}{5}=\dfrac{5x-y}{3}-\dfrac{x+y}{2}, \\ 3(x-1)=5(y+1). \end{cases}$

(b) $\begin{cases} \dfrac{1}{2}y+\dfrac{x}{4}-\dfrac{1}{5}(x+2)=1.1, \\ x-2y+4=\dfrac{1}{4}\left(2x+3\left(y-\dfrac{1}{2}\right)\right). \end{cases}$

8. Given two equations in two variables: $x - y = 2$ and $x + y = 8$. Find the pair of numbers which is: (a) a solution to the first equation, but is not a solution to the second equation; (b) a solution to the second equation, but is not a solution to the first equation; (c) a solution to both equations; (d) not a solution to both equations.

9. Find some of the integral solutions of the equation $(x + y)/5 - (x - 1)/10 = (x - 2)/4$.

10. Write the system of two linear equations in two variables if it is known that the solution of this system is the given pair of numbers: (a) (0, 8); (b) (−3, −4).

11. The sum of the numerator and the denominator of a proper fraction is equal to 13. If 6 is added to each of them, then the fraction 2/3 is obtained. Find the original fraction.

12. The difference of two numbers is 52. The division of the first number by the second yields 3 as the quotient and 4 as the remainder. Find these numbers.

13. The division of a two-digit number by the sum of its digits yields 7 as the quotient and 3 as the remainder. Find this number if it is known that interchanging its digits results in a number which is by 36 less than the required number.

14. The sum of the digits of a two-digit number is 7. Adding 2 to each of the digits yields a number which is by 3 less than twice the original number. What was the original number?

15. The sum of lengths of two pieces of cable was 70 m. After 10 m were cut away rom one piece, the length of the remaining piece became three times the length of the second piece. Find the original length of each piece of cable.

16. The sports section of a Young Pioneer Palace planned to purchase fifteen sets of chess games and twelve sets of draughts, the total sum amounting to 74 roubles and 40 kopecks. Since the number of Young Pioneers who joined the section exceeded the expected number, they purchased ten chess sets and three draughts sets more than it had been planned before, and therefore the expenses amounted to 118 roubles. What is the price of one set of chess games and one set of draughts?

17. Two pedestrians started out at the same time from points A and B which are 30 km apart and met in 3 hours and 45 minutes. If the first pedestrian started out 2 hours earlier than the second, they would meet 2 hours and 30 minutes after the departure of the second pedestrian. Find the speed with which each pedestrian moved.

18. One of the angles of a triangle is 75°, the difference between two other angles being equal to 45°. Determine the size of each angle of the triangle.

19. The distance between the points A and B is 301 m. A body moves uniformly from A to B. On reaching the point B, it immediately returns to A. A second body, leaving the point B for A 11 seconds later than the first body, also moves uniformly but with a lower speed. On its way from B to A it meets the first body twice: 10 seconds and 45 seconds after its start from B. Find the speed of each body.

§ 1.7. Inequalities in One Variable

1. Basic Notions Connected with Solving Inequalities. This section deals with inequalities in one variable x, that is, with inequalities of the form $f(x) > g(x)$ (or $f(x) < g(x)$, $f(x) \geqslant g(x)$, and $f(x) \leqslant g(x)$).

If the variable x is assigned a numerical value, then a numerical inequality is obtained which expresses either true or false proposition. Let, for example, the inequality $5x - 1 > 3x + 2$ be given. For $x = 2$ we get $5 \cdot 2 - 1 > 3 \cdot 2 + 2$ which is a true proposition (a correct numerical inequality); for $x = 0$ we obtain $5 \cdot 0 - 1 > 3 \cdot 0 + 2$ which is a false proposition.

Any value of the variable for which a given inequality with a variable turns into a correct numerical inequality is called a *solution of the inequality*. To solve an inequality with a variable means to find all its solutions. The set of all particular solutions forms the solution of an inequality.

Two inequalities with the same variable x are said to be *equivalent* if the solutions of these inequalities coincide.

The main idea of solving an inequality consists in the following: we replace a given inequality by a simpler one which is equivalent to it. Such replacements are realized on the basis of the following statements:

Theorem 1.7.1. *If any term of an inequality in one variable is transposed from one side of the inequality to the other with an opposite sign, the sense of the inequality remaining unchanged, then an inequality is obtained which is equivalent to the given.*

Theorem 1.7.2. *If both sides of an inequality in one variable are multiplied or divided by the same positive number with the sense of the inequality left unchanged, then an inequality is obtained which is equivalent to the given.*

Theorem 1.7.3. *If both sides of an inequality in one variable are multiplied or divided by the same negative number with the sense of the inequality reversed, then a new inequality is obtained which is equivalent to the given.*

□ Let us, for example, prove the last statement. Let there be given an inequality

$$f(x) > g(x) \tag{1}$$

and let m be a negative number. We are going to prove that the inequality

$$mf(x) < mg(x) \tag{2}$$

is equivalent to inequality (1).

Let a be a solution of inequality (1); then $f(a) > g(a)$ is a correct numerical inequality. If we multiply both of its sides by the

same negative number m and change the sense of the inequality, then we get $mf(a) < mg(a)$ which is a correct numerical inequality. This means that a is a solution of inequality (2).

Thus, any solution of inequality (1) is at the same time a solution of inequality (2). Let us now prove that the converse is also true, that is, any solution of inequality (2) is a solution of inequality (1).

Let b be a solution of inequality (2); then $mf(b) < mg(b)$ is a correct numerical inequality. Dividing both sides of this inequality by a negative number m and reversing the inequality sign, we get $f(b) > g(b)$ which is a correct numerical inequality. This just means that b is a solution of inequality (1).

Thus, the solutions of inequalities (1) and (2) coincide; hence, inequalities (1) and (2) are equivalent. ■

The inequality $f(x) > g(x)$ is said to be solved if, as a result of equivalent transformations, it is reduced to simplest form: $x < a$ $(x > a,\ x \leqslant a,\ x \geqslant a)$ or $a < x < b$ $(a \leqslant x \leqslant b,\ a \leqslant x < b,\ a < x \leqslant b.)$

2. Solving Linear Inequalities in One Variable. *Linear* inequalities are defined as inequalities of the form $ax + b > 0$ (also $ax + b < 0$, $ax + b \geqslant 0$, $ax + b \leqslant 0$), where a and b are real numbers, and $a \neq 0$. The solution of such inequalities is based on the three equivalence theorems from the preceding item.

Example 1. Solve the inequality $2(x - 3) + 5(1 - x) \geqslant 3(2x - 5)$.

○ Removing the parentheses, we get

$$2x - 6 + 5 - 5x \geqslant 6x - 15 \text{ or } -3x - 1 \geqslant 6x - 15.$$

Further, we have $-3x - 6x \geqslant -15 + 1$, that is,

$$-9x \geqslant -14. \tag{3}$$

By Theorem 1.7.1, inequality (3) is equivalent to the given inequality.

Let us now divide both sides of inequality (3) by the negative number -9 and reverse the sense of the inequality. Then, according to Theorem 1.7.3, we get an inequality equivalent to inequality (3): $x \leqslant 14/9$. The answer can also be written in the form of the number interval $(-\infty, 14/9]$. ●

Example 2. Solve the inequality

$$12x - \frac{x-2}{3} + 2(x+1) > 5(3x-1) - \frac{2x+3}{2} - \frac{x}{3}.$$

○ Multiplying both sides of the inequality by the positive number 6 and leaving the sense of the inequality unchanged, we clear the fractions:

$$72x - 2(x - 2) + 12(x + 1) > 30(3x - 1) - 3(2x + 3) - 2x.$$

Further, we get in succession

$$72x - 2x + 4 + 12x + 12 > 90x - 30 - 6x - 9 - 2x,$$

$$82x + 16 > 82x - 39,\ 82x - 82x > -39 - 16,\ 0 \cdot x > -55.$$

The last inequality is correct for any value of x, since for any value of the variable x we get a true proposition: $0 > -55$. Therefore its solution, and hence, the solution of the given inequality is represented by the entire number line: $(-\infty, +\infty)$. ●

3. Systems and Collections of Inequalities. Several inequalities in one variable are said to form a *system* if it is required to find all the general solutions of the given inequalities. The value of the variable for which each of the inequalities turns into a correct numerical inequality is called a *solution of the system of inequalities.*

Inequalities forming a system are united by a brace. For example, the notation

$$\begin{cases} 2x - 1 > 3, \\ 3x - 2 < 11 \end{cases}$$

means that the inequalities $2x - 1 > 3$ and $3x - 2 < 11$ form a system.

Sometimes, the notation in the form of a double inequality is used. For instance, the system of inequalities

$$\begin{cases} 2x + 1 > 1, \\ 2x + 1 < 5 \end{cases}$$

may be written as follows: $1 < 2x + 1 < 5$.

Several inequalities in one variable are said to form a *collection* if it is required to find all values of the variable such that each of them is a solution of at least one of the given inequalities. The value of the variable for which at least one of the inequalities forming a collection turns into a correct numerical inequality is called a *solution of the collection of inequalities.*

The inequalities forming a collection are written "in line" and separated from each other by a semicolon. Sometimes, a collection of inequalities is denoted by a square bracket. Thus, if the inequalities $2x + 3 < x$ and $3x + 1 < x - 2$ form a collection, then we may write them either as $2x + 3 < x$; $3x + 1 < x - 2$ or as

$$\begin{cases} 2x + 3 < x, \\ 3x + 1 < x - 2. \end{cases}$$

To solve systems and collections of inequalities, it is convenient to represent them graphically. Let, for instance, an open interval (a, b) be the solution of the inequality $f_1(x) > g_1(x)$ and an open interval (c, d) the solution of the inequality $f_2(x) > g_2(x)$. Let us

mark these intervals on a common number line, using the hatching above the line for the first interval and below the line for the second as in Fig. 21. Then the solution of the system of inequalities $\begin{cases} f_1(x) > g_1(x) \\ f_2(x) > g_2(x) \end{cases}$ is represented by the interval at which both hatchings coincide (if they coincide nowhere, then a system has no solution). The solution of the collection of inequalities $f_1(x) > g_1(x)$, $f_2(x) > g_2(x)$ is represented by the part of the

Fig. 21

Fig. 22

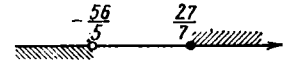


Fig. 23

straight line hatched at least once. In our case, the solution of the system is the interval (c, d), while the solution of the collection the interval (a, d).

Consider a number of examples of solving systems and collections. In this section, we shall confine ourselves to linear inequalities and inequalities reducible to linear ones.

Example 1. Solve the system of inequalities

$$\begin{cases} 5x + 2 > 3x - 1, \\ 3x + 1 > 7x - 4. \end{cases}$$

○ The first inequality of the given system is transformed into an equivalent inequality $x > -3/2$, the second into the inequality $x < 5/4$. Thus, we get the system $\begin{cases} x > -3/2, \\ x < 5/4. \end{cases}$ Its solution is represented by the open interval $(-3/2, 5/4)$ (Fig. 22). ●

Example 2. Solve the system of inequalities

$$\begin{cases} 3(x+1) - \frac{x-2}{4} < 5x - 7\frac{x+3}{2}, \\ 2x - \frac{x}{3} + 6 \leqslant 4x - 3. \end{cases}$$

○ Transforming each of the inequalities, we get the system $\begin{cases} x < -56/5, \\ x \geqslant 27/7, \end{cases}$ which has no solution (Fig. 23). ●

Example 3. Solve the system of inequalities

$$\begin{cases} \frac{3x-1}{3} < \frac{5x+1}{5}, \\ \frac{7x}{3} \geqslant 2(x+1). \end{cases}$$

○ After transformations we get the system $\begin{cases} 0 \cdot x < 8 \\ x \geqslant 6 \end{cases}$. The solution of the first inequality of this system is the entire number line, while the solution of the second inequality is the interval $[6, +\infty)$. This interval is the solution of the given system. ●

Example 4. Solve the system of inequalities

$$\begin{cases} 6x - 1 > x - 3 + 5(x + 1), \\ 4x + \frac{x}{3} < -3x + 8. \end{cases}$$

○ After obvious transformations, we get the system $\begin{cases} 0 \cdot x > 3 \\ x < 12/11 \end{cases}$. Since the first inequality has no solution, the system has no solution either. ●

Example 5. Solve the collection of inequalities

$$\begin{cases} \frac{2x - 3x}{5} > \frac{3x - 2}{2}, \\ \frac{x}{3} + 1 > \frac{3x}{2}. \end{cases}$$

○ Transforming each of the given inequalities, we get the collection $\begin{cases} x < 4/11, \\ x < 6/7, \end{cases}$ whose solution is the ray $(-\infty, 6/7)$(Fig. 24). ●

Example 6. Solve the inequality $\frac{2x - 3}{3x - 7} > 0$.

○ The fraction is positive if and only if its numerator and denominator are of the same sign, that is, if

$$\begin{cases} 2x - 3 > 0, \\ 3x - 7 > 0, \end{cases} \text{ or } \begin{cases} 2x - 3 < 0, \\ 3x - 7 < 0. \end{cases}$$

We may say that the given inequality is equivalent to the collection of two systems

$$\begin{cases} 2x - 3 > 0, \\ 3x - 7 > 0, \end{cases} \quad \begin{cases} 2x - 3 < 0, \\ 3x - 7 < 0. \end{cases}$$

The solution of the first system is the interval $(7/3, +\infty)$, the solution of the second system is the interval $(-\infty, 3/2)$. Uniting these sets, we get the solution of the collection of systems and, at the same time, the solution of the given inequality $x < 3/2$, $x > 7/3$. ●

Example 7. Solve the inequality $\frac{3x + 2}{2x + 3} \geqslant 4$.

○ We first transform the given inequality

$$\frac{3x+2}{2x+3}-4\geqslant 0,\quad \frac{-5x-10}{2x+3}\geqslant 0.$$

Let us then divide both sides of the last inequality by -5: $\frac{x+2}{2x+3}\leqslant 0$. The value of a fraction is negative if and only if the values of its numerator and denominator have opposite signs; a fraction vanishes if its numerator is zero, and the denominator is nonzero. Taking advantage of this note, we arrive at the following collection of two systems:

$$\begin{cases} x+2\geqslant 0, \\ 2x+3<0, \end{cases} \quad \begin{cases} x+2\leqslant 0, \\ 2x+3>0. \end{cases}$$

The solution of the first system is the interval $[-2, -3/2)$, while the second system has no solution. Hence, the solution of the collection, and therefore of the given inequality is the interval $[-2, -3/2)$. ●

4. Inequalities Containing Absolute Values. By way of examples, let us show various methods of solving inequalities containing an absolute value of the variable.

Example 1. Solve the inequality $|2x+5|\geqslant 7$.

○ *First method* (geometrical, see § 1.3, Item 7). We have $|x+2.5|\geqslant 3.5$. We have to indicate on the number line all such

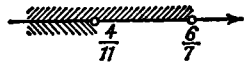


Fig. 24

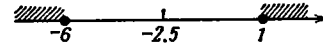


Fig. 25

points x whose distance from the point -2.5 is greater than, or equal to 3.5 (Fig. 25). Thus, we get $x\leqslant -6$, $x\geqslant 1$.

Second method (using the definition of absolute value). We have $|a|=a$ if $a\geqslant 0$; $|a|=-a$ if $a<0$.

The given inequality is equivalent to the following collection of systems of inequalities:

$$\begin{cases} 2x+5\geqslant 0, \\ 2x+5\geqslant 7, \end{cases} \quad \begin{cases} 2x+5<0, \\ -(2x+5)\geqslant 7. \end{cases}$$

It follows from the first system that $x\geqslant 1$, and from the second system that $x\leqslant -6$. ●

Example 2. Solve the inequality $|2x+4|\leqslant 3x+2$.

○ If $2x+4\geqslant 0$, then $|2x+4|=2x+4$ and, consequently, the given inequality takes the form $2x+4\leqslant 3x+2$. If

$2x + 4 < 0$, then $|2x + 4| = -(2x + 4)$, and it takes the form $-(2x + 4) \leqslant 3x + 2$. Thus, the given inequality is equivalent to the collection of two systems:

$$\begin{cases} 2x+4 \geqslant 0, \\ 2x+4 \leqslant 3x+2, \end{cases} \qquad \begin{cases} 2x+4 < 0, \\ -(2x+4) \leqslant 3x+2. \end{cases}$$

The solution of the first system is represented by the interval $[2, +\infty)$, the second system has no solution. Consequently, the solution of the collection of systems, and, hence, of the given inequality is $x \geqslant 2$. ●

Exercises

1. Show that: (a) the value $x = 3$ satisfies the inequality $(x + 5)^2 > x^2 + 25$, (b) the value $x = 2$ is not a solution of the inequality $x^4 + x + 1 < 36/x$.

2. Prove that for any values of x the following inequality holds true:

$$(x - 5)(x - 4) \geqslant (x - 7)(x - 2).$$

In Exercises 3 to 5, solve the given inequalities:

3. (a) $6(2x + 7) < 15(x + 2)$, (b) $0.01(1 - 3x) > 0.02x + 3.01$.

4. (a) $5(x+3) - \dfrac{x-7}{8} > \dfrac{11}{2}(x-2)$,

(b) $\dfrac{5}{3}x + \dfrac{7-3x}{2} \leqslant x - \dfrac{7+3x}{3} + \dfrac{x}{6}$.

5. (a) $x(x + 1) + x(x + 3) < (2x - 1)(x - 5)$, (b) $(x - 3)(x - 5) > (x + 1)(x - 4) - 5x$.

In Exercises 6 to 10, solve the given systems of inequalities:

6. (a) $\begin{cases} 5x+3 > 0, \\ 2x-1 \geqslant 0, \end{cases}$ (b) $\begin{cases} 5(x+1) - 9x - 3 > -6(x+2), \\ 3(3+2x) < 7x - 2(x-8). \end{cases}$

7. $\begin{cases} \dfrac{x-9}{4} - x < \dfrac{x-1}{2} - \dfrac{x-2}{3}, \\ 2 - x > 2x - 8. \end{cases}$

8. $\begin{cases} \dfrac{2x-11}{4} + \dfrac{19-2x}{2} < 2x, \\ \dfrac{2x+15}{9} > \dfrac{1}{5}(x-1) + \dfrac{x}{3}. \end{cases}$

9. $\begin{cases} x + 2 < \dfrac{2x-8}{6} - \dfrac{18-4x}{3}, \\ 9 - \left(\dfrac{x-2}{4} + \dfrac{2}{3}\right) > x. \end{cases}$

10. $\begin{cases} \dfrac{x^2-1}{5} + \dfrac{x-1}{2} < \dfrac{2x^2+3}{10} + \dfrac{x}{2} + 3, \\ 1 - x > \dfrac{0.5(x-3)}{2} - \dfrac{2x+3.5}{3}. \end{cases}$

In Exercises 11 to 13, find all the integral solutions for each of the following systems of inequalities:

11. $\begin{cases} 1-\dfrac{3x-88}{7} > 5x, \\ 4x+5 < 5x+4.5. \end{cases}$

12. $\begin{cases} \dfrac{3}{4}x+\dfrac{7}{8} \leqslant \dfrac{1}{4}x+\dfrac{5}{2}, \\ \dfrac{x+1}{4}-\dfrac{2x-1}{3} < 2. \end{cases}$

13. $\begin{cases} x+2 > \dfrac{2x-8}{6}-\dfrac{18-4x}{3}, \\ \dfrac{2-x}{4}+9\dfrac{2}{3} > x. \end{cases}$

14. If 1/4 of a number is added to this number, then 2/3 of the number obtained will be greater than 40, but less than 60. Find the original number if it is known that it is divisible by 19.

15. For what values of y do the fractions $-\dfrac{1}{5-y}$, $\dfrac{y^2+3}{y+4}$, $\dfrac{7}{y^4+1}$ take on: (a) positive values; (b) negative values?

16. Solve the following inequalities:

(a) $\dfrac{-5}{-6x+3} > 0$, (b) $\dfrac{3}{2x-3} \leqslant 0$, (c) $\dfrac{x^2+3}{-5x-2} \geqslant 0$.

17. Find the values of the variable c for which the inequality $9 < -2c + 5 \leqslant 15$ holds true.

18. Is the number -5 a solution to the given systems of inequalities:

(a) $\begin{cases} 100-5a > 80a-40, \\ 5a+13 > 4a+16, \end{cases}$ $\begin{cases} x^2+2x+3 < -7x, \\ 2x^2-15 > -2x? \end{cases}$

19. Solve the following collections of inequalities:

(a) $5x+3 > 0$, $2x-1 > 0$, (b) $\dfrac{2x-3}{5} < x-2$, $5x-7 < x-6$.

20. Solve the collection of systems of inequalities:

$$\begin{cases} x-1 < x+1.5, \\ 2x-8 > 3-0.5x, \end{cases} \qquad \begin{cases} 5x-6 < 26, \\ 3x+4 > 24. \end{cases}$$

In Exercises 21 to 34, solve the given inequalities:

21. (a) $\dfrac{2x-1}{x} > 0$, (b) $\dfrac{3-4x}{x+1} < 0$, (c) $\dfrac{2x+1}{x-2} \geqslant 0$, (d) $\dfrac{x+4}{3x-5} \leqslant 0$.

22. (a) $\dfrac{3x+2}{x-2} > 1$, (b) $\dfrac{1}{x} < \dfrac{1}{3}$.

23. $\dfrac{9}{x}-\dfrac{4}{9} > \dfrac{10}{x}-\dfrac{1}{2}$.

24. $|2x-5| < 3$.
25. $|3x-1| \geqslant 5$.
26. $|1-2x| > 3-x$.
27. $|x+8|+1 \leqslant 3x$.
28. $x \geqslant 2-|4-3x|$.
29. $|2x-7| > 0$.
30. $-1 < |x-2| < 3$.
31. $|2x-10| \leqslant 0$.
32. $|3x-5| < -2$.
33. $|x-2| > |x-3|$.
34. $1 < |2x-3| < 7$.

Chapter 2

TRANSFORMATIONS OF ALGEBRAIC EXPRESSIONS

§ 2.1. Polynomials

1. Identities for Short-cut Multiplication. In § 1.4, we introduced the notions of monomial and polynomial and considered the simplest transformations of integral expressions. Let us now dwell in more detail on various operations with polynomials and also on transformations of polynomials.

In some cases, polynomials are reduced to standard form by means of the so-called **identities for short-cut multiplication** given below:

I. $(a - b)(a + b) = a^2 - b^2$,
II. $(a + b)^2 = a^2 + 2ab + b^2$,
III. $(a - b)^2 = a^2 - 2ab + b^2$,
IV. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$,
V. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$,
VI. $(a + b)(a^2 - ab + b^2) = a^3 + b^3$,
VII. $(a - b)(a^2 + ab + b^2) = a^3 - b^3$.

In Examples 1 to 5, reduce the given expressions to standard form:

1. $(3x^2 - 4y^3)(3x^2 + 4y^3)$.

○ Using identity 1, we get

$$(3x^2 - 4y^3)(3x^2 + 4y^3) = (3x^2)^2 - (4y^3)^2 = 9x^4 - 16y^6. \quad \bullet$$

2. $(a + b - c)(a + b + c)$.

○ We have

$$(a + b - c)(a + b + c) = ((a + b) - c)((a + b) + c)$$
$$= (a + b)^2 - c^2 = a^2 + 2ab + b^2 - c^2. \quad \bullet$$

3. $(3a^2 - 5b^3)^2$.

○ By virtue of identity III, we get

$$(3a^2 - 5b^3)^2 = (3a^2)^2 - 2(3a^2)(5b^3) + (5b^3)^2 = 9a^4 - 30a^2b^3 + 25b^6. \quad \bullet$$

4. $(x - 2)^3 - x(x + 3)^2$.

○ Using identities V and II, we find

$$(x - 2)^3 - x(x + 3)^2 = (x^3 - 3x^2 \cdot 2 + 3x \cdot 2^2 - 2^3) - x(x^2 + 6x + 9)$$
$$= x^3 - 6x^2 + 12x - 8 - x^3 - 6x^2 - 9x$$
$$= -12x^2 + 3x - 8. \quad \bullet$$

5. $(3a + 1)(9a^2 - 3a + 1)$.

○ Using identity VI, we get

$$(3a + 1)(9a^2 - 3a + 1) = (3a)^3 + 1^3 = 27a^3 + 1. \bullet$$

2. Operations with Polynomials in One Variable. In what follows we shall consider polynomials in one variable.

Let $P(x)$ be a polynomial representing the sum of the following monomials: a_0x^n, a_1x^{n-1}, a_2x^{n-2}, . . ., $a_{n-1}x$, a_n, where n is a natural number, and a_0, a_1, a_2, . . ., a_{n-1}, a_n are real numbers, $a_0 \neq 0$. Let us agree to arrange these monomials in decreasing powers of the variable

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$$

and call this notation the *standard form of a polynomial* $P(x)$. The number n is termed the *degree* of the polynomial, the numbers a_0, a_1, a_2, . . ., a_{n-1}, a_n are its *coefficients*. The monomial a_0x^n is referred to as the *leading term*, while a_0 as the leading coefficient (if $a_0 = 1$, then the polynomial is said to be *reduced*). The monomial a_n is called the *absolute* (or *constant*) term of the polynomial. If $n = 2$, then we have a polynomial of the second degree or a *quadratic trinomial*.

The following theorem will be of help for our further considerations (the proof is left to the student).

Theorem 2.1.1. *Two polynomials in one variable (reduced to standard form) are identical if and only if they are of the same degree, and the coefficients of like powers of the variable in both polynomials coincide.*

We may perform the following operations with polynomials: addition, subtraction, multiplication, and raising to a natural power. Let us agree, on performing the indicated arithmetical operations on given polynomials to transform the result obtained to standard form.

In some cases, the operation of division is also feasible. Namely, if there is a polynomial $S(x)$ such that $P(x) = Q(x) \cdot S(x)$, then the polynomial $P(x)$ is said to be **divisible** by the polynomial $Q(x)$; $P(x)$ is called the *dividend*, $Q(x)$ the *divisor*, and $S(x)$ the *quotient*.

For example, the polynomial $P(x) = x^3 - 3x^2 + 5x - 15$ is divisible by the polynomial $Q(x) = x^2 + 5$ since

$$x^3 - 3x^2 + 5x - 15 = (x^2 + 5)(x - 3).$$

In this case, we get the polynomial $S(x) = x - 3$ as the quotient.

If $P(x) = Q(x) \cdot S(x)$, then the leading term of the polynomial $P(x)$ is the product of the leading terms of the polynomials $Q(x)$ and $S(x)$. Hence it follows, in particular, that the degree of the polynomial $P(x)$ is equal to the sum of the degrees of the polyno-

mials $Q(x)$ and $S(x)$. Consequently, if $P(x)$ is exactly divisible by $Q(x)$ then the degree of the polynomial $P(x)$ is greater than that of the polynomial $Q(x)$; however, if the number $c \neq 0$ is assumed to be a polynomial of degree zero, then the degree of the dividend can be equal to the degree of the divisor. For instance, the polynomial $P(x) = 6x^2 - 12x + 18$ is divisible by the polynomial $Q(x) = x^2 - 2x + 3$ since $6x^2 - 12x + 18 = (x^2 - 2x + 3) \cdot 6$; the quotient is $S(x) = 6$, which is a polynomial of degree zero.

If $P(x)$ is not divisible by $Q(x)$, then we consider the **division with a remainder** whose possibility follows from the theorem given below without proof.

Theorem 2.1.2. *For any two polynomials $P(x)$ and $Q(x)$ such that the degree of $P(x)$ is not less than that of $Q(x)$ there is one and only one pair of polynomials $S(x)$ and $R(x)$ such that the identity*

$$P(x) = Q(x) \cdot S(x) + R(x)$$

is satisfied, the degree of the polynomial $R(x)$ being less than that of the polynomial $Q(x)$ (here $P(x)$ is the dividend, $Q(x)$ the divisor, $S(x)$ the quotient, and $R(x)$ the remainder).

For dividing polynomials (of course, reduced to standard form), we use the rule for long division which, in a sense, is similar to the rule for dividing many-digit numbers.

In order to get the leading term of the quotient, we divide the leading term of the dividend by that of the divisor. The found term of the quotient is then multiplied by the divisor and the polynomial thus obtained is subtracted from the dividend. Dividing the leading term of the difference obtained first by the leading term of the divisor, we find the second term of the quotient. The second found term of the quotient is multiplied by the divisor, the polynomial thus obtained is subtracted from the first difference, the leading term of the second difference is divided by the leading term of the divisor, and so on. As a result of this process the polynomial $P(x)$ will be either exactly divided by the polynomial $Q(x)$ or a difference will be obtained, at a certain step, of a degree less than the degree of the divisor. Then this difference is the remainder $R(x)$ (the degree of the remainder is less than that of the divisor).

Example. Divide the polynomial $P(x) = 3x^4 + 2x^3 + 70x^2 + 3x - 4$ by the polynomial $Q(x) = x^2 + 5x + 1$.

○ We have

$$\begin{array}{rl}
 & x^2+5x+1 \\
\text{quotient } 3x^2-13x+132) & \overline{3x^4+2x^3+70x^2+3x-4} \\
 & 3x^4+15x^3+3x^2 \\
\text{first difference} & \overline{-13x^3+67x^2+3x-4} \\
 & -13x^3-65x^2-13x \\
\text{second difference} & \overline{132x^2+16x-4} \\
 & 132x^2+660x+132 \\
\text{remainder} & \overline{-644x-136}
\end{array}$$

•

Thus, $3x^2 - 13x + 132$ is the quotient, $-644x - 136$ the remainder. In this case, we have the following identity:

$$3x^4 + 2x^3 + 70x^2 + 3x - 4$$
$$= (x^2 + 5x + 1)(3x^2 - 13x + 132) + (-644x - 136). \bullet$$

3. Roots of a Polynomial. Let there be given a polynomial $P(x)$. If the variable x is assigned numerical values, then the polynomial $P(x)$ will take on numerical values. If, for example, $P(x) = x^3 - 3x + 2$, then for $x = 2$ we have $P(2) = 2^3 - 3 \cdot 2 + 2 = 4$, for $x = 1$ we find $P(1) = 1^3 - 3 \cdot 1 + 2 = 0$, for $x = -\sqrt{3}$ we get $P(-\sqrt{3}) = (-\sqrt{3})^3 - 3(-\sqrt{3}) + 2 = 2$.

If for a certain value of $x = x_0$ a polynomial $P(x)$ vanishes, that is, $P(x_0) = 0$, then the number x_0 is called the *root of the polynomial*. Thus, the number $x = 1$ serves as a root of the polynomial $P(x)$.

Theorem 2.1.3. *Let there be given a polynomial*

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n,$$

whose all coefficients are whole numbers ($a_0 \neq 0$). *If a whole number* $x = b$ *is a root of the polynomial* $P(x)$, *then it serves as the divisor of the absolute term* a_n.

□ By hypothesis $P(b) = 0$, that is,

$$a_0b^n + a_1b^{n-1} + \ldots + a_{n-1}b + a_n = 0.$$

Hence it follows that

$$a_n = -(a_0b^n + a_1b^{n-1} + \ldots + a_{n-1}b),$$

that is,

$$a_n = b(-(a_0b^{n-1} + a_1b^{n-2} + \ldots + a_{n-1})).$$

Let us denote the number $(-(a_0b^{n-1} + a_1b^{n-2} + \ldots + a_{n-1}))$ by m. It is clear that m is a whole number, and we have $a_n = bm$. But this means that the number b is a divisor of the number a_n. ■

Example. It is known that the polynomial $P(x) = x^3 - 3x^2 + 5x - 15$ has only one root. Find this root.

○ According to Theorem 2.1.3, the integral roots of a polynomial may be represented only by such numbers which are divisors of the absolute term. In our case, the absolute term has the following divisors: ± 1, ± 3, ± 5, ± 15. Hence, the integral root must be sought among these eight numbers. We have $P(1) = -12 \neq 0$, $P(-1) = -24 \neq 0$, $P(3) = 0$. Thus, $x = 3$ is an integer root of the polynomial $P(x)$. ●

4. Factorizing Polynomials. Sometimes, we succeed in splitting a polynomial into a product of several simpler polynomials or monomials. Such identical transformation is called **factorization.** As we shall see, factorization is used rather often; for example, when

performing operations with algebraic fractions (reducing fractions, reducing to a common denominator), when solving equations, and so forth. There is a theorem which is given without proof.

Theorem 2.1.4. *Any polynomial $P(x)$ of degree $n \geqslant 2$ can be represented as a product of irreducible polynomials of degree not higher than 2.*

Consider some methods of factoring polynomials.

(1) **Taking outside a common factor.** This transformation is an immediate consequence of the distributive law; for clarity we have only to rewrite this law "from right to left": $ac + bc = (a + b)\,c$.

Example 1. Factor the polynomial $28x^3 - 35x^4$.

○ We have

$$28x^3 - 35x^4 = 7x^3 \cdot 4 - 7x^3 \cdot 5x = 7x^3 (4 - 5x). \bullet$$

This example shows that a common factor for the terms of a polynomial in one variable x is a monomial whose coefficient is the greatest common divisor of all the coefficients of the polynomial, and the power of the variable x of the monomial is equal to the least of the powers of the polynomial's terms.

(2) **Grouping.** It is sometimes possible to carry out factorization by various grouping methods. This is a more difficult technique requiring practical habits and quick wits. It consists in breaking the terms of a polynomial into several groups, taking outside a factor in each group, and finding the same expression in all the sets of brackets. This expression can then be factored out. The laws of addition enable us to group the terms of a polynomial in any manner.

Example 2. Factor the polynomial $x^3 - 3x^2 + 5x - 15$.

○ Let us group the terms in the following way:

$$(x^3 - 3x^2) + (5x - 15).$$

In the first group, we take out the common factor x^2, in the second the common factor 5. We get $x^2 (x - 3) + 5 (x - 3)$. Now we take out the polynomial $(x - 3)$ as a common factor: $(x - 3) \times (x^2 + 5)$. Thus

$$x^3 - 3x^2 + 5x - 15 = (x - 3)(x^2 + 5). \bullet$$

Example 3. Factor the polynomial $x^2 - 5x + 6$.

○ Here, no grouping will isolate one and the same polynomial in all possible groups. In such cases, it is sometimes useful to represent some terms of the given polynomial in the form of a sum which allows a successful grouping. In this example it is advisable to represent the term $-5x$ as a sum $-2x - 3x$:

$$x^2 - 5x + 6 = (x^2 - 2x) + (-3x + 6) = x(x - 2) - 3(x - 2)$$
$$= (x - 2)(x - 3).$$

(3) **Using short-cut multiplication identities.** Identities I-VII, when read "from right to left", in many cases turn out to be successfully used for factoring polynomials.

Example 4. Factor the polynomial $x^6 - 1$.

○ We have $x^6 - 1 = (x^3)^2 - (-1)^2$. Applying Identity I ("difference of squares"), we get $(x^3+1)(x^3 - 1)$. Further, using Identities VI and VII ("sum of cubes" and "difference of cubes"), we have $(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$. Thus,

$$x^6 - 1 = (x + 1)(x - 1)(x^2 - x + 1)(x^2 + x + 1). \quad \bullet$$

(4) **Factoring a polynomial using its roots.**

Theorem 2.1.5. *Let a polynomial $P(x)$ have a root x_1. Then this polynomial can be factored in the following way:*

$$P(x) = (x - x_1) \cdot S(x),$$

where $S(x)$ is a polynomial whose degree is by unity less than the degree of the polynomial $P(x)$.

Hence, by the way, it follows that if x_1 is a root of the polynomial $P(x)$, then $P(x)$ is exactly divisible by the binomial $x - x_1$.

□ We shall carry out the proof for a particular case of a polynomial of degree 4. Let the polynomial

$$P(x) = ax^4 + bx^3 + cx^2 + dx + m$$

have a root x_1. Then $P(x_1) = 0$, that is, $ax_1^4 + bx_1^3 + cx_1^2 + dx_1 + m = 0$. Therefore

$$\begin{aligned} P(x) = P(x) - P(x_1) &= (ax^4 + bx^3 + cx^2 + dx + m) \\ &- (ax_1^4 + bx_1^3 + cx_1^2 + dx_1 + m) = a(x^4 - x_1^4) \\ &+ b(x^3 - x_1^3) + c(x^2 - x_1^2) + d(x - x_1) \\ &= (x - x_1)(a(x + x_1)(x^2 + x_1^2) + b(x^2 + xx_1 + x_1^2) \\ &+ c(x + x_1) + d). \end{aligned}$$

Denoting the factor following $x - x_1$ by $S(x)$ (this is a polynomial of degree 3), we get

$$P(x) = (x - x_1) \cdot S(x). \quad \blacksquare$$

For the time being we are not going to raise the problem how to find the roots of a polynomial; this will be discussed later, when solving equations. Here, we confine ourselves to finding the roots by trial and error, using, in particular, Theorem 2.1.3.

Example 5. Factor the polynomial

$$P(x) = x^3 - 3x^2 - 10x + 24.$$

○ Let us try to find the integral root of this polynomial. According to Theorem 2.1.3, it should be sought (if any) among the divi-

sors of the number 24. Let us write out the divisors which may possibly be the roots of the given polynomial: ± 1, ± 2, ± 3, ± 4, ± 6, ± 8, ± 12, ± 24. We shall substitute, in turn, these values into the expression for $P(x)$. We have $P(1) = 12$, that is, $x = 1$ is not a root of the polynomial. Further, $P(-1) = 30$, that is, $x = -1$ is not a root of the polynomial either. Finally, $P(2) = 0$, that is, $x = 2$ is a root of the given polynomial. We then divide the polynomial $P(x)$ by $x - 2$ (it was stated above that if $x = x_1$ is a root of the polynomial $P(x)$, then the latter is exactly divisible by $x - x_1$):

$$\begin{array}{r|l} & x^2-x-12 \\ \hline x-2 & x^3-3x^2-10x+24 \\ & x^3-2x^2 \\ \hline & -x^2-10x+24 \\ & -x^2+2x \\ \hline & -12x+24 \\ & -12x+24 \\ \hline & 0 \end{array}$$

Thus, $P(x) = (x - 2)(x^2 - x - 12)$.

Let us now factor the quadratic trinomial $x^2 - x - 12$; we have

$$x^2 - x - 12 = x^2 + 3x - 4x - 12 = x(x + 3) - 4(x + 3) = (x + 3)(x - 4).$$

As a result, we get

$$x^3 - 3x^2 - 10x + 24 = (x - 2)(x + 3)(x - 4). \bullet$$

5. Polynomials in Several Variables. The same as polynomials in one variable, polynomials in several variables can be added, subtracted, multiplied, and raised to a natural power.

Factorization is an important identical transformation of a polynomial in several variables. Here, as in the case of a polynomial in one variable, we use such methods of factorization as taking out a common factor, grouping, using short-cut multiplication identities, completing the square, and introducing auxiliary variables.

Example 1. Factor the polynomial

$$P(x, y) = 2x^5 - 128x^2y^3.$$

○ Taking outside the common factor $2x^2$, we get $P(x, y) = 2x^2(x^3 - 64y^3)$. The polynomial between parentheses is factored by the formula for the difference of cubes:

$$x^3 - 64y^3 = (x - 4y)(x^2 + 4xy + 16y^2).$$

Thus,

$$P(x, y) = 2x^2(x - 4y)(x^2 + 4xy + 16y^2). \bullet$$

Example 2. Factor the polynomial

$$P(x, y, z) = 20x^2 + 3yz - 15xy - 4xz.$$

○ Applying the method of grouping, we get

$$(x, y, z) = (20x^2 - 15xy) - (4xz - 3yz)$$
$$= 5x(4x - 3y) - z(4x - 3y) = (4x - 3y)(5x - z). \bigcirc$$

Example 3. Factor the polynomial $P(x, y) = x^4 + 4y^4$.

○ Let us complete the square by adding and subtracting the monomial $4x^2y^2$. We then get

$$P(x, y) = (x^4 + 4y^4 + 4x^2y^2) - 4x^2y^2$$

and, further,

$$P(x, y) = (x^2 + 2y^2)^2 - (2xy)^2$$
$$= (x^2 + 2y^2 - 2xy)(x^2 + 2y^2 + 2xy). \bullet$$

Exercises

1. Given two polynomials $P(x) = x^2 - 5x + 2$ and $Q(x) = x^3 + 3x + 1$. Find: (a) $P(x) + Q(x)$; (b) $2P(x) - 3Q(x)$; (c) $P(x) \cdot Q(x)$; (d) $(P(x))^2$.

2. Evaluate the given expressions:

(a) $-2xy^2 + 4x + 11xy^2$ if $x = 11/4$, $y = -1/3$,

(b) $(5.7a^2b - 3.1ab + 8b^3) - (6.9ab - 2.3a^2b + 8b^3)$ if $a = -2$, $b = 5$.

3. Prove that the value of the expression $a(2a + 1) - a^2(a + 2) + (a^3 - a + 3)$ is independent of the variable a.

4. Prove that the difference between a two-digit number and the number consisting of the same digits taken in reverse order is divisible by 9.

5. Find the coefficient B in the product $(x^2 - Bx + 2x^3) \cdot (0.1x - 0.25x^2 + 25x^3)$ if, on removing the parentheses, the coefficient of x^3 is equal to 1.8.

6. Represent, if possible, the given product in the form of the difference of squares: (a) $a^2(x - y)(x + y)$, (b) $(a + b + 1)(a + b - 1)$, (c) $x^3(x - 1) \times (x^2 + x)$.

In Exercises 7 to 10, perform the indicated operations using the formulas for abridged multiplication:

7. (a) $(a + 5)^2$, (b) $(3a - 7)^2$, (c) $(a^n + b^n)^2$.

8. (a) $(1 - x)^3$, (b) $(3 + y)^3$, (c) $\left(3a - \frac{1}{3}b\right)^3$.

9. (a) $(4 - a)(4 + a)$, (b) $(8 + a^2)(a^2 - 8)$, (c) $(a^n b^k - a^k b^n)(a^n b^k + a^k b^n)$.

10. (a) $(a - 2b)(a^2 + 2ab + 4b^2)$, (b) $(a + 3)(a^2 - 3a + 9)$, (c) $(a^2 + b^2)(a^4 - a^2b^2 + b^4)$.

In Exercises 11 to 14, compute the given expressions using the abridged multiplication identities:

11. (a) $48 \cdot 52$, (b) $93 \cdot 107$, (c) $24 \cdot 26$.

12. (a) $53^2 - 43^2$, (b) $22.4^2 - 22.3^2$, (c) $\left(6\frac{1}{3}\right)^2 - \left(5\frac{1}{3}\right)^2$.

13. (a) $(30 + 2)^2$, (b) 51^2, (c) 99^2.

14. (a) $17^2 + 2 \cdot 17 \cdot 23 + 23^2$, (b) $85^2 - 2 \cdot 85 \cdot 15 + 15^2$.

In Exercises 15 to 18, prove the given identities:

15. $(x - 2)(x + 2)(x^2 + 4) = x^4 - 16$.

16. $(x^2 + 1)^3 - 3(x^2 - x + 1)^2 + 6x(x - 1) + 11 = (x^3 + 3)^2$.

17. $(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2$.

18. $(a^2 + b^2)(x^2 + y^2) = (ax - by)^2 + (bx + ay)^2$.

In Exercises 19 to 26, factor the given polynomials:

19. (a) $3x(a-2)-a+2$, (b) $ac+7bc-3a-21b$.

20. (a) $(63m^4n^3+27m^3n^4-45m^5n^7$, (b) $5b^2c^3-2bc^2k-5bck+2k^2$, (c) $2x^3y^2+3xyz^2-2x^2yz-3z^3$, (d) $24ax-38bx-12a+19b$.

21. (a) $25a^2-\frac{1}{4}b^2q^2$, (b) $9(5a-4b)^2-64a^2$, (c) $121n^2-(3n+2t)^2$.

22. (a) $4t^2-20tn+25n^2-36$, (b) $-p^4+6p^2k-9k^2$, (c) $-16p^3q^8+72p^4q^7-81p^5q^6$.

23. (a) $6x^3-36x^2+72x-48$, (b) $15ax^3+45ax^2+45ax+15a$.

24. (a) $9a^{3n-1}-4.5a^{2n-1}$, (b) $4a^7b^2-32a^4b^5$, (c) $5p^{2n}q^n+15p^{5n}q^{2n}$.

25. (a) $(7x^2-4y^2)^2-(3x^2+8y^2)^2$, (b) $1000t^3+27t^6$.

26. (a) $7-56a^6b^3$, (b) $128a^4-96a^3+24a^2-2a$.

27. Prove that the difference of the squares of the sum and difference of two natural numbers is divisible by 4.

28. The difference of the squares of two successive natural numbers is 23. Find these numbers.

29. Prove that for any natural n the expression $\frac{n^3}{6}+\frac{n^2}{2}+\frac{n}{3}$ is a natural number.

30. Prove that the difference between the cube of an odd number and the number itself is divisible by 24.

31. Reduce the given fractions: (a) $\frac{53^2-27^2}{79^2-51^2}$, (b) $\frac{101^2-31^2}{139^2-29^2}$, (c) $\frac{100.6^2-80.2^2}{180.8}$.

32. For what value of m is it possible to represent the given expressions in the form of the square of a binomial: (a) $64p^2-mpq+9q^2$, (b) $25x^2+30x+m$?

33. Solve the following equations: (a) $(2x-1)^2-36=0$, (b) $4x^2-12x+9=49$.

34. Prove that: (a) if, when divided by 5, a natural number yields 2 as a remainder, then its cube, when divided by 5, yields 3 as a remainder; (b) if, when divided by 4, a natural number yields 3 as a remainder, then the sum of its cube and square is divisible by 4.

35. Compute the given expressions using the simplest way possible:

(a) $\frac{59^3-41^3}{18}+59\cdot41$, (b) $\frac{67^3+52^3}{119}-67\cdot52$,

(c) $(28.6^3-14.6^3)\div(28.6^2+28.6\cdot14.6+14.6^2)$.

36. Find the quotient and the remainder after the division of the polynomial $P(x)$ by the polynomial $Q(x)$: (a) $P(x)=2x^4-x^3+5$, $Q(x)=x^3-9x$, (b) $P(x)=2x^3+2$, $Q(x)=x^3+2x^2+x$, (c) $P(x)=x^6-1$, $Q(x)=x^4-4x^2$.

37. Prove that the polynomial x^2+2x+2 has no roots.

38. Find the roots of the given polynomials: (a) x^3-4x, (b) x^3+10x^2-x-10.

39. Factor the given polynomials: (a) $6a^2(a-5)+(5-a)^3$, (b) $x^2(x-3)(2x+3)^2-4x^3(3-x)^2$, (c) a^2-4a-5, (d) $x^3-6x^2-11x-6$.

40. Solve the given equations by completing the square of a binomial: (a) $x^2+2x-3=0$, (b) $x^2-13x+30=0$.

41. Prove that: (a) $7^{10}-7^9-7^8$ is divisible by 41, (b) $81^7-27^9-9^{13}$ is divisible by 45, (c) 11^3+12^3 is divisible by 23, (d) 94^3-9^3 is divisible by 17.

42. Prove that: (a) for any integral n the value of the expression $(5n+7)^2-(1-n)^2$ is divisible by 48; for any even n tbe value of the expression $(7n+3)^2-(3-n)^2$ is divisible by 96.

§ 2.2. Transformations of Fractional Expressions

1. Basic Notions. It was noted above that if a rational expression does not involve a division by an expression with variables, then it is called integral. If, when composing a rational expression, we use a division by an expression with variables, then such a rational expression is called *fractional*. Given below are a few examples of fractional expressions:

$$\frac{1+2y+y^2}{y^2-4}, \quad \frac{a+b+c}{a}-\frac{1}{c}, \quad \frac{\frac{1}{x}+\frac{1}{y}}{2}.$$

On the other hand, the expression $\frac{3x^2+y^2+2z^3}{7}$ is not fractional, although its notation looks like a fraction. This is an integral expression which can be reduced to standard form of a polynomial:

$$\frac{3}{7}x^2+\frac{1}{7}y^2+\frac{2}{7}z^3.$$

Fraction is defined as an expression of the form $\frac{a}{b}$, where the letters a and b denote numerical expressions or expressions with variables; a is the numerator of the fraction, b its denominator. All of the examples considered above represent fractions except for the fractional expression $\frac{a+b+c}{a}-\frac{1}{c}$.

One of the main problems of identical transformations of fractional expressions is to represent a given expression in the form of a fraction whose numerator and denominator are integral expressions (to separate such fractions out of the set of all fractions, let us agree to call them *algebraic*). Such a transformation, as we shall see, is always feasible.

The *domain of definition* of an expression in one variable is the set of values of the variable for which this expression has sense. For example, the domain of definition of the expression $\frac{1+2y+y^2}{y^2-4}$ consists of all numbers except -2 and 2.

If an expression in two variables x and y is given, then the domain of its definition is represented by a set of numerical pairs of the form (x, y) for which the expression is satisfied. Thus, the expression $\frac{1/x+1/y}{2}$ is defined on the set of all pairs (x, y) except for pairs of the form $(0, y)$ and $(x, 0)$.

Integral expressions with variables are defined for any values of the variables. *To find the domain of definition of a given algebraic fraction, it is necessary to find the values of the variables for which the denominator vanishes and to discard such values.*

2. Fundamental Property of Algebraic Fractions. As is known, the numerator and denominator of a common fraction may be

multiplied or divided by the same nonzero number. A natural generalization of this fact yields the **fundamental property of an algebraic fraction** expressed by the following identity:

$$\frac{ac}{bc}=\frac{a}{b}.$$

The fundamental property of fractions has various applications. Thus, if both the numerator and denominator of a fraction are polynomials with fractional coefficients, then, to simplify the notation, it is advisible to multiply the numerator and denominator of the fraction by the least common multiple of the denominators of all coefficients. This multiplication is valid by virtue of the fundamental property of algebraic fractions.

Example. Simplify the fraction

$$\frac{\frac{1}{3}x^2-\frac{1}{2}x+1}{\frac{1}{4}x^2+\frac{1}{6}x+\frac{1}{2}}.$$

○ In this case, the least common multiple of the denominators of all the coefficients is the number 12. Multiplying both the numerator and denominator of the fraction by 12, we get

$$\frac{\frac{1}{3}x^2-\frac{1}{2}x+1}{\frac{1}{4}x^2+\frac{1}{6}x+\frac{1}{2}}=\frac{12\left(\frac{1}{3}x^2-\frac{1}{2}x+1\right)}{12\left(\frac{1}{4}x^2+\frac{1}{6}x+\frac{1}{2}\right)}=\frac{4x^2-6x+12}{3x^2+2x+6}. \quad \bullet$$

The fundamental property of fractions is used to change signs of the terms entering the fraction. Multiplying the numerator and denominator of the fraction $\frac{a}{b}$ by -1, we get $\frac{a}{b}=\frac{-a}{-b}$. Thus, *the value of a fraction remains unchanged if the signs of the numerator and denominator are changed simultaneously.* If we change the sign of either the numerator or the denominator, then the sign of the fraction also changes:

$$\frac{-a}{b}=-\frac{a}{b}, \quad \frac{a}{-b}=-\frac{a}{b}.$$

Changing the signs of both sides in the last identities, we get

$$\frac{a}{b}=-\frac{-a}{b}, \quad \frac{a}{b}=-\frac{a}{-b}.$$

Consequently, if we want to change the sign only of the numerator or denominator, then we have to change the sign of the fraction itself. For example,

$$\frac{3x-2}{3x+4}=-\frac{-(3x-2)}{3x+4}=-\frac{2-3x}{3x+4}.$$

3. Reducing Algebraic Fractions. To reduce a fraction means to divide both the numerator and denominator of the fraction by a common factor. The possibility of such reduction is stipulated by the fundamental property of fractions.

To reduce an algebraic fraction, we have to factor its numerator and denominator. If it turns out that the numerator and denominator have common factors, then they can be cancelled out. If there are no common factors, then it is impossible to simplify the fraction by reduction.

Example. Reduce the fraction $\frac{x^2-5xy+6y^2}{9y^2-x^2}$.

○ To factor the numerator, we first represent the monomial $-5xy$ in the form of a sum $(-2xy) + (-3xy)$ and then use the method of grouping. We have

$$x^2 - 5xy + 6y^2 = (x^2 - 2xy) - (3xy - 6y^2)$$
$$= x(x - 2y) - 3y(x - 2y) = (x - 2y)(x - 3y).$$

To factor the denominator, we use the formula for the difference of squares:

$$9y^2 - x^2 = (3y + x)(3y - x).$$

Thus,

$$\frac{x^2-5xy+6y^2}{9y^2-x^2} = \frac{(x-2y)(x-3y)}{(3y+x)(3y-x)}.$$

Further, changing the sign in the denominator, we get

$$\frac{(x-2y)(x-3y)}{(3y+x)(3y-x)} = -\frac{(x-2y)(x-3y)}{(3y+x)(x-3y)} = -\frac{x-2y}{3y+x} = \frac{2y-x}{3y+x}. \bullet$$

4. Reducing Algebraic Fractions to a Common Denominator. A *common denominator* of two or more algebraic fractions is a polynomial divisible by the denominator of each fraction.

For example, the common denominator of the fractions $\frac{x}{x+2}$ and $\frac{3x-1}{x-2}$ is the polynomial $(x + 2)(x - 2)$ since it is divisible both by $x + 2$ and $x - 2$. It should be noted, however, that this polynomial is not unique. The polynomials $3(x + 2)(x - 2)$, $x(x + 2) \times (x - 2)$, $5(x + 2)^2(x - 2)^3$, etc. are also common denominators of the given fractions. We usually take a common denominator such that any other common denominator is exactly divisible by the chosen one. Such a denominator is called the *least common denominator.*

In the example considered, the least common denominator is $(x+2)(x-2)$. We have:

$$\frac{x}{x+2} = \frac{x(x-2)}{(x+2)(x-2)}; \quad \frac{3x-1}{x-2} = \frac{(3x-1)(x+2)}{(x-2)(x+2)}.$$

We have succeeded in reducing the given fractions to the least common denominator. This is achieved by multiplying the numerator and denominator of the first fraction by $x - 2$ and the numerator and denominator of the second fraction by $x + 2$. The polynomials $x - 2$ and $x + 2$ are called *multipliers* for the first and the second fraction, respectively. It is easy to grasp that a multiplier for a given fraction is equal to the quotient of the division of the least common denominator by the denominator of the given fraction.

Example 1. Reduce the fractions $\frac{1}{2(x-y)}$ and $\frac{1}{6(x-y)^2}$ to the least common denominator.

○ In this case, the least common denominator is equal to $6(x-y)^2$. In order to rewrite the given fractions with the denominator $6(x-y)^2$, we have to find the multipliers. For the first fraction, the multiplier is $3(x-y)$. As far as the second fraction is concerned, its denominator coincides with the least common denominator. In such a case, the multiplier is said to be equal to 1. Hence,

$$\frac{1}{2(x-y)} = \frac{3(x-y)}{2(x-y)\cdot 3(x-y)} = \frac{3(x-y)}{6(x-y)^2}.$$

The second fraction is left unchanged. Thus, the given fractions have been reduced to the least common denominator. ●

In the general case, to reduce two or more fractions to the least common denominator, we have to factor out all the denominators, to take all the factors from the first denominator and to add from the remaining denominators only those factors which are absent from the first denominator. Let us consider an example to show how this is done.

Example 2. Reduce the following fractions to the least common denominator:

$$\frac{a+2}{2a^3-4a^2}, \quad \frac{a+3}{6a-3a^2} \text{ and } \frac{a+1}{4a^5-16a^3}.$$

○ Factor each denominator:

$$2a^3 - 4a^2 = 2a^2(a-2), \quad 6a - 3a^2 = 3a(2-a) = -3a(a-2),$$
$$4a^5 - 16a^3 = 4a^3(a^2-4) = 4a^3(a-2)(a+2).$$

Then the least common denominator is found. To this end, we take the first denominator $2a^2(a-2)$. From the second denominator we take the factor 3 since it is absent from the first denominator, and from the third the factor $2a(a+2)$. Thus, the least common denominator is equal to $12a^3(a-2)(a+2)$.

Now, we find the multipliers. Dividing the least common denominator by the denominator of the first fraction, we get the multiplier for the first fraction: $6a(a+2)$. Similarly, dividing the least com-

mon denominator by the denominator of the second fraction, we get the multiplier for the second fraction: $-4a^2(a+2)$. Finally, dividing the least common denominator by the denominator of the third fraction, we find the multiplier for the third fraction, 3. We have

$$\frac{a+2}{2a^3-4a^2}=\frac{(a+2)\,6a\,(a+2)}{12a^3(a-2)(a+2)}=\frac{6a(a+2)^2}{12a^3(a-2)(a+2)},$$

$$\frac{a+3}{6a-3a^2}=-\frac{4a^2(a+3)(a+2)}{12a^3(a-2)(a+2)},$$

$$\frac{a+1}{4a^5-16a^3}=\frac{3(a+1)}{12a^3(a-2)(a+2)}.$$ ●

We recommend to stick to the following table for finding least common denominators and multipliers (based on the above example):

Denominators	Least common denominator	Multipliers
$2a^2(a-2)$ $-3a(a-2)$ $4a^3(a-2)(a+2)$	$12a^3(a-2)(a+2)$	$6a(a+2)$ $-4a^2(a+3)$ 3

5. Multiplying and Dividing Algebraic Fractions. Common fractions are multiplied as follows:

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

The following identity is a natural generalization of this equality:

$$\frac{P_1(x,\,y)}{Q_1(x,\,y)}\cdot\frac{P_2(x,\,y)}{Q_2(x,\,y)}=\frac{P_1(x,\,y)\cdot P_2(x,\,y)}{Q_1(x,\,y)\cdot Q_2(x,\,y)},$$

where P_1, P_2, Q_1, and Q_2 are integral algebraic expressions. Hence, *the product of two* (and, in general, of any finite number of) *algebraic fractions is identically equal to a fraction whose numerator is the product of the numerators, and the denominator is the product of the denominators of the fractions being multiplied.*

Common fractions are divided in the following way:

$$\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}.$$

The following identity is a natural generalization of this equality:

$$\frac{P_1(x,\,y)}{Q_1(x,\,y)}\div\frac{P_2(x,\,y)}{Q_2(x,\,y)}=\frac{P_1(x,\,y)\cdot Q_2(x,\,y)}{Q_1(x,\,y)\cdot P_2(x,\,y)}.$$

Hence, *the quotient of the division of two algebraic fractions is identically equal to a fraction whose numerator is the product of the numera-*

tor of the first fraction by the denominator of the second, and whose de nominator is the product of the denominator of the first fraction by the numerator of the second.

Example 1. Change the given product into a fraction:

$$\frac{x^2+2x+1}{18x^3}\cdot\frac{9x^4}{x^2-1}.$$

○ Using the rule for multiplying fractions, we get

$$\frac{x^2+2x+1}{18x^3}\cdot\frac{9x^4}{x^2-1}=\frac{(x^2+2x+1)\cdot 9x^4}{18x^3(x^2-1)}.$$

Simplifying the obtained fraction, we have

$$\frac{9x^4(x+1)^2}{18x^3(x+1)(x-1)}=\frac{x(x+1)}{2(x-1)}.$$ ●

Example 2. Change the given quotient into a fraction:

$$\frac{a^3-2a^2}{3a+3}\div\frac{a^2-4}{3a^2+6a+3}.$$

○ Using the rule for dividing fractions, we get

$$\frac{a^3-2a^2}{3a+3}\div\frac{a^2-4}{3a^2+6a+3}=\frac{(a^3-2a^2)(3a^2+6a+3)}{(3a+3)(a^2-4)}.$$

What remains is to simplify, thus:

$$\frac{(a^3-2a^2)(3a^2+6a+3)}{(3a+3)(a^2-4)}=\frac{a^2(a-2)\cdot 3(a^2+2a+1)}{3(a+1)(a-2)(a+2)}$$

$$=\frac{3a^2(a-2)(a+1)^2}{3(a+1)(a-2)(a+2)}=\frac{a^2(a+1)}{a+2}.$$ ●

To conclude this subsection, we should like to note the following:

(1) Taking into consideration the possibility of reducing the algebraic fraction obtained as a result of multiplication or division of algebraic fractions, prior to performing these operations, we try to factor both the numerators and denominators of the original fractions.

(2) The given rules for multiplying and dividing algebraic fractions are also extended to the case of multiplying or dividing by a polynomial: it suffices to write this polynomial in the form of a fraction with denominator 1.

6. **Raising an Algebraic Fraction to a Natural Power.** Let there be given an algebraic fraction $P(x, y)/Q(x, y)$. Raising it to the nth power, we get

$$\left(\frac{P}{Q}\right)^n=\underbrace{\frac{P}{Q}\cdot\frac{P}{Q}\cdots\frac{P}{Q}}_{n\text{ times}}=\underbrace{\frac{P\cdot P\ldots P}{Q\cdot Q\ldots Q}}_{n\text{ times}}=\frac{P^n}{Q^n}.$$

Thus,

$$\left(\frac{P(x, y)}{Q(x, y)}\right)^n = \frac{(P(x, y))^n}{(Q(x, y))^n}.$$

Hence, *the power of a fraction is identically equal to a fraction in which the numerator is the power of the numerator and the denominator the power of the denominator.*

Example. Transform the power $\left(\frac{2x^2y^3}{3z^5}\right)^3$ into a fraction.

○ We have

$$\left(\frac{2x^2y^3}{3z^5}\right)^3 = \frac{(2x^2y^3)^2}{(3z^5)^3} = \frac{8x^6y^9}{27z^{15}}. \quad \bullet$$

7. Adding and Subtracting Algebraic Fractions. For common fractions the following equality is valid:

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}.$$

The following identity is a natural generalization of this equality:

$$\frac{P_1(x, y)}{Q(x, y)} + \frac{P_2(x, y)}{Q(x, y)} = \frac{P_1(x, y) + P_2(x, y)}{Q(x, y)}$$

It means that the *sum of two* (and, in general, any finite number of) *algebraic fractions with a common denominator is identically equal to an algebraic fraction with the same denominator and with the numerator equal to the sum of the numerators of the added fractions.*

In the case of subtracting fractions, we have a similar thing. Indeed,

$$\frac{P_1}{Q} - \frac{P_2}{Q} = \frac{P_1}{Q} + \frac{-P_2}{Q} = \frac{P_1 + (-P_2)}{Q} = \frac{P_1 - P_2}{Q}.$$

In order to add and subtract algebraic fractions with different denominators, we have first of all to reduce the given fractions to the least common denominator and then to perform operations on the obtained fractions with the same denominator.

Example. Simplify (that is, transform into a fraction) the expression

$$\frac{4b}{4b^2-1} + \frac{2b+1}{3-6b} - \frac{1-2b}{4b+2}.$$

○ We find the least common denominator. To this end, let us first factor the denominators:

$$(1) \quad 4b^2 - 1 = (2b - 1)(2b + 1), \quad (2)\ 3 - 6b = -3(2b - 1),$$
$$(3)\ 4b + 2 = 2(2b + 1).$$

Further, we have

Denominators	Least common denominator	Multipliers
(1) $(2b-1)(2b+1)$ (2) $-3(2b-1)$ (3) $2(2b+1)$	$6(2b-1)(2b+1)$	(1) 6 (2) $-2(2b+1)$ (3) $3(2b-1)$

We now rewrite the given algebraic sum of fractions using the multipliers which reduce the fractions to a common denominator:

$$\frac{4b}{4b^2-1}\overset{|6}{+}\frac{2b+1}{3-6b}\overset{|-2(2b+1)}{-}\frac{1-2b}{4b+2}\overset{|3(2b-1)}{\cdot}.$$

It is common practice to omit multipliers and to write the obtained fraction whose denominator is the least common denominator, while the numerator represents the sum of the products of the numerators of the original fractions by the corresponding multipliers. In our case, we have

$$\frac{4b\cdot 6+(2b+1)(-2(2b+1))-(1-2b)\cdot 3(2b-1)}{6(2b-1)(2b+1)}$$

and, further,

$$\frac{24b-2(4b^2+4b+1)+3(4b^2-4b+1)}{6(2b-1)(2b+1)}$$

$$=\frac{24b-8b^2-8b-2+12b^2-12b+3}{6(2b-1)(2b+1)}=\frac{4b^2+4b+1}{6(2b-1)(2b+1)}$$

$$=\frac{(2b+1)^2}{6(2b-1)(2b+1)}=\frac{2b+1}{6(2b-1)}. \bullet$$

8. **Rational Fraction Represented as the Sum of Partial Fractions.** Let $P(x)$ and $Q(x)$ be polynomials. Then the quotient $P(x)/Q(x)$ is called the *rational fraction*. In other words, a fraction whose numerator and denominator are both polynomials is a rational fraction; it is a *proper fraction* if the numerator (that is, the polynomial $P(x)$) is of lower degree than the denominator (that is, the polynomial $Q(x)$). If the degree of the numerator is equal to, or greater than, that of the denominator, then the fraction is said to be *improper*.

For example, $\frac{x^2+2x-3}{x^3+3x+4}$ is a proper fraction (the numerator is of a lower degree than the denominator), $\frac{x^4-1}{100x^4+x^3+2}$ is an improper fraction (both the numerator and denominator are of the same degree), $\frac{x^5-3x^2+x-8}{x^3+x+2}$ is an improper fraction (the numerator is of a higher degree than the denominator).

An improper fraction can always be represented as a sum of a polynomial and a proper fraction. This is achieved by dividing the polynomial in the numerator of the fraction by the polynomial in the denominator. Let $P(x)$ be the numerator and $Q(x)$ the denominator of an improper fraction. In § 2 (Item 2),

we noted that the following identity is satisfied

$$P(x) = Q(x) \cdot S(x) + R(x),$$

where the degree of the remainder $R(x)$ is less than that of the divisor $Q(x)$. Dividing termwise by $Q(x)$, we get

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}.$$

Here $S(x)$ is a polynomial (in particular, it may be a polynomial of degree zero, that is, a constant number; this happens when the numerator and denominator of a rational fraction are of the same degree), and $R(x)/Q(x)$ is a proper fraction (since the degree of the remainder $R(x)$ is less than that of the divisor $Q(x)$).

Example. Represent the improper rational fraction

$$\frac{x^5 - 3x^4 + x^3 + 2x^2 - 3x + 4}{x^4 - 4x^3 + 4x^2}$$

as a sum of a polynomial and a proper fraction.

○ Let us divide the numerator by the denominator using the rule for dividing a polynomial by a polynomial (see § 2.1 (Item 2)):

$$\begin{array}{r|l} & x+1 \\ \cline{2-2} x^4 - 4x^3 + 4x^2 & x^5 - 3x^4 + x^3 + 2x^2 - 3x + 4 \\ & \underline{x^5 - 4x^4 + 4x^3} \\ & \quad x^4 - 3x^3 + 2x^2 - 3x + 4 \\ & \quad \underline{x^4 - 4x^3 + 4x^2} \\ & \quad\quad x^3 - 2x^2 - 3x + 4 \end{array}$$

We have obtained the quotient $S(x) = x + 1$ and the remainder $R(x) = x^3 - 2x^2 - 3x + 4$. Hence,

$$\frac{x^5 - 3x^4 + x^3 + 2x^2 - 3x + 4}{x^4 - 4x^3 + 4x^2} = x + 1 + \frac{x^3 - 2x^2 - 3x + 4}{x^4 - 4x^3 + 4x^2}. \quad \bullet$$

The above transformation resembles reduction of an improper common fraction to a whole or mixed number. Therefore, it is also called the reduction of an improper rational fraction.

Fractions of the form $\frac{A}{(x-a)^n}$, where A and a are real numbers, and $n \in \mathbf{N}$ are called *partial* rational fractions. The latter are exemplified by the following fractions:

$$\frac{3}{x-2}, \quad \frac{2.3}{x+1}, \quad \frac{-4}{(x-3)^2}, \quad \frac{\sqrt{3}}{(x+5.5)^{17}}.$$

We can show that the sum of any finite number of partial fractions is a proper rational fraction. For example,

$$\frac{2}{x-1} + \frac{3}{x+1} - \frac{4}{x+2} = \frac{x^2 + 9x + 2}{(x-1)(x+1)(x+2)},$$

$$\frac{3}{(x-2)^3} + \frac{2}{(x-2)^2} + \frac{1}{x-2} - \frac{2}{x+3} = \frac{-x^3 + 13x^2 - 27x + 25}{(x-2)^3(x+3)}$$

(check this!).

Many proper fractions can be represented as a sum of partial fractions. The possibility of such a representation is stipulated by the following theorem (given without proof!).

Theorem. *Let there be given a proper fraction* $P(x)/Q(x)$, *and*

$$Q(x) = (x-a)^n(x-b)^k \ldots (x-c)^m,$$

where $a, b, \ldots, c$ *are real numbers, and* $n, k, \ldots, m$ *natural numbers. Then the fraction* $P(x)/Q(x)$ *can be represented as a sum of partial fractions, each factor of the denominator of the form* $(x-a)^n$, *where* $n > 1$, *being associated with the following sum of partial fractions:*

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A^3}{(x-a)^3} + \ldots + \frac{A_n}{(x-a)^n}$$

($A_1, A_2, \ldots, A_n$ *real numbers*), *and each factor of the form* $x-b$ *being associated with a fraction* $\frac{B}{x-b}$. Thus (see above),

$$\frac{x^2+9x+2}{(x-1)(x+1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+1} - \frac{4}{x+2},$$

$$\frac{-x^3+13x^2-27x+25}{(x+3)(x-2)^3} = \frac{-2}{x+3} + \frac{1}{x-2} + \frac{2}{(x-2)^2} + \frac{3}{(x-2)^3}.$$

Example 1. Decompose the proper rational fraction $\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}$ into partial fractions.

○ According to the above theorem, the given fraction can be expressed as the following sum of partial fractions:

$$\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{x-4}.$$

Here A, B, and C are some undetermined coefficients. To find them, multiply both sides of the identity by the denominator of its left-hand side:

$$2x^2 + 41x - 91 \equiv A(x+3)(x-4) + B(x-1)(x-4) + C(x-1)(x+3). \quad (1)$$

The obtained identity is valid for any x. We shall assign to the variable x such values which make all the terms, but one, of the right-hand side vanish. Such values are, obviously, $x = 1$, $x = -3$, and $x = 4$.

For $x = 1$ all the terms of the right-hand side except the first one will vanish, the left-hand side for $x = 1$ will be equal to -48, and we shall get

$$-48 = A(1+3)(1-4) \text{ or } -48 = -12A, \text{ that is, } A = 4.$$

For $x = 4$ the left-hand side of identity (1) takes on the value 105, and all the terms of the right-hand side, except the third one, vanish

$$105 = C(4-1)(4+3), \text{ that is, } C = 5.$$

Finally, for $x = -3$ we get

$$-196 = B(-3-1)(-3-4), \text{ that is, } B = -7.$$

Thus,

$$\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} = \frac{4}{x-1} - \frac{7}{x+3} + \frac{5}{x-4}. \bullet$$

Example 2. Represent the improper fraction

$$\frac{x^5-3x^4+x^3+2x^2-3x+4}{x^4-4x^3+4x^2}$$

in the form of a sum of a polynomial and partial fractions.

○ We have already reduced the given fraction into the integral and fractional parts (see the beginning of this subsection):

$$\frac{x^5-3x^4+x^3+2x^2-3x+4}{x^4-4x^3+4x^3}=x+1+\frac{x^3-2x^2-3x+4}{x^4-4x^3+4x^2}.$$

Let us now decompose the proper fraction into partial fractions

$$\frac{x^3-2x^2-3x+4}{x^4-4x^3+4x^2}.$$

We first factor the denominator:

$$x^4-4x^3+4x^2=x^2(x^2-4x+4)=x^2(x-2)^2.$$

According to the above theorem, the given rational fraction can be expressed as a sum of partial fractions in the following way:

$$\frac{x^3-2x^2-3x+4}{x^2(x-2)^2}\equiv\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}+\frac{D}{(x-2)^2}.$$

We then multiply both sides of this identity by the denominator of the left-hand side:

$$x^3-2x^2-3x+4\equiv Ax(x-2)^2+B(x-2)^2+Cx^2(x-2)+Dx^2. \quad (2)$$

For $x=0$ we get $4=4B$, that is, $B=1$, for $x=2$ we have: $-2=4D$, that is, $D=-1/2$.

Unfortunately, in this example we have no more such values of x from which the undetermined coefficients A and C can be computed at once. To find these coefficients, we are going to use the theorem of the identity of two polynomials (see page 72) according to which two polynomials in one variable are identical if and only if they are of the same degree, and the coefficients of like powers of the variable in both polynomials coincide.

Removing the parentheses from the right-hand side of identity (2) and collecting like terms, we get

$$x^3-2x^2-3x+4\equiv(A+C)x^3+(-4A+B-2C+D)x^2+(4A-4B)x+4B.$$

From the comparison of the terms containing x in the first power, we get $-3=4A-4B$. But $B=1$, consequently, $4A=1$, whence $A=1/4$. From the comparison of terms containing x^3, we have $1=A+C$. Since $A=1/4$, we get $C=3/4$. Thus,

$$\frac{x^3-2x^2-3x+4}{x^2(x-2)^2}\equiv\frac{1/4}{x}+\frac{1}{x^2}+\frac{3/4}{x-2}-\frac{1/2}{(x-2)^2}.$$

Finally, we get

$$\frac{x^5-3x^4+x^3+2x^2-3x+4}{x^4-4x^3+4x^2}\equiv x+1+\frac{1/4}{x}+\frac{1}{x^2}+\frac{3/4}{x-2}-\frac{1/2}{(x-2)^2}. \bullet$$

Exercises

1. Find the domain of definition of the given fractions:

(a) $\frac{x+1}{x}$, (b) $\frac{x}{x^2-4}$, (c) $\frac{x^2+5}{x^2+6}$,

(d) $\frac{x^4+x^2+2}{x^2-6x}$, (e) $\frac{x}{x^2-7x+10}$.

2. Think of a fraction $f(x)$ such that for $x = 1$: (a) $f(x) = 0$, (b) $f(x) > 0$, (c) $f(x) < 0$, (d) $f(x) > 2$, (e) $f(x) < -3$, (f) $f(x) = 4$.

3. Give an example of a fraction $f(x)$ which vanishes for: (a) $x = 0$, (b) $x = 9$, (c) $x = -3$, (d) $x = 3$, $x = 0$, (e) $x = 4$, $x = -4$.

4. Give an example of a fraction $f(x)$ which is not defined for: (a) $x = 8$, (b) $x = 1$, $x = -1$, (c) $x = 0$, $x = -8$.

5. Find the numerical value of algebraic fraction $f(x, y) = \dfrac{2x+3y}{4x-y}$ if $x = 1/8$ and $y = -1/3$.

In Exercises 6 and 7, reduce the given fractions:

6. (a) $\dfrac{63x^2y^2}{77y^4}$, (b) $\dfrac{2x^3-8x^5}{12x^4}$,

(c) $\dfrac{(x-4)(x-5)}{(x+4)(5-x)}$, (d) $\dfrac{5a^2b^7-4ab^3}{10a^4b^6-8a^3b^2}$.

7. (a) $\dfrac{a^2+5a+25}{2a^4-250a}$, (b) $\dfrac{1-3x+3x^2-x^3}{x^2-1}$,

(c) $\dfrac{x^2+yz+xz-y^2}{x^2-yz-xz-y^2}$, (d) $\dfrac{x^2-3x+2}{x^2-1}$.

8. Simplify the fraction $\dfrac{\frac{x^2}{60}-0.1x+\frac{1}{12}}{\frac{x^2}{15}-0.2x+\frac{2}{15}}$.

9. Using the most rational method, find the numerical value of the given fractions:

(a) $\dfrac{1-6a+9a^2}{9a^2-1}$ for $a=\dfrac{5}{12}$,

(b) $\dfrac{4xy^3-4x^3y}{6xy^3-12x^2y^2+6x^3y}$ for $x=4$, $y=3$.

In Exercises 10 to 25, simplify the given expressions:

10. (a) $\dfrac{8x^5}{33x^4}\left(-\dfrac{55x^5}{12x^6}\right)$, (b) $\dfrac{3a^{n+5}}{44a^2} \div \dfrac{a^{n-5}}{22a}$,

(c) $(-48a^5x^{k-1}) \div \dfrac{36x^k}{a^3}$.

11. $\dfrac{25a^5}{6a^3-6} \div \dfrac{10a^4}{9a^2+9a+9}$.

12. $\dfrac{a^3+8}{4a^2-1} \cdot \dfrac{1-2a}{2a^2-4a+8} \cdot \dfrac{6a+3}{4+4a+a^2}$.

13. $\dfrac{a^4-64a}{a^2-2a+1} \cdot \dfrac{a^2-1}{a^2-16} \div \dfrac{a^3+4a^2+16a}{a+4}$.

14. (a) $\left(-\dfrac{(x-2)^3}{2x-1}\right)^5$, (b) $\left(-\dfrac{(2x+1)^k}{(2x-1)^n}\right)^{2m}$.

15. (a) $\dfrac{(3a+2)^3}{a+2} - \dfrac{36a+8}{2+a}$, (b) $\dfrac{(3a-4)^2}{2-3a} + \dfrac{9a^2}{3a-2}$.

16. $\frac{x^3+y^3}{(x-y)^2}+\frac{3xy^2-y^3}{(y-x)^2}+\frac{3x^2y+y^3}{2xy-x^2-y^2}$.

17. $\frac{x^3+27}{x-3}-x^2-3x-9$.

18. (a) $\frac{5}{14x^3}-\frac{1}{42x}+\frac{1}{16x}$, (b) $\frac{1+y^{n+2}}{y^{2n+1}}-\frac{1}{y^{n-1}}$.

19. (a) $\frac{15a-13}{3a+1}-\frac{10a-1}{2a}$, (b) $\frac{x}{9x-3}+\frac{x-1}{18x-6}-\frac{x+2}{4-12x}$.

20. (a) $\frac{ad-bc}{2cd(c+d)}+\frac{ad+bc}{2cd(c-d)}$, (b) $\frac{1}{4x^2-9}+\frac{2x+3}{2x-3}$.

21. $\frac{2}{x+3}+\frac{x+15}{x^2-9}+\frac{3}{3-x}$.

22. $\frac{3a(16-3a)}{9a^2-4}+\frac{3(1+2a)}{2-3a}-\frac{2-9a}{3a+2}$.

23. $\frac{x^2}{x-3}+\frac{18+2x^2}{x^3+3x+9}+\frac{3x(x^2+x+15)}{27-x^3}$.

24. $\frac{x+2y}{x^2+2xy+y^2}-\frac{x-2y}{x^2-y^2}$.

25. $\frac{x^2}{x^2+5x+6}-\frac{4}{x+2}+\frac{9}{x+3}$.

In Exercises 26 to 28, prove the given identities:

26. $\frac{5}{18y}-\frac{2+3y}{3y^2}+\frac{y+2}{3y^2}-\frac{y-3}{9y^2}=\frac{y-2}{6y^2}$.

27. $\frac{1}{2z^2+5z}-\frac{2}{25-10z}-\frac{4}{4z^2-25}=\frac{1}{5z}$.

28. $\frac{\frac{a^2}{b^2}-\frac{a}{b}+\frac{b^2}{a^2}-\frac{b}{a}}{\left(\frac{1}{a}-\frac{1}{b}\right)^2\left(\frac{a}{b}+\frac{b}{a}+1\right)}=ab$.

29. Solve the given equations:

(a) $\frac{x^2}{x-6}+\frac{6x}{6-x}=0$,

(b) $\frac{3}{2x-1}-\frac{7}{5x-6}=0$,

(c) $\frac{y+1}{y-1}-\frac{y-5}{y-3}=0$.

30. Evaluate the expression $\frac{3-x}{x-2}-\frac{x^2+4}{2x^3-8x}+\frac{x+3}{x+2}-\frac{x}{x^2-4}$ for $x=-0.05$.

In Exercises 31 to 38, simplify the given expressions:

31. $\left(\frac{6a+3}{a^2-18a}+\frac{6a-3}{a^2+18a}\right)\cdot\frac{a^2-324}{a^2+9}$.

32. $\left(\frac{a-3}{a^2-3a+9}-\frac{2a-6}{a^3+27}\right)\div\frac{a+1}{2a^3+54}$.

33. $\left(1+\frac{x^2-2x+4}{x^2-4}\right)\cdot\left(\frac{3x}{2x-2}-\frac{3x+2}{x}+1\right)$.

34. $\left(\frac{m-n}{mn}+\frac{3m+n}{m^2-mn}-\frac{3n+m}{n^2-mn}\right)\div\frac{2m+2n}{mn}+\frac{2m}{n-m}$.

35. $\left(\frac{x^2}{x^2-y^2}-\frac{x^2y}{x^3+y^3}\cdot\left(\frac{x}{xy+y^2}+\frac{y}{x^2+xy}\right)\right)\div\frac{y}{x-y}$.

36. $\left(2x+1-\frac{1}{1-2x}\right)\div\left(2x-\frac{4x^2}{2x-1}\right)$.

37. $\frac{a-2}{a^2+2a}\div\left(\frac{a}{a^2-2a}-\frac{a^2+4}{a^3-4a}-\frac{1}{a^2+2a}\right)$.

38. $\left(\frac{25}{a^2+5a+25}-\frac{2a}{5-a}-\frac{a^3+25a^2}{a^3-125}\right)\left(a-5+\frac{15a}{a-5}\right)$.

39. Compute:

(a) $f(x)=\left(\frac{x}{x^2-4}+\frac{2}{2-x}+\frac{1}{2+x}\right)\div\left(\frac{x^3-4x}{x+1}\div\frac{x^2+2x}{x+1}-\frac{x^2-10}{x+2}\right)$

for $x=-10.25$.

(b) $f(x)=\frac{2-x}{5}+\left(\frac{1}{1-2x}\right)^2\div\left(\frac{x+2}{4x^3-4x^2+x}-\frac{2-x}{1-8x^3}\cdot\frac{4x^2+2x+1}{2x^2+x}\right)$

for $x=\frac{537}{4151}$.

40. Prove that for any value of x greater than 2 the numerical value of the expression

$$\left(\frac{x+1}{2}+\frac{4}{x+3}-2\right)\div\frac{x+1}{x+3}-\frac{x^2-5x+3}{2x}$$

is negative.

41. Make the substitution and simplify the expression $\frac{ax}{x+a}+\frac{bx}{x-b}$, where $x=\frac{ab}{a-b}$.

42. Represent the given proper fractions as a sum of partial fractions:

(a) $\frac{5x}{(x-2)(x+3)}$, (b) $\frac{5x+31}{x^2-3x-10}$, (c) $\frac{x^2+2}{(x-1)^3}$.

43. Represent the given improper fractions as a sum of a polynomial and partial fractions:

(a) $\frac{x^3+x+2}{x^2-7x+12}$, (b) $\frac{2x^4-x^3-9x^2+9x+81}{x^3-9x}$.

§ 2.3. Arithmetic Root and Its Properties

1. Arithmetic Root. Let a be a real number and n a natural number greater than 1. Let us pose the following problem: find a number x such that the following equality is satisfield:

$$x^n = a. \tag{1}$$

Let us consider a few examples.

Let $a = 16$, and $n = 2$. Then equality (1) takes the form $x^2 = 16$, whence $x_1 = 4$, $x_2 = -4$.

Let now $a = 27$, and $n = 3$. Then equality (1) takes the form $x^3 = 27$, whence $x = 3$.

Let, further, $a = -16$, $n = 4$. Then equality (1) takes the form $x^4 = -16$ which is satisfied for no real value of x.

Let, finally, $a = -32$, $n = 5$. Then equality (1) takes the form $x^5 = -32$, whence $x = -2$.

These examples show that the set problem can have two solutions (for an even n and $a > 0$), one solution (for an odd n), and no solution (for an even n and $a < 0$). If a problem has a solution, that is, the equality $x^n = a$ is fulfilled for some values of x, then these values are called the *nth roots of the number a.*

The above examples imply that there are two square roots of the number 16: the numbers 4 and -4; there is one cube root of the number 27: the number 3; there is no fourth root of the number -16; there is one fifth root of the number -32: the number -2.

Consider the case of finding the nth root of a nonnegative number. It is possible to prove (this will be done later) *that if* $a \geqslant 0$ *and* $n > 1$, *then there is one and only one nonnegative number* x *such that* $x^n = a$. It is called the *nth arithmetic root* of the nonnegative number a.

The nth arithmetic root of the number a is customarily denoted by the $\sqrt[n]{a}$, where a is the *radicand*, and n the *index*. If $n = 2$, then we usually write $\sqrt{a}$ rather than $\sqrt[2]{a}$, which is called the square root. The term "root" is frequently substituted by the term "radical".

Thus, according to the definition, the notation $\sqrt[n]{a} = x$, where $a \geqslant 0$, means, firstly, that $x \geqslant 0$, and, secondly, that $x^n = a$, that is, $(\sqrt[n]{a})^n = a$. For example, $\sqrt{36} = 6$, $\sqrt[3]{27} = 3$, $\sqrt[4]{16} = 2$, $\sqrt[n]{0} = 0$.

From the aforesaid it follows that, for instance, the equality $\sqrt{36} = 6$ is correct, whereas the equalities $\sqrt{36} = -6$ or $\sqrt{36} = \pm 6$ are incorrect.

2. Properties of Arithmetic Roots. Let us first agree on the following: all variables which are encountered in formulating the properties and in the examples considered in the present and subsequent subsections will be regarded as nonnegative values. Besides, we shall consider only arithmetic roots and, therefore, for the sake of brevity, we shall not mention the word "arithmetic" each time it is meant. When proving the properties of roots, we shall use the properties of a power with natural exponent (see § 1.4, Item 1).

1°. *The nth root of the product of two nonnegative factors is equal to the product of the roots of these factors*:

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

□ We know that $\sqrt[n]{ab}$ is a nonnegative number such that, when being raised to a power of n, yields the integrand ab. It is clear that $\sqrt[n]{a}\cdot\sqrt[n]{b}$ is a nonnegative number. Hence, if we show that $(\sqrt[n]{a}\cdot\sqrt[n]{b})^n = ab$, then this will mean that $\sqrt[n]{ab} = \sqrt[n]{a}\cdot\sqrt[n]{b}$.

Thus, let us consider the expression $(\sqrt[n]{a}\cdot\sqrt[n]{b})^n$. According to Property 4° of a power with a natural exponent, we have $(\sqrt[n]{a}\cdot\sqrt[n]{b})^n = (\sqrt[n]{a})^n\cdot(\sqrt[n]{b})^n$. But, by the definition of the arithmetic root $(\sqrt[n]{a})^n = a$, $(\sqrt[n]{b})^n = b$. Consequently, $(\sqrt[n]{a}\cdot\sqrt[n]{b})^n = ab$. ■

This property is extended to the product of any finite set of nonnegative numbers. For example, $\sqrt[3]{8\cdot 27\cdot 125} = \sqrt[3]{8}\cdot\sqrt[3]{27}\cdot\sqrt[3]{125} = 2\cdot 3\cdot 5 = 30$.

2°. *The nth root of a fraction whose numerator is nonnegative and the denominator is positive is equal to the root of the numerator divided by the root of the denominator*:

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad (b \neq 0).$$

This property is proved in the same manner as the preceding one.

3°. *In order to raise the nth root to a natural power k, it suffices to raise the integrand to this power and to extract the nth root of the result obtained*:

$$(\sqrt[n]{a})^k = \sqrt[n]{a^k}.$$

□ Using Property 1°, we have

$$(\sqrt[n]{a})^k = \underbrace{\sqrt[n]{a}\cdot\sqrt[n]{a}\ \ldots\ \sqrt[n]{a}}_{k \text{ factors}} = \sqrt[n]{\underbrace{a\cdot a\ \ldots\ a}_{k \text{ factors}}} = \sqrt[n]{a^k}. \quad ■$$

For example $(\sqrt[3]{m^2})^4 = \sqrt[3]{(m^2)^4} = \sqrt[3]{m^8}$.

4°. *In order to take the root of a root, it is necessary to multiply the indices of the roots, leaving the integrand unchanged*:

$$\sqrt[n]{\sqrt[k]{a}} = \sqrt[nk]{a}.$$

□ Since

$$\left(\sqrt[n]{\sqrt[k]{a}}\right)^{nk} = \left(\left(\sqrt[n]{\sqrt[k]{a}}\right)^n\right)^k = (\sqrt[k]{a})^k = a,$$

we have

$$\sqrt[nk]{a} = \sqrt[n]{\sqrt[k]{a}}. \quad ■$$

For example, $\sqrt[3]{\sqrt{5}} = \sqrt[3\cdot 2]{5} = \sqrt[6]{5}$.

5°. *If the index of the root and the exponent of the integrand are multiplied or divided by the same natural number, then the value of the root remains unchanged:*

$$\sqrt[nm]{a^{km}} = \sqrt[n]{a^k}.$$

□ By the definition of the root, $\sqrt[nm]{a^{km}}$ is such a nonnegative number which, being raised to the power nm, yields a^{km}. Hence, it suffices to show that $(\sqrt[n]{a^k})^{nm} = a^{km}$. We have

$$(\sqrt[n]{a^k})^{nm} = ((\sqrt[n]{a^k})^n)^m = (a^k)^m = a^{km}. \blacksquare$$

For example, $\sqrt[16]{a^{12}} = \sqrt[4]{a^3}$.

Example 1. Extract the root of the product $\sqrt{a^2b^6}$.

○ Using Property 1° of the arithmetic root, we get

$$\sqrt{a^2b^6} = \sqrt{a^2} \cdot \sqrt{b^6} = ab^3$$

(recall that at the beginning of this subsection we agreed to regard all the variables as attaining only nonnegative values; if it were not so, we would have no right to write $\sqrt{a^2} = a$, since for $a < 0$ it is incorrect; the same holds true for the equality $\sqrt{b^6} = b^3$). ●

Example 2. Take the root of the fraction $\sqrt{7\frac{19}{32}}$.

○ Let us convert the mixed number $7\frac{19}{32}$ into an improper fraction: $7\frac{19}{32} = \frac{7 \cdot 32 + 19}{32} = \frac{243}{32}$. Hence, $\sqrt[5]{7\frac{19}{32}} = \sqrt[5]{\frac{243}{32}}$. According to Property 2°, we get $\sqrt[5]{\frac{243}{32}} = \frac{\sqrt[5]{243}}{\sqrt[5]{32}} = \frac{3}{2}$. ●

Example 3. Extract $\sqrt{45a^5}$, simplifying the radicand.

○ Let us represent the radicand $45a^5$ in the form $9 \cdot a^4 \cdot 5a$ and apply Property 1° of the arithmetic root; we then get

$$\sqrt{45a^5} = \sqrt{9a^4 \cdot 5a} = \sqrt{9} \cdot \sqrt{a^4} \cdot \sqrt{5a} = 3a^2\sqrt{5a}. \bullet$$

Such a change is called the **removing a factor from under the radical sign.**

In some cases, it turns out to be convenient to use a transformation which is, in a way, the inverse of the change just considered, namely, **introducing a factor into the radicand.**

Suppose that it is required to find out which of the two numbers is greater: $2\sqrt{5}$ or $3\sqrt{2}$. Consider the number $2\sqrt{5}$. Let us introduce the factor 2 under the radical sign. This is done as follows: $2\sqrt{5} = \sqrt{2^2} \cdot \sqrt{5} = \sqrt{20}$. The number $3\sqrt{2}$ is transformed in a similar way:

$3\sqrt{2} = \sqrt{3^2}\cdot\sqrt{2} = \sqrt{18}$. Since $\sqrt{20} > \sqrt{18}$, we have $2\sqrt{5} > 3\sqrt{2}$.

In the above examples, we made use only of the definition of the root and Properties 1^0 and 2^0. Let us now consider a few examples of applying Properties 3^0 and 4^0.

Example 4. Simplify $(\sqrt[3]{a^2})^5$.

○ According to Property 3^0, we have $\sqrt[3]{(a^2)^5} = (\sqrt[3]{(a^2)^5} = \sqrt[3]{a^{10}}$. We usually try to simplify the radicand by removing a factor from it, we find

$$\sqrt[3]{a^{10}} = \sqrt[3]{a^9\cdot a} = \sqrt[3]{a^9}\cdot\sqrt[3]{a} = a^3\sqrt[3]{a}.$$

Thus, $(\sqrt[3]{a^2})^5 = a^3\sqrt[3]{a}$. ●

Example 5. Simplify $\sqrt[4]{x^2\sqrt[3]{x}}$.

○ We transform the expression $x^2\sqrt[3]{x}$, by bringing the factor x^2 under the radical sign

$$x^2\sqrt[3]{x} = \sqrt{(x^2)^3}\cdot\sqrt[3]{x} = \sqrt[3]{x^6\cdot x} = \sqrt[3]{x^7}.$$

According to Property 4^0, we have

$$\sqrt[4]{\sqrt[3]{x^7}} = \sqrt[4\cdot 3]{x^7} = \sqrt[12]{x^7}.$$ ●

Finally, consider some examples in which Property 5^0 is used.

Example 6. Simplify $\sqrt[30]{2^9}$.

○ According to Property 5^0, the index of the root and the exponent of the radicand may be divided by the same natural number. Dividing them by 3, we get $\sqrt[30]{2^9} = \sqrt[10]{2^3} = \sqrt[10]{8}$. ●

Example 7. Simplify the given expressions:

(1) $\sqrt[5]{a}\cdot\sqrt[5]{a^2}$, (2) $\sqrt[3]{a^4} \div \sqrt[3]{a^2}$, (3) $\sqrt[3]{a}\cdot\sqrt[6]{a}$, (4) $\sqrt[8]{x^3}\cdot\sqrt[12]{x^7}$.

○ (1) Property 1^0 implies that to multiply roots having the same index, it suffices to multiply the radicands and to take the same root of the result obtained. Hence $\sqrt[5]{a}\cdot\sqrt[5]{a^2} = \sqrt[5]{a\cdot a^2} = \sqrt[5]{a^3}$.

(2) It follows from Property 2^0 that, to divide roots of the same index, it suffices to divide the radicand under the first radical sign by the radicand under the second radical sign and take the same root of the quotient obtained. Consequently, $\sqrt[3]{a^4} \div \sqrt[3]{a^2} = \sqrt[3]{\frac{a^4}{a^2}} = \sqrt{a^2}$.

(3) We know how to multiply roots having the same index. Here it is required to multiply the roots with different indices. Hence, we have first to reduce the given radicals to a common index. According to Property 5^0, we have to multiply the index of the root and the exponent of the radicand by the same natural number. Therefore

$\sqrt[3]{a} = \sqrt[3\cdot 2]{a^{1\cdot 2}} = \sqrt[6]{a^2}$. Further, we have

$$\sqrt[3]{a}\cdot\sqrt[6]{a} = \sqrt[6]{a^2}\cdot\sqrt[6]{a} = \sqrt[6]{a^3}.$$

Dividing now both the index and exponent by 3, we get $\sqrt{a^3} = \sqrt{a}$. Thus, $\sqrt[3]{a}\cdot\sqrt[6]{a} = \sqrt{a}$.

(4) We reduce the radicals to a common index. To this end, obviously, it is necessary to find the least common multiple of the numbers 8 and 12. Since the L.C.M. (8, 12) = 24, the index of the root and the exponent of the radicand of the first of the given radicals must be multiplied by 3, and of the second by 2. We then get

$$\sqrt[8]{x^3}\cdot\sqrt[12]{x^7} = \sqrt[12]{x^9}\cdot\sqrt[24]{x^{14}} = \sqrt[24]{x^9\cdot x^{14}} = \sqrt[24]{x^{23}}. \quad \bullet$$

3. The Identity $\sqrt{a^2} = |a|$. In the present subsection, we answer the following question: if the variable a takes on both nonnegative and negative values, then what is $\sqrt{a^2}$ equal to?

Two cases are possible here: $a \geqslant 0$ and $a < 0$. If $a \geqslant 0$, then $\sqrt{a^2} = a$, for example, $\sqrt{2^2} = 2$, $\sqrt{27^2} = 27$, $\sqrt{0^2} = 0$. And if $a < 0$, then $\sqrt{a^2} = -a$, for example, $\sqrt{(-2)^2} = -(-2) = 2$. Thus,

$$\sqrt{a^2} = \begin{cases} a \text{ for } a \geqslant 0, \\ -a \text{ for } a < 0. \end{cases}$$

But just in the same way the modulus (absolute value) of the real number a is defined:

$$|a| = \begin{cases} a \text{ for } a \geqslant 0, \\ -a \text{ for } a < 0. \end{cases}$$

Thus,

$$\sqrt{a^2} = |a|.$$

For example, $\sqrt{3^2} = |3| = 3$, $\sqrt{(-5)^2} = |-5| = 5$, $\sqrt{(2-\sqrt{7})^2} = |2-\sqrt{7}| = \sqrt{7}-2$.

To prove the identity $\sqrt{a^2} = |a|$ it suffices to note that, firstly, $|a| \geqslant 0$, and, secondly, $|a|^2 = a^2$. Hence, $|a|$ is such a nonnegative number which, when squared, yields a^2, that is, $|a| = \sqrt{a^2}$.

In general, if n is an even number, that is, $n = 2k$, then

$$\sqrt[2k]{a^{2k}} = |a|.$$

Thus, if in Example 1 of Item 2 the requirement for the variables to be nonnegative is removed, then the solution will be as follows:

$$\sqrt{a^2b^6} = \sqrt{a^2}\cdot\sqrt{b^6} = |a|\cdot|b^3| = |ab^3|.$$

4. Additional Notes on Properties of Radicals. Let $a < 0$, and n be a natural number greater than 1. If n is an even number, then the equality $x^n = a$ is satisfied for no real value of x. This means that it is impossible to define an even-indexed root of a negative number in the domain of real numbers. If n is an odd number, then there is one and only one real number x such that $x^n = a$. We shall denote this number by $\sqrt[n]{a}$.

For example, $\sqrt[3]{-8} = -2$, since $(-2)^3 = -8$, $\sqrt[5]{-243} = -3$, since $(-3)^5 = -243$.

In Item 2 we considered five properties of arithmetic roots (radicals) which are unconditionally true for nonnegative radicands. But when performing operations with radicals, one should keep in mind that negative values of the variables may be found under the radical sign. Obviously, all the mentioned properties are extended to the case of radicals having odd indices.

It is somewhat different with radicals having even indices. Let a and b be negative numbers, and let n be an even number. In this case, we have no right to write $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, since the right-hand side of such an "equality" has no sense (for instance, it is impossible to write $\sqrt{(-5)(-6)} = \sqrt{-5} \cdot \sqrt{-6}$). Here, we may reason in such a way: a and b are negative numbers, consequently, $ab > 0$. But then $ab = |ab| = |a| \cdot |b|$. Hence $\sqrt[n]{ab} = \sqrt[n]{|ab|} = \sqrt[n]{|a| \cdot |b|}$. Since $|a| > 0$ and $|b| > 0$, by making use of Property 1^0 of arithmetic roots, we get

$$\sqrt[n]{|a| \cdot |b|} = \sqrt[n]{|a|} \cdot \sqrt[n]{|b|}.$$

Thus, if n is an even number, and the numbers a and b have like signs, then $\sqrt[n]{ab} = \sqrt[n]{|a|} \cdot \sqrt[n]{|b|}$ and, similarly $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{|a|}}{\sqrt[n]{|b|}}$.

Property 5^0 should be considered very attentively. Suppose we have to simplify the expression $\sqrt[4]{(\sqrt{3}-5)^2}$. Dividing the index of the radical and the exponent of the radicand by 2, we arrive at the expression $\sqrt{\sqrt{3}-5}$ which has no sense since the radicand of an even root is a negative number. In this case, a correct equality is represented by

$$\sqrt[4]{(\sqrt{3}-5)^2} = \sqrt{|\sqrt{3}-5|} = \sqrt{5-\sqrt{3}}.$$

Indeed, $(\sqrt{3}-5)^2 = |\sqrt{3}-5|^2$ and, consequently,

$$\sqrt[4]{(\sqrt{3}-5)^2}=\sqrt[4]{|\sqrt{3}-5|^2}=\sqrt{|\sqrt{3}-5|}.$$

5. Square Root Algorithm. The examples given in this subsection will explain the algorithm* (method) of taking the square root of a natural number used in practice.

Example 1. Compute $\sqrt{138\,384}$.

○ We first subdivide the number 138 384 from right to left into two digit groups: 13'83'84. We then seek for a maximum number whose square does not exceed the number 13 which represents the first group. This number is 3 (since $3^2 = 9 < 13$, while $4^2 = 16 > 13$). This is the first digit of the desired result.

We then square the number 3 and subtract the result from the first group: $13 - 3^2 = 4$. To the difference thus found, we add on the right the second two-digit group, thus getting the number 483.

Further, we double the first digit of the result ($3 \cdot 2 = 6$) and attach to 6 a greatest digit a at the right such that the product of the number $\overline{6a}$ by a does not exceed the number 483. In our case, the number a is equal to 7, since $67 \cdot 7 = 469 < 483$, whereas $68 \cdot 8 = 544 > 483$. We write 7 after the first obtained digit 3 in the result. It will be the second digit in the answer. We now subtract 469 from 483. To the difference thus obtained (14) we attach from the right the third two-digit group 1484. We then double the number 37 (that is, the first two digits obtained for the result) and attach from the right the greatest possible digit b such that the product of the number $\overline{37b}$ by b does not exceed the number 1484. In our case, $b = 2$ since $742 \cdot 2 = 1484$. The digit 2 thus found will be the third digit of the desired result. Since $1484 - 1484 = 0$, the extraction of the root has been completed.

The algorithm under consideration is usually written in the following way:

$$\sqrt{13'83'84} = 372$$

$$\begin{array}{r|l} & 9 \\ \hline \times\ 67 & 483 \\ \times\ 7 & -469 \\ \hline \times\ 742 & 1484 \\ \times\ 2 & -1484 \\ \hline & 0 \end{array}$$ ●

Example 2. Compute $\sqrt{45\,369}$.

* Some special process of solving a certain type of problem, particularly a method that continually repeats some basic process—*Tr.*

○ We have

$$\sqrt{4'53'59} = 213$$

$$\begin{array}{r|l} & 4 \\ \hline \times\ 41 & -\ 53 \\ \quad\ 1 & -\ 41 \\ \hline \times\ 423 & -\ 1269 \\ \quad\ 3 & -\ 1269 \\ \hline & 0 \end{array}$$

●

If the root cannot be extracted exactly, then the last digit of the number is followed by the decimal point, thus forming a new group of two. In this case, the process of extracting the root is endless; it is ceased when the required accuracy is achieved.

Example 3. Compute $\sqrt{5}$ to within 0.01.

○ We have

$$\sqrt{5'.00'00'00'} \ldots = 2.236 \ldots$$

$$\begin{array}{r|l} & 4 \\ \hline \times\ 42 & -\ 1\,00 \\ \quad\ 2 & -\ 84 \\ \hline \times\ 443 & -\ 1600 \\ \quad\ 3 & -\ 1329 \\ \hline \times\ 4466 & -\ 27100 \\ \quad\ 6 & -\ 26796 \\ \hline & 304 \end{array}$$

$$\ldots$$

Thus, to within 0.01, we get $\sqrt{5} \approx 2.24$. ●

Exercises

1. Simplify the expression $\sqrt{(1-\sqrt{3})^2} - \sqrt[6]{27}$.
2. Prove that $a+5-\sqrt{(a-5)^2} = 2a$ if $a \leqslant 5$.
3. Simplify the given expressions: (a) $\sqrt{x^2}$ if $x < -1$, (b) $\sqrt{(a+3)^2}$ if $a < -3$, (c) $\sqrt{(x-5)^2}$ if $x \geqslant 5$, (d) $\sqrt{(a^2-1)^2}$ if $a < -1$.
4. Solve the following equations: (a) $\sqrt{y^2} = 2.4$, (b) $\sqrt{(y+3)^2} = 6$, (c) $\sqrt{a^2} = a$, (d) $\sqrt{x^2} = -2$.
5. Compute: (a) $\sqrt{625 \cdot 9 \cdot 36}$, (b) $\sqrt{6^6 \cdot 2^4}$, (c) $\sqrt{\dfrac{64 \cdot 324}{529 \cdot 49}}$.

6. Remove a factor from the radicand: (a) $\sqrt{9a^2b}$, where $a<0$, $b>0$, (b) $\sqrt{25a^2b^3}$, where $a>0$, $b>0$, (c) $\sqrt{144a^3b^3}$, where $a<0$, $b<0$, (d) $\sqrt{32a^4b^3}$, where $a<0$, $b>0$, (e) $\sqrt{75a^3b^6}$, where $a>0$, $b<0$, (f) $\sqrt{8a^5b^7}$, where $a<0$, $b<0$.

7. Bring the indicated factor under the radical sign: (a) $a\sqrt{3}$, where $a\geqslant 0$, (b) $a\sqrt{3}$, where $a<0$, (c) $-b\sqrt{3}$, where $b\geqslant 0$, (d) $-b\sqrt{3}$, where $b<0$, (e) $x\sqrt{\frac{2}{x}}$, (f) $x\sqrt{\frac{-2}{x}}$, (g) $y\sqrt{3}$, (h) $-c\sqrt{7a}$.

8. Which of the two numbers is greater: (a) $2\sqrt{45}$ or $\sqrt{125}$, (b) $\frac{1}{4}\sqrt{112}$ or $\frac{1}{2}\sqrt{32}$?

9. Prove that the value of the given expression is a rational number: (a) $(2\sqrt{3}+5)(5-2\sqrt{3})+(4-\sqrt{5})^2+8\sqrt{5}$, (b) $\left(\frac{3}{\sqrt{3}}+\frac{7}{\sqrt{7}}\right)\cdot(\sqrt{3}-\sqrt{7})$.

10. Simplify the given expressions:

(a) $3\sqrt{48}-\sqrt{75}+\frac{1}{7}\sqrt{147}$, (b) $\sqrt{1300}-2\sqrt{52}-12\sqrt{1\frac{4}{9}}+5\sqrt{3\frac{1}{4}}$.

In Exercises 11 to 16, perform the indicated operations:

11. (a) $\frac{3}{4}\sqrt[3]{2\frac{1}{2}a}\cdot\sqrt[3]{\frac{0.4}{a}}$, (b) $9\sqrt{\frac{1}{45}}\div\frac{3}{2}\sqrt{2\frac{2}{3}}$.

12. (a) $\sqrt[3]{2a^2}\cdot\sqrt[6]{32a^4}$, (b) $\sqrt[5]{\sqrt{5}-2}\cdot\sqrt[10]{\sqrt{5}+2}$.

13. $(4x\sqrt[3]{x^2}-5y\sqrt[3]{xy}+xy\sqrt[3]{y^2})\cdot 2xy\sqrt[3]{xy}$.

14. $\left(\frac{a}{b^3}\sqrt{ab}-6a^3b^2\sqrt[3]{a^2b}+\sqrt[8]{a^4b^3}\right)\div\frac{a^2}{b}\sqrt[6]{ab^2}$.

15. (a) $\left(-0.1\cdot\sqrt[5]{a^4b^2}\right)^3$, (b) $\left(\frac{x}{2y}\sqrt[4]{\frac{y}{x}}\right)^5$, (c) $\left(\frac{2}{3}\sqrt{18}+\frac{3}{4}\sqrt{2}\right)^2$.

16. (a) $\sqrt[3]{\sqrt[4]{m^3}}$, (b) $\sqrt[5]{a\sqrt[4]{a}}$, (c) $\sqrt{\frac{m}{n}\sqrt[3]{\frac{n}{m}}}$.

17. Compute: (a) $\sqrt{841}$, (b) $\sqrt{1225}$, (c) $\sqrt{5476}$, (d) $\sqrt{10\,609}$, (e) $\sqrt{21\,609}$, (e) $\sqrt{126\,736}$.

18. Factor: (a) $10+\sqrt{5}$, (b) $12-5\sqrt{3}$, (c) $4a^2-5$, (d) $3b^2-16$.

19. Reduce the following fractions:

(a) $\frac{10-y^2}{y-\sqrt{10}}$, (b) $\frac{x-9}{(\sqrt{x}-3)^2}$, (c) $\frac{\sqrt{12}-\sqrt{6}}{\sqrt{30}-\sqrt{15}}$.

20. For what values of the variable do the given expressions have sense:

(a) $\frac{1}{\sqrt{x-2}}$, (b) $\frac{2}{\sqrt{x}-2}$, (c) $\sqrt{2x+4}+\sqrt{15-x}$,

(d) $\sqrt{-x^2-5}$, (e) $\sqrt{x^2+9}$, $\sqrt{|x|-1}$?

21. Simplify the given expressions:

(a) $\sqrt[8]{(2-\sqrt{7})^4}$, (b) $\sqrt[6]{(3-\pi)^2}$,

(c) $\sqrt[3]{1-\sqrt{2}}\cdot\sqrt[6]{3+2\sqrt{2}}$.

22. Compute: (a) $\sqrt{\frac{\sqrt{25^2-24^2}}{21.5^2-14.5^2}}$, (b) $\sqrt{\frac{149^2-76^2}{457^2-384^2}}$.

23. Find the numerical values of the following expressions: (a) x^2-4x+3 for $x=2+\sqrt{3}$, (b) $9x^3+3x^2-13x-6$ for $x=(1-2\sqrt{2})/2$.

§ 2.4. Generalizing the Notion of Exponent

1. Problem Statement. In § 1.4, Item 1, we introduced the notion of power with natural exponent and considered its properties. Let us recall this definition and the properties implied.

Definition 1. $a^n=\underbrace{a\cdot a\ \ldots\ a}_{n\text{ factors}}$, $a^1=a$.

Powers with a natural exponent possess the following properties:

1^0. $a^m\cdot a^n=a^{m+n}$.

2^0. $a^m\div a^n=a^{m-n}$ if $m>n$.

3^0. $(a^m)^n=a^{mn}$.

4^0. $(ab)^n=a^nb^n$.

5^0. $\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}$ $(b\neq 0)$.

In what follows, we shall give definitions of powers with any rational exponent.

First, we shall define a power with a positive fractional exponent (Item 2), then a power with a zero exponent (Item 3), and finally, the power with a negative rational exponent (Item 4). It is clear that the definition given above is not extended to any of these cases (for example, $a^{3/5}$ cannot be defined as the product of the number a by itself 3/5 times). Therefore, each time we shall have to introduce a new definition. When choosing a new definition, we will be guided by the demand that properties similar to 1^0-5^0 for a power with a natural exponent be extended to each new case.

2. Power with a Positive Fractional Exponent. Let $a\geqslant 0$. It is required to define $a^{p/q}$ so that, for instance, the equality $(a^{p/q})^q=a^{(p/q)\cdot q}=a^p$ is fulfilled, that is, so that when raising a power to a power, the exponents are multiplied.

This equality is possible only if $a^{p/q}=\sqrt[q]{a^p}$. Quite a natural idea appears to define $a^{p/q}$ as $\sqrt[q]{a^p}$. But will such a definition be correct, that is, will the properties similar to Properties 1^0-5^0 be satisfied in this case? Let us verify this.

1^0. $a^{p/q}\cdot a^{m/n}=a^{p/q+m/n}$.

□ Consider the expression $a^{p/q} \cdot a^{m/n}$. According to the introduced definition of the power with a positive fractional exponent, we have

$$a^{p/q} \cdot a^{m/n} = \sqrt[q]{a^p} \cdot \sqrt[n]{a^m}.$$

Using now the properties of radicals, we find

$$\sqrt[q]{a^p} \cdot \sqrt[n]{a^m} = \sqrt[nq]{a^{np}} \cdot \sqrt[nq]{a^{mq}} = \sqrt[nq]{a^{np} \cdot a^{mq}}$$

$$= \sqrt[nq]{a^{np+mq}} = a^{(np+mq)/(nq)} = a^{p/q+m/n}.$$

Thus, $a^{p/q} \cdot a^{m/n} = a^{p/q+m/n}$. ■

3^0. $(a^{p/q})^{m/n} = a^{(p/q)\cdot(m/n)}$.

□ We have

$$(a^{p/q})^{m/n} = \sqrt[n]{(a^{p/q})^m} = \sqrt[n]{(\sqrt[q]{(a^p)^m}} = \sqrt[n]{\sqrt[q]{(a^p)^m}} = \sqrt[nq]{a^{pm}}$$

$$= a^{(pm)/(nq)} = a^{(p/q)\cdot(m/n)}. \blacksquare$$

Proceeding in a similar manner, we can show that Properties 2^0, 4^0, and 5^0 are also fulfilled:

2^0. $a^{p/q} \div a^{m/n} = a^{p/q-m/n}$ if $p/q > m/n$,

4^0. $(ab)^{p/q} = a^{p/q} \cdot b^{p/q}$,

5^0. $\left(\frac{a}{b}\right)^{p/q} = \frac{a^{p/q}}{b^{p/q}}$.

Thus, for the introduced definition of a power with a positive fractional exponent basic properties, similar to Properties 1^0-5^0 of the power with a natural exponent, turn out to be satisfied. Hence, the definition can be accepted.

Definition 2. *If* $a \geqslant 0$, *then* $a^{p/q} = \sqrt[q]{a^p}$.

For example, $8^{1/3} = 2$ since $\sqrt[3]{8} = 2$, $27^{2/3} = 9$, since $27^{2/3} = \sqrt[3]{27^2} = \sqrt[3]{3^6} = 3^2 = 9$.

Note that the power with a fractional exponent is defined only for **nonnegative** bases. Thus, for instance, the expression $(-8)^{1/3}$ is meaningless, although, as we saw above, the notation $\sqrt[3]{-8}$ is quite rightful ($\sqrt[3]{-8} = -2$).

In practice, when performing operations with radicals, we often pass to fractional exponents.

Example 1. Carry out the multiplication $\sqrt[8]{x^3} \cdot \sqrt[12]{x^7}$.

○ We have

$$\sqrt[8]{x^3} \cdot \sqrt[12]{x^7} = x^{3/8} \cdot x^{7/12} = x^{3/8+7/12} = x^{23/24} = \sqrt[24]{x^{23}}. \bullet$$

Example 2. Factor: $\sqrt[6]{a^2x^3} - \sqrt[4]{a^3x}$.

○ We find

$$\sqrt[6]{a^2x^3} - \sqrt[4]{a^3x} = \sqrt[6]{a^2} \cdot \sqrt[6]{x^3} - \sqrt[4]{a^3} \cdot \sqrt{x}$$
$$= a^{2/6} \cdot x^{3/6} - a^{3/4} \cdot x^{1/4} = a^{1/3} \cdot x^{1/2} - a^{3/4} \cdot x^{1/4}$$
$$= a^{1/3}x^{1/4}(x^{1/4} - a^{5/12}) = \sqrt[3]{a} \cdot \sqrt[4]{x}(\sqrt[4]{x} - \sqrt[12]{a^5}). \bullet$$

3. Power with a Zero Exponent. As in the preceding subsection, when choosing the definition, we shall also be guided by the requirement that Properties 1°-5° of the power with a natural exponent be extended to the power with a zero exponent (however, now we have the right to speak of the extension of the properties of the power with a positive rational exponent). In particular, as powers with the same base are multiplied, their exponents must be added together, that is, the equality $a^n \cdot a^0 = a^n$ must be satisfied (since $n + 0 = n$; here n is a natural number).

For $a \neq 0$ this equality is possible only if $a^0 = 1$. Therefore, there appears an idea of defining a^0 as 1. It is not difficult to verify that for such a definition properties similar to Properties 1°-5° of the power with a natural exponent are fulfilled, and, hence, this definition may be accepted.

Definition 3. *If* $a \neq 0$, *then* $a^0 = 1$.

For example, $5^0 = 1$, $(-2.3)^0 = 1$, $(1/7)^0 = 1$.

4. Power with a Negative Rational Exponent. Let $a > 0$, and r be a positive rational number. It is required to define a^{-r} so that, for instance, the following equality is satisfied:

$$a^r \cdot a^{-r} = a^0. \tag{1}$$

Since $a^0 = 1$, equality (1) is possible only if a^{-r} is defined as $1/a^r$. It can be easily shown that for such a definition the properties similar to Properties 1°-5° will be fulfilled.

Let us show, for instance, that $(a^{-r_1})^{r_2} = a^{-r_1 r_2}$. Indeed,

$$(a^{-r_1})^{r_2} = \left(\frac{1}{a^{r_1}}\right)^{r_2} = \frac{1^{r_2}}{(a^{r_1})^{r_2}} = \frac{1}{a^{r_1 r_2}} = a^{-r_1 r_2}.$$

The other properties are verified in a similar way.

Definition 4. *If* $a > 0$, *then* $a^{-r} = 1/a^r$.

For example, $8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{\sqrt[3]{8^2}} = \frac{1}{4}$, $\left(\frac{3}{5}\right)^{-1} = \frac{1}{3/5} = \frac{5}{3}$.

Note that if r is an integer, then this definition is also extended to the case of negative bases: if $a < 0$ and r is an **integer**, then $a^{-r} = 1/a^r$.

For example, $(-5)^{-3} = 1/(-5)^3 = 1/(-125) = -1/125$.

5. Power with Any Rational Exponent. In the previous subsections, we defined the notion of a power with any rational exponent. This power possesses the following properties (we hold that $a > 0$,

$b > 0$, r_1 and r_2 are arbitrary rational numbers):

1^0. $a^{r_1} \cdot a^{r_2} = a^{r_1+r_2}$.

2^0. $a^{r_1} \div a^{r_2} = a^{r_1-r_2}$.

3^0. $(a^{r_1})^{r_2} = a^{r_1 r_2}$.

4^0. $(ab)^{r_1} = a^{r_1} b^{r_1}$.

5^0. $\left(\frac{a}{b}\right)^{r_1} = \frac{a^{r_1}}{b^{r_1}}$.

Note that having introduced a zero and a negative exponent we have the right not to make the reservation $r_1 > r_2$ in Property 2^0.

Exercises

1. Evaluate the given expressions: (a) $\frac{4^{-3} \cdot 8^{-5}}{16^{-6}}$, (b) $\frac{2 \cdot 5^{20} - 9 \cdot 5^{19}}{25^9}$, (c) $\frac{|125^{-4}}{5^3 \cdot 25^{-7}}$, (d) $100^{1/2}$, (e) $0.01^{-1/2}$, (f) $0^{5/6}$, (g) $0.01^{-2.5}$, (h) $\left(3\frac{3}{8}\right)^{-2/3}$.

2. Find the domain of definition for each of the given expressions: (a) $x^{1/2}$, (b) $(y-1)^{1/3}$, (c) $(a+2)^{-3/5}$, (d) $(c-5)^{-3}$.

3. Compute:

(a) $(6.25)^{0.5} \cdot \left(\frac{1}{16}\right)^{0.25} - (-4)^{-1} \cdot (0.343)^0$,

(b) $(16^{-0.25} - (2\sqrt{2})^{1/3}) \cdot (16^{-0.25} + (2\sqrt{2})^{1/3})$,

(c) $\left(\left(\frac{3}{5}\right)^{-4}\right)^{-0.75} \cdot 0.09^{-0.5} (-3)^0 \cdot 0.1^{-4}$.

4. Transform the following expressions so that they contain no negative exponents:

(a) $2a^{-2}b$, (b) $\frac{2a^2}{b^{-1}x^{-3}}$, (c) $\frac{a^{-1}b^{-3/4}}{cd^{-2/3}}$, (d) $\left(\frac{a^2bc^{-1}}{d}\right)^{-1/3}$.

In Exercises 5 to 10, simplify the given expressions:

5. (a) $\frac{4}{7} m^7 n^{-3} x^2 \div 1\frac{3}{4} m^{-3} n^{-2} x^3$, (b) $0.2a^{2/5}b^{-7/8} \div 0.1a^{-3/5}b^{1/8}$.

6. $(x^{-2.5}y^{1.25})^{-0.4}$, (b) $(0.027m^{-3/5}n^{-1})^{-5/3}$, (c) $(0.2m^{-2}n^3)^{-3}$.

7. (a) $\left(\frac{a}{b^2}\right)^{2/3}$, (b) $\left(\frac{32x^{-3/4}}{a^3y^5}\right)^{-0.2}$.

8. (a) $\sqrt[3]{\frac{64a^{-12}b^{15}}{125c^{-6}d^{-3}}}$, (b) $(x^{-3}y^2 \sqrt[5]{2x^3y^{-2}})^{-4}$.

9. (a) $(a^{-1} - b^{-1})(a^{-1} + b^{-1})$,
(b) $(a^{-2} - b^{-2})(a^{-4} + a^{-2}b^{-2} + b^{-4})$.

10. (a) $\frac{a^{1/2}b^{1/2}}{c^{1/6}} \div \left(\frac{c^{-1/2}}{a^{-1/3}b^{-1/3}} \cdot \frac{a^{-5/6}c^{-2/3}}{b^{5/6}}\right)$,

(b) $\left(\frac{a^{-1/2}b^{-1/3}}{a^{-3/4}b^{-5/6}} \div \sqrt[4]{a^{-3}b^{-5}}\right)^{2/7}$.

11. Solve the following equations: (a) $x^{1/2} = 3$, (b) $x^{-1/3} = 2$, (c) $x^{-3} = 8$, (d) $x^{3/2} = 0$, (e) $2(x^{1/2} + 1) = 18$, (f) $x^{1/4} + 7 = 0$, (g) $x^{-7/3} \cdot x^{-2/3} + 27 = 0$, (h) $0.5y^{19/6} \cdot y^{-1/6} = 4$.

12. Using the identity $a^2 - b^2 = (a - b)(a + b)$, factor the given expressions: (a) $y^{2/5} - 9$, (b) $25 - p^{4/7}$, (c) $x^{1/3} - y^5$.

13. Using the identities $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$, represent the following expressions in the form of a product: (a) $p^{3/2} + 1$, (b) $125 - b$, (c) $x^{1/4} - y^{1/2}$.

14. Reduce the following fractions:

(a) $\dfrac{y-1}{1+y^{1/2}}$, (b) $\dfrac{a^{2/3}-a^{1/3}b^{1/3}+b^{2/3}}{a+b}$,

(c) $\dfrac{n}{n-n^{1/2}}$, (d) $\dfrac{x+2x^{1/2}}{2x}$.

15. Evaluate the given expressions:

(a) $\dfrac{a-4a^{1/2}}{a^{3/4}+2a^{1/2}}$ for $a=81$,

(b) $\dfrac{a^{1/3}+b^{1/3}}{a^{2/3}-a^{1/3}b^{1/3}+b^{2/3}} - \dfrac{a^{2/3}+2a^{1/3}b^{1/3}}{a+b}$

for $a = 63$, $b = 27$.

16. Find the relation between the variables x and y if: (a) $x = t^{1/2}$, $y = t^{-1/2}$, (b) $x = t^{1/3}$, $y = t^{1/6}$, (c) $x = 3t^{1/2}$, $y = 2t^{-1/3}$, (d) $x = (t + 1)^{1/2}$, $y = (t - 1)^{1/2}$.

§ 2.5. Transformations of Irrational Expressions

An expression with variables is said to be *irrational* if it involves taking the root of a variable or raising a variable to a fractional power.

As a rule, identical transformations of irrational expressions are performed on the set of nonnegative numbers. This follows from the aforegoing definitions. Let us, for example, reduce the fraction $\dfrac{a-4}{a^{1/2}+2}$. For $a \geqslant 0$ the expression $a - 4$ can be represented as the difference of the expressions $(a^{1/2})^2$ and 2^2, and then the given fraction is reduced:

$$\frac{a-4}{a^{1/2}+2} = \frac{(a^{1/2})^2-2^2}{a^{1/2}+2} = \frac{(a^{1/2}-2)(a^{1/2}+2)}{a^{1/2}+2} = a^{1/2}-2.$$

Such an identical transformation is valid for nonnegative numbers, that is, for $a \geqslant 0$. Hereafter, this will be implied, but not specially stipulated.

Example 1. Perform the indicated operations:

$$\frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{\sqrt{2}-1}{\sqrt{2}+1} - \frac{\sqrt{2}+3}{\sqrt{2}}.$$

○ Here, it is advisable to use the method of **rationalizing the denominator.** To this end, we multiply both the numerator and denominator of the first fraction by $\sqrt{2} + 1$ (this expression is said to be the conjugate of $\sqrt{2} - 1$):

$$\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{2+2\sqrt{2}+1}{2-1} = 3+2\sqrt{2}.$$

The second fraction is dealt with in a similar way (now, the conjugate expression for the denominator is $\sqrt{2}-1$):

$$\frac{\sqrt{2}-1}{\sqrt{2}+1}=\frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)}=\frac{2-2\sqrt{2}+1}{2-1}=3-2\sqrt{2}.$$

In order to rationalize the denominator of the third fraction, we multiply both the numerator and denominator by $\sqrt{2}$:

$$\frac{\sqrt{2}+3}{\sqrt{2}}=\frac{(\sqrt{2}+3)\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}=\frac{2+3\sqrt{2}}{2}=1+\frac{3}{2}\sqrt{2}.$$

Thus, we have

$$\frac{\sqrt{2}+1}{\sqrt{2}-1}+\frac{\sqrt{2}-1}{\sqrt{2}+1}-\frac{\sqrt{2}+3}{\sqrt{2}}=3+2\sqrt{2}+3-2\sqrt{2}-\left(1+\frac{3}{2}\sqrt{2}\right)$$
$$=5-\frac{3}{2}\sqrt{2}. \bullet$$

Example 2. Perform the indicated operations:

$$\left(\frac{a\sqrt{a}+b\sqrt{b}}{\sqrt{a}+\sqrt{b}}-\sqrt{ab}\right)\div(a-b)+\frac{2\sqrt{b}}{\sqrt{a}+\sqrt{b}}.$$

○ Let us first reduce the first fraction. Its numerator can be transformed in the following way: $a\sqrt{a}+b\sqrt{b}=(\sqrt{a})^3+(\sqrt{b})^3$, therefore

$$\frac{a\sqrt{a}+b\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{(\sqrt{a})^3+(\sqrt{b})^3}{\sqrt{a}+\sqrt{b}}=\frac{(\sqrt{a}+\sqrt{b})((\sqrt{a})^2-\sqrt{a}\cdot\sqrt{b}+(\sqrt{b})^2)}{\sqrt{a}+\sqrt{b}}$$
$$=a-\sqrt{ab}+b.$$

Further, we have

$$\frac{a\sqrt{a}+b\sqrt{b}}{\sqrt{a}+\sqrt{b}}-\sqrt{ab}=(a-\sqrt{ab}+b)-\sqrt{ab}=a-2\sqrt{ab}+b$$
$$=(\sqrt{a}-\sqrt{b})^2,$$
$$\frac{(\sqrt{a}-\sqrt{b})^2}{a-b}=\frac{(\sqrt{a}-\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}=\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}},$$
$$\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}+\frac{2\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}=1. \bullet$$

Note the necessity of timely reducing fractions in carrying out identical transformations. This will be illustrated by Example 3.

The above example can be solved in a different way, namely, by using the method of **introducing new variables** widely applied in algebra. If we introduce new variables by setting $\sqrt{a}=x$, $\sqrt{b}=y$,

then the expression given in Example 2 will take the form

$$\left(\frac{x^3+y^3}{x+y}-xy\right)(x^2-y^2)+\frac{2x}{x+y},$$

where the transformations are performed much more easier. We have

$$((x^2-xy+y^2)-xy)(x^2-y^2)+\frac{2y}{x+y}=\frac{(x-y)^2}{x^2-y^2}+\frac{2y}{x+y}$$
$$=\frac{x-y}{x+y}+\frac{2y}{x+y}=\frac{x+y}{x+y}=1.$$

From now on we shall use this method rather often.

Example 3. Simplify the following expression:

$$\frac{\frac{a+x}{\sqrt[3]{a^2}-\sqrt[3]{x^2}}+\frac{\sqrt[3]{ax^2}-\sqrt[3]{a^2x}}{\sqrt[3]{a^2}-2\sqrt[3]{ax}+\sqrt[3]{a^2}}}{\sqrt[6]{a}-\sqrt[6]{x}}-\sqrt{x}.$$

○ We introduce new variables, setting $\sqrt[6]{a}=b$, $\sqrt[6]{x}=c$. Then $a=b^6$, $x=c^6$, $\sqrt{a}=b^2$, $\sqrt[3]{x}=c^2$ and the given expression takes the form

$$\frac{\frac{b^6+c^6}{b^4-c^4}+\frac{b^2c^4-b^4c^2}{b^4-2b^2c^2+c^4}}{b-c}-c.$$

We then find in succession:

$$\frac{b^6+c^6}{b^4-c^4}=\frac{(b^2+c^2)(b^4-b^2c^2+c^4)}{(b^2+c^2)(b^2-c^2)}=\frac{b^4-b^2c^2+c^4}{b^2-c^2},$$

$$\frac{b^2c^4-b^4c^2}{b^4-2b^2c^2+c^4}=\frac{b^2c^2(c^2-b^2)}{(c^2-b^2)^2}=\frac{b^2c^2}{c^2-b^2},$$

$$\frac{b^4-b^2c^2+c^4}{b^2-c^2}+\frac{b^2c^2}{c^2-b^2}=\frac{b^4-b^2c^2+c^4-b^2c^2}{b^2-c^2}=\frac{(b^2-c^2)^2}{b^2-c^2}=b^2-c^2,$$

$$\frac{b^2-c^2}{b-c}=b+c,\ (b+c)-c=b.$$

Returning to the original variables, we get the answer: $\sqrt[3]{a}$. ●

Example 4. Prove the identity

$$a^{1/2}-\frac{a-a^{-2}}{a^{1/2}-a^{-1/2}}+\frac{1-a^{-2}}{a^{1/2}+a^{-1/2}}+\frac{2}{a^{3/2}}=0\quad(a>0).$$

○ Consider the fraction $\frac{a-a^{-2}}{a^{1/2}-a^{-1/2}}$. In order to reduce this fraction, we factor out a^{-2} from the numerator and $a^{-1/2}$ from the denominator. Then, we get

$$\frac{a-a^{-2}}{a^{1/2}-a^{-1/2}}=\frac{a^{-2}(a^3-1)}{a^{-1/2}(a-1)}=\frac{a^{-2}(a-1)(a^2+a+1)}{a^{-1/2}(a-1)}$$
$$=a^{-3/2}(a^2+a+1)=a^{1/2}+a^{-1/2}+a^{-3/2}.$$

Similarly, we transform the second fraction:

$$\frac{1-a^{-2}}{a^{1/2}+a^{-1/2}}=\frac{a^{-2}(a^2-1)}{a^{-1/2}(a+1)}=\frac{a^{-2}(a-1)(a+1)}{a^{-1/2}(a+1)}$$
$$=a^{-3/2}(a-1)=a^{-1/2}-a^{-3/2}.$$

Finally, we have

$$a^{1/2}-(a^{1/2}+a^{-1/2}+a^{-3/2})+(a^{-1/2}-a^{-3/2})+2a^{-3/2}=0. \quad \bullet$$

Example 5. Simplify the expression

$$\sqrt{\frac{a+x^2}{x}-2\sqrt{a}}+\sqrt{\frac{a+x^2}{x}+2\sqrt{a}}.$$

○ Consider the expression $\frac{a+x^2}{x}-2\sqrt{a}$. It is transformed to the form $\frac{a-2x\sqrt{a}+x^2}{x}$. We note that $a-2x\sqrt{a}+x^2=(\sqrt{a}-x)^2$. Thus, $\frac{a+x^2}{x}-2\sqrt{a}=\frac{(\sqrt{a}-x)^2}{x}$. Similarly, $\frac{a+x^2}{x}+2\sqrt{a}=\frac{(\sqrt{a}+x)^2}{x}$.

Now, we may rewrite the given expression as follows:

$$\sqrt{\frac{(\sqrt{a}-x)^2}{x}}+\sqrt{\frac{(\sqrt{a}+x)^2}{x}} \text{ or } \frac{\sqrt{(\sqrt{a}-x)^2}}{\sqrt{x}}+\frac{\sqrt{(\sqrt{a}+x)^2}}{\sqrt{x}}.$$

In § 2.3, Item 3 we noted that $\sqrt{a^2}=|a|$. Hence,

$$\frac{\sqrt{(\sqrt{a}-x)^2}}{\sqrt{x}}+\frac{\sqrt{(\sqrt{a}+x)^2}}{\sqrt{x}}=\frac{|\sqrt{a}-x|+|\sqrt{a}+x|}{\sqrt{x}}.$$

It is obvious that $x>0$ (since the given expression contains $\sqrt{x}$) and $\sqrt{a}\geqslant 0$. Hence, $\sqrt{a}+x>0$, and therefore $|\sqrt{a}+x|=\sqrt{a}+x$. Thus, we arrive at the expression

$$\frac{|\sqrt{a}-x|+\sqrt{a}+x}{\sqrt{x}}.$$

Now, we have to consider two cases: $\sqrt{a}-x\geqslant 0$ and $\sqrt{a}-x<0$. In the first case $|\sqrt{a}-x|=\sqrt{a}-x$, and in the second $|\sqrt{a}-x|=x-\sqrt{a}$. Therefore if $\sqrt{a}-x\geqslant 0$, then

$$\frac{\sqrt{a}-x+\sqrt{a}+x}{\sqrt{x}}=\frac{2\sqrt{a}}{\sqrt{x}}=\frac{2\sqrt{ax}}{x}.$$

And if $\sqrt{a}-x<0$, then

$$\frac{x-\sqrt{a}+\sqrt{a}+x}{\sqrt{x}}=\frac{2x}{\sqrt{x}}=2\sqrt{x}.$$

Thus, we get the answer: $\frac{2\sqrt{ax}}{x}$ if $\sqrt{a}\geqslant x$, $2\sqrt{x}$ if $\sqrt{a}<x$. ●

Exercises

In Exercises 1 to 22, simplify the given expressions:

1. $\frac{\sqrt{ab}\sqrt[4]{a}}{(a+2)\sqrt[4]{a^{-1}b^2}}-\frac{a^2+4}{a^2-4}$.

2. $\frac{a^{3/2}+b^{3/2}}{b^{1/3}(a^2-ab)}\div\frac{\sqrt[3]{a^{-3}b^{-1}}}{a\sqrt{a}-b\sqrt{b}}-ab-\frac{1}{(a^2+b^2)^{-1}}$.

3. $\frac{1}{2}(\sqrt{a^3b^{-3}}-\sqrt{b^3a^{-3}})\div\left(\frac{a^2+b^2}{ab}+1\right)\frac{2(a-b)^{-1}}{(ab)^{-1/2}}$, $a>0$, $b>0$.

4. $\left(\frac{\sqrt{y}}{x+\sqrt{xy}}-\frac{\sqrt{y}}{x-\sqrt{xy}}\right)\frac{x-y}{2\sqrt{xy}}$.

5. $\sqrt{\frac{a-b}{a+b}-\frac{2a\sqrt{a^2-b^2}}{b^2(ab^{-1}+1)^2}}\,\frac{1}{1+\frac{1-ba^{-1}}{1+ba^{-1}}}$, $a+b>0$.

6. $\frac{(\sqrt{a}-\sqrt{b})^3+2a^{3/2}+b^{3/2}}{a\sqrt{a}+b\sqrt{b}}+\frac{3\sqrt{ab}-3b}{a-b}$.

7. $\sqrt{\left(\frac{1}{2}\left(\frac{x}{y}\right)^{-1/2}-\frac{1}{2}\left(\frac{y}{x}\right)^{-1/2}\right)^{-2}+1}$.

8. $\frac{a+\sqrt{ab}}{a+b}\left(a^{1/2}(a^{1/2}-b^{1/2})^{-1}-\left(\frac{a^{1/2}+b^{1/2}}{b^{1/2}}\right)^{-1}\right)$.

9. $\frac{2x^{-1/3}}{x^{2/3}-3x^{-1/3}}-\frac{x^{2/3}}{x^{5/3}-x^{2/3}}-\frac{x+1}{x^2-4x+3}$.

10. $(a^{-2/3}-b^{-2/3})\,ab\,(\sqrt[3]{a}-\sqrt[3]{b})^{-1}+\sqrt[3]{ab^2}$, $a\neq b$.

11. $\frac{x-1}{x+\sqrt{x}+1}\div\frac{x^{0.5}+1}{x^{1.5}-1}+\frac{2}{x^{-0.5}}$.

12. $\left(\frac{\sqrt[4]{a^3}-1}{\sqrt[4]{a}-1}+\sqrt[4]{a}\right)^{\frac{1}{2}}\left(\frac{\sqrt[4]{a^3}+1}{\sqrt[4]{a}+1}-\sqrt{a}\right)(a-\sqrt{a^3})^{-1}$.

13. $\left(\frac{\sqrt[4]{x^3}-\sqrt[4]{x}}{1-\sqrt{x}}+\frac{1+\sqrt{x}}{\sqrt[4]{x}}\right)^2\left(1+\frac{2}{\sqrt{x}}+\frac{1}{x}\right)^{-1/2}$.

14. $\sqrt[4]{(1-2a+a^2)(a^2-1)(a-1)}\div\frac{a^2+2a-3}{\sqrt[4]{a+1}}$.

15. $\left(\frac{(x+\sqrt[3]{2ax^2})(2a+\sqrt[3]{4a^2x})^{-1}-1}{\sqrt[3]{x}-\sqrt[3]{2a}}-(2a)^{-1/3}\right)^{-6}$, $x\neq 0$, $x\neq 2a$.

16. $\dfrac{\sqrt{\sqrt{5}-2}\cdot\sqrt[4]{9+4\sqrt{5}}+\sqrt[3]{a^2}-\sqrt[3]{a}}{\sqrt{\sqrt{5}+2}\cdot\sqrt{9-4\sqrt{5}}+a}$.

17. (a) $\dfrac{\sqrt{x}+1}{x\sqrt{x}+x+\sqrt{x}}\div\dfrac{1}{x^2-\sqrt{x}}$, (b) $\dfrac{\sqrt{(x+2)^2-8x}}{\sqrt{x}-\dfrac{2}{\sqrt{x}}}$.

18. (a) $\dfrac{\sqrt[4]{x^5}+\sqrt[4]{xy^4}-\sqrt[4]{x^4y}-\sqrt[4]{y^5}}{\sqrt{x}-\sqrt{y}}\cdot(\sqrt[4]{y}+\sqrt[4]{x})$,

(b) $\dfrac{(a^2-b^2)/(\sqrt[3]{a}-\sqrt[3]{b})}{\sqrt[3]{a^4}+\sqrt[3]{ab^3}-\sqrt[3]{a^3b}-\sqrt[3]{b^4}}$.

19. $\left(1-\dfrac{x^{-3}-1}{x^{-1}-1}\div\dfrac{1+x+x^2}{x}\right)\cdot\left(\dfrac{1}{1-x^{-0,5}}-\dfrac{1}{1-x^{-1}}\right)$.

20. $\dfrac{a}{a^{1/2}b^{1/2}+b}+\dfrac{b}{a^{1/2}b^{1/2}-a}-\dfrac{a+b}{a^{1/2}b^{1/2}}$.

21. $\left(\dfrac{x^{1/2}+3y^{1/2}}{x-2x^{1/2}y^{1/2}+y}+\dfrac{x^{1/2}-3y^{1/2}}{x-y}\right)\cdot\dfrac{x^{1/2}-y^{1/2}}{2}$.

22. $\left(\dfrac{49}{c+27}-\dfrac{c^{1/3}+3}{c^{2/3}-3c^{1/3}+9}\right)\cdot\dfrac{c^{4/3}+27c^{1/3}}{16-c^{2/3}}+\dfrac{40-c^{2/3}}{4+c^{1/3}}$.

23. Evaluate the expression $2x^2-5ax+2a^2$ for $x=\sqrt{6}+\sqrt{5}$, $a=\sqrt{6}-\sqrt{5}$.

24. Rationalize the denominator in the given expressions:

(a) $\dfrac{1}{\sqrt[3]{a^2b}}$, (b) $\dfrac{1}{2\sqrt{5}-\sqrt{7}}$, (c) $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$,

(d) $\dfrac{1}{\sqrt[3]{5}-\sqrt[3]{2}}$, (e) $\dfrac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}}$.

25. Prove the identity: $\dfrac{1}{\sqrt{6}-\sqrt{5}}=\dfrac{3}{\sqrt{5}-\sqrt{2}}+\dfrac{4}{\sqrt{6}+\sqrt{2}}$.

In Exercises 26 to 28, perform the indicated operations:

26. $\dfrac{5}{4-\sqrt{11}}+\dfrac{1}{3+\sqrt{7}}-\dfrac{6}{\sqrt{7}-2}-\dfrac{\sqrt{7}-5}{2}$.

27. $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}+\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}-\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$.

28. $4\sqrt{7\dfrac{1}{2}}-\dfrac{2\sqrt{10}}{2\sqrt{3}-\sqrt{10}}$.

Chapter 3

FUNCTIONS

§ 3.1. Basic Notions

1. Function, Domain of Definition and Range. If by a known value of one quantity it is possible to find the value of another quantity, then we say that there is a *functional dependence* between these quantities. For instance, if the length x of the side of a square is known, then its area can be found by the formula $S = x^2$. Thus, the functional dependence between the length of the side of a square and its area is thereby specified.

In order to specify a *numerical function*, one has to specify a set of numbers X and a rule f, according to which every number x from this set is associated with a certain number (the value of the function). An independent variable taking on values from the set X is called the *argument* of the function. If the function is denoted by the letter f and $a \in X^*$, then the value of the function for the argument a is denoted by $f(a)$.

The set X on which a function f is specified is called the *domain of definition* of this function and is denoted by $D(f)$. The set of all values of the function f (the range) is denoted by $E(f)$.

Example. Find the domain of definition and the range of the function describing the dependence of the area of a square on the length of its side.

○ Since the length of the side of a square is always positive, the domain of definition of the given function is the set $\mathbf{R}_+$ of positive numbers: $D(f) = \mathbf{R}_+$. The area of a square is also always positive. Hence, any positive number S is the area of a square (with side equal to $\sqrt{S}$). Thus, the range of the considered function f is also represented by $\mathbf{R}_+$, that is, $E(f) = \mathbf{R}_+$. This function assigns a number x^2 to every number x from $\mathbf{R}_+$.

The detailed notation of a function that associates every positive number x with its square has the form $f\colon x \to x^2$, $x \in \mathbf{R}_+$. We shall use the following notation: x^2, where $x \in \mathbf{R}_+$ or $y = x^2$, $x \geqslant 0$.

2. Coordinate Plane. It was noted above that the set $\mathbf{R}$ of real numbers is called the number line. A set of pairs of real numbers is called a *number plane* and is denoted by the symbol $\mathbf{R}^2$.

* The notation $a \in X$ means: "the number a is an element of the set X" or "the number a belongs to the set X".

As is known, the set $\mathbf{R}$ can be represented geometrically as a horizontal number line. A geometrical representation of the set $\mathbf{R}^2$ is the *coordinate plane xOy* defined as two perpendicular number lines with a common origin O and the same scale. The point O is called the *origin of coordinates* (Fig. 26).

A one-to-one correspondence is established between the set of points of the coordinate plane and the set $\mathbf{R}^2$ in the following way:

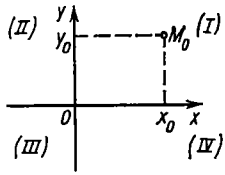


Fig. 26

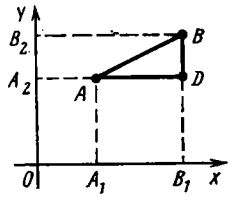


Fig. 27

if M_0 is a point in the coordinate plane, then by projecting it on the coordinate lines Ox and Oy, we find the coordinates of the projections x_0 and y_0. The coordinates are called respectively the *abscissa* and the *ordinate* of the point M_0, and the straight lines Ox and Oy the *axis of abscissas* and the *axis of ordinates*, respectively. Thus, to the point M_0 there corresponds one pair of numbers (x_0, y_0). Conversely, given a pair of numbers (x_0, y_0), we mark the points x_0 and y_0 on the coordinate lines Ox and Oy, respectively, and, drawing through these points straight lines parallel to the coordinate lines Oy and Ox, we find the point of their intersection M_0. To this unique point there corresponds the pair of numbers (x_0, y_0). The relevant notation is $M_0(x_0; y_0)$.

The coordinate axes divide the plane into four regions called the *quadrants*: Quadrant I is located to the right of the axis of ordinates and above the axis of abscissas. The remaining three quadrants are numbered anticlockwise as is shown in Fig. 26.

Theorem 3.1.1. *The distance* $\rho(A, B)$ *between the points* $A(x_1, y_1)$ *and* $B(x_2, y_2)$ *of the coordinate plane is computed by the formula*

$$\rho(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

□ Let A_1B_1 and A_2B_2 be projections of the line segment AB on the axes Ox and Oy, respectively (Fig. 27). Then $A_1B_1 = |x_2 - x_1|$, $A_2B_2 = |y_2 - y_1|$ (see § 1.3, Item 7). Further, we have

$$\begin{aligned}\rho(A, B) = AB &= \sqrt{AD^2 + BD^2} \\ &= \sqrt{A_1B_1^2 + A_2B_2^2} = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \ \blacksquare\end{aligned}$$

Example. Find the distance between the points A (2, —5) and B (—3, 7).

○ We find

$$\rho(A, B) = \sqrt{(-3-2)^2+(7+5)^2} = \sqrt{25+144} = \sqrt{169} = 13. \bullet$$

3. Ways of Representing Functions. Consider three most commonly used modes of representing functions: (a) analytical, (b) graphical, and (c) tabular.

(a) Analytical representation of a function. If we are given a collection of operations which must be performed with the argument x to obtain a function value, then the function is said to be represented by an *analytical expression*. This can be exemplified by the following functions: $y = x^2 - 5x + 1$, $x \in [0, 1]$, $y = x^2 - 5x + 1$, $x \in [1, 3]$, $y = x^2 - 5x + 1$, $x \in (-\infty, +\infty)$.

As we see, the analytical expressions of these functions are the same in form, nevertheless, we have three different functions, since they are given on three different sets.

Sometimes, a function may be specified on distinct sets of numbers by different analytical expressions; for example,

$$f(x) = \begin{cases} 2x+3 \text{ for } -1 \leqslant x \leqslant 0, \\ x+2 \text{ for } 0 < x \leqslant 1. \end{cases}$$

This function is defined on the interval [—1, 1]. To evaluate the function, it is necessary to find out what analytical expression should be used for each assigned value of the argument. For instance, if we have to compute f (0.5), then, taking into account that $0 < 0.5 \leqslant 1$, we make use of the equality $f(x) = x + 2$. We get f (0.5) = 2.5. And if it is necessary to evaluate f (—0.5), then, taking into consideration that $-1 \leqslant 0.5 \leqslant 0$, we make use of the equality $f(x) = 2x + 3$ and get f (—0.5) = 2.

Consider the function $y = \sqrt{x-1}$, $x \in [1, +\infty)$. Let us find the domain of definition of the analytical expression $\sqrt{x-1}$, that is, the set of all the values of the variable x for which the expression $\sqrt{x-1}$ makes sense. Since the operation of taking the square root is defined only on the set of nonnegative numbers, we get the inequality $x - 1 \geqslant 0$, whence $x \geqslant 1$.

Note that in this example the domain of the analytical expression of the function coincides with the given domain of the function. In such cases, the function is said to be given on the *natural* domain of definition. Let us agree that if the domain of definition of a function is natural, then, for the sake of brevity, we shall not mention this fact. For example, we shall speak of the function $y = \sqrt{x-1}$, meaning the function $y = \sqrt{x-1}$, $x \in [1, +\infty)$.

Example 1. Find the domain of the function

$$y = \sqrt{9-x} + \sqrt[4]{x-4} + \frac{1}{x-5}.$$

○ Since $\sqrt{9-x}$ and $\sqrt[4]{x-4}$ are defined only for nonnegative values of the radicands, we get the system of inequalities

$$\begin{cases} 9-x \geqslant 0, \\ x-4 \geqslant 0, \end{cases}$$ whence we find $4 \leqslant x \leqslant 9$.

Further, since the expression $1/(x-5)$ is defined for all x's, except for $x = 5$, we finally get that the domain of the given function is the closed interval $[4, 9]$ with the point $x = 5$ removed. ●

Example 2. Given the function $f(x) = \frac{2x+1}{x^2-1}$. Find: (1) $f(2)$, (2) $f(2a)$, (3) $f(a+3)$, (4) $f(-x)$, (5) $f(|x|)$.

○ (1) $f(2) = \frac{2 \cdot 2a+1}{2^2-1} = \frac{5}{3}$,

(2) $f(2a) = \frac{2 \cdot 2a+1}{(2a)^2-1} = \frac{4a+1}{4a^2-1}$,

(3) $f(a+3) = \frac{2(a+3)+1}{(a+3)^2-1} = \frac{2a+7}{a^2+6a+8}$,

(4) $f(-x) = \frac{2 \cdot (-x)+1}{(-x)^2-1} = \frac{-2x+1}{x^2-1}$,

(5) $f(|x|) = \frac{2 \cdot |x|+1}{|x|^2-1} = \frac{2|x|+1}{x^2-1}$. ●

(b) Graphical representation of a function. Consider the set F of points of the coordinate plane having the following property: any straight line parallel to the axis of ordinates intersects this set at no more than one point (Fig. 28). Let us take an arbitrary number x from the closed interval $[a, b]$ and draw through the point x a straight line parallel to the axis of ordinates. This line intersects F at the point M. Projecting the point M on the axis of ordinates, we find the number $f(x)$ corresponding to the number x. Thus, the function $y = f(x)$, $x \in [a, b]$, is specified by a graph. Such a way of representing a function is called *graphical*. In practice, we use various instruments and devices that record automatically variations of some quantities with time (thermographs, barographs, etc.). They supply a graphical representation of these quantities as functions of time.

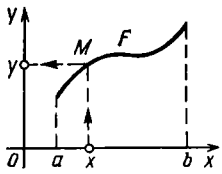


Fig. 28

(c) Tabular representation of a function. In this case, we are given a table of values (usually approximate ones) of a function for

a finite set of values of the argument. The tables of squares, cubes, square roots, etc. may serve as examples.

In many cases, tabular representation of a function turns out to be convenient. It enables us to find the needed value of the function for the corresponding values of the argument without any computations. In practice, the dependence of one quantity on the other is found experimentally. In this case, one quantity is assigned certain values, and then the value of the second quantity is found experimentally for each of the assigned values. In such a way a table of func-

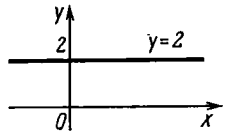


Fig. 29

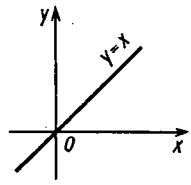


Fig. 30

tion values is compiled. There are methods enabling the user to choose an analytical expression for the function from such a table (to a certain accuracy).

4. Graph of a Function. Let a function f be given by an analytical expression $f(x)$, that is, $y = f(x)$, $x \in X$. The *graph of the function* f is defined as a set of points in the coordinate plane with the coordinates $(x, f(x))$, where $x \in X$.

Example 1. Construct the graph of the function $y = 2$.

○ The function is given by the analytical expression $f(x) = 2$. The domain of the function is not indicated; hence, by virtue of the established agreement, the natural domain is meant, that is, the domain of the expression $f(x)$. Since the expression $f(x) = 2$ is defined for any real values of x, the given function is defined on the entire set **R** of real numbers.

According to the hypothesis, to every $x \in \mathbf{R}$ there corresponds the value $y = 2$. Hence, the graph of the given function is the set of all points of the coordinate plane which have the form $(x, 2)$. Geometrically, this set is represented by a straight line parallel to the axis of abscissas (Fig. 29). ●

Example 2. Construct the graph of the function $y = x$.

○ Here, as in the preceding example, the domain of the given function is the set **R**. Since $f(x) = x$, the graph of the function is the set of points of the form (x, x), that is, the set of such points of the coordinate plane whose abscissa and ordinate are equal. If a point has the same abscissa and ordinate, then it is equidistant from the lines Ox and Oy. The set of all such points coincides with the set of points on the angle bisector of Quadrants I and III. This straight

line (the angle bisector) is just the graph of the function $y = x$ (Fig. 30). ●

Example 3. Construct the graph of the function $y = [x]$ (recall that $[x]$ is the integral part of the number x).

○ If n is an integer, then on the interval $[n, n + 1)$ we have $[x] = n$. For example, $[x] = 0$ on the interval $[0, 1)$, $[x] = 1$ on the interval $[1, 2)$, $[x] = -1$ on the interval $[-1, 0)$. The graph of the given function (see Fig. 31) consists of horizontal line segments whose right-hand end points (indicated by arrows) are excluded. ●

In practice, in order to sketch the graph of a function, one has first to compile a table of corresponding values of the function for some

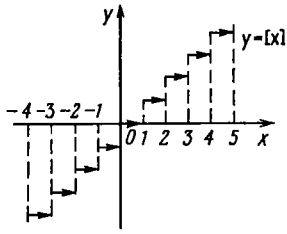


Fig. 31

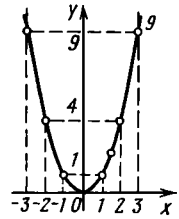


Fig. 32

values of the argument, then to plot the corresponding points in the plane and to connect them by a curve. The graph of the function in this case is supposed to be a smooth curve, and the found points give a sufficiently exact picture of the variation of the function.

Example 4. Graph the function $y = x^2$.

○ First compile a table of function values:

x	0	1	−1	2	−2	3	−3
y	0	1	1	4	4	9	9

Then plot the found points (0, 0), (1, 1), (−1, 1), (2, 4), (−2, 4), (3, 9), and (−3, 9) on the coordinate plane and connect them by a smooth curve to get the graph (or more precisely, the sketch) of the function $y = x^2$ (Fig. 32). This curve is called the *parabola*. ●

There are functions whose graphs cannot be represented; for example, the function

$$D(x) = \begin{cases} 1 \text{ if } x \text{ is a rational number,} \\ 0 \text{ if } x \text{ is an irrational number,} \end{cases}$$

which is called the *Dirichlet function*. Its graph consists of points of the axis of abscissas with irrational abscissas and of points of the straight line $y = 1$ with rational abscissas. Since there are both rational and irrational points on an arbitrarily small segment of the number line, it is impossible to draw such a graph.

5. Even and Odd Functions. A set X of numbers is said to be *symmetric* if, along with each of its elements x, it also contains the opposite element $-x$. Here are some examples of symmetric sets: the closed interval $[-5, 5]$, the open interval $(-3, 3)$, the number line $(-\infty, +\infty)$.

Let us give some examples of asymmetric sets: the closed interval $[-5, 4]$, the open interval $(-2, 3)$, the ray $[-10, +\infty)$. The set $[-2, 2)$ is also asymmetric, since -2 belongs to this set, while the opposite number 2 does not.

Definition. A function $y = f(x)$, $x \in X$ is said to be *even* if: (1) X is a symmetric set; (2) for any x from X the equality $f(-x) = f(x)$ holds true.

A function $y = f(x)$, $x \in X$ is said to be *odd* if: (1) X is a symmetric set; (2) for any $x \in X$ the equality $f(-x) = -f(x)$ is valid.

Example 1. Determine whether or not each of the indicated functions is even, odd, or neither:

$$(1)\ y = x^4,\ (2)\ y = x^5,\ (3)\ y = \frac{x-4}{x^2-9},\ (4)\ y = \sqrt{x-3}.$$

○ (1) The domain of definition of the function $y = x^4$ is the number line $(-\infty, +\infty)$ which is a symmetric set. Further, we have $f(x) = x^4$, $f(-x) = (-x)^4 = x^4$. Thus, $f(-x) = f(x)$, that is, the given function is even.

(2) The domain of the function $y = x^5$ is the number line $(-\infty, +\infty)$ which is a symmetric set. Further, we have $f(x) = x^5$, $f(-x) = (-x)^5 = -x^5$. Consequently, $f(-x) = -f(x)$, that is, the given function is odd.

(3) The function is not defined for those values of x for which the denominator $x^2 - 9$ vanishes, that is, at the points -3 and 3. Hence, the domain of the given function is a symmetric set. Further, we have

$$f(x) = \frac{x-4}{x^2-9},\quad f(-x) = \frac{(-x)-4}{(-x)^2-9} = -\frac{x+4}{x^2-9}.$$

Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, the given function is neither even nor odd.

(4) The domain of the function $y = \sqrt{x-3}$ is represented by the ray $[3, +\infty)$ which is an asymmetric set. Hence, the function is neither even nor odd. ●

Thus, in order to determine whether an analytically represented function is even or odd, one should find out whether the given or natural domain of definition is symmetric. If this domain is an asymmetric set, then the given function is neiher even nor odd. And if this domain is a symmetric set, then one must replace x by $-x$ in the analytical expression of $f(x)$, carry out all possible simplifications, and compare the result with $f(x)$. If it turns out that $f(-x) =$

$f(x)$, then the given function is even; if it appears that $f(-x) = -f(x)$, then the function is odd, and if $f(-x)$ differs both from $f(x)$ and $-f(x)$, then the given function is neither even nor odd.

The following theorems illustrate the properties of graphs of even and odd functions.

Theorem 3.1.2. *If a function $y = f(x)$, $x \in X$ is even, then its graph is symmetric with respect to the axis of ordinates.*

□ Let $M(x, f(x))$, be a point belonging to the graph of the function under consideration. Since, by hypothesis, the function is even,

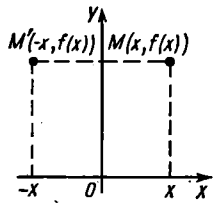


Fig. 33

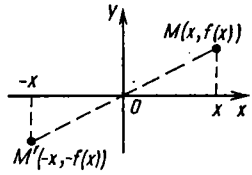


Fig. 34

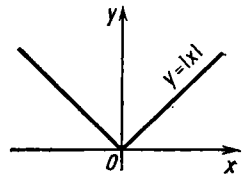


Fig. 35

we have, firstly, $(-x) \in X$ and, secondly, $f(-x) = f(x)$. Hence, the point $M'(-x, f(x))$ also belongs to the graph of the given function. But the points M and M' are symmetric about the axis of ordinates (Fig. 33). Thus, the graph of an even function along with each of its points contains a point symmetric to it with respect to the axis of ordinates. Therefore the graph of an even function is symmetric with respect to the axis of ordinates. ■

Theorem 3.1.2. *If $y = f(x)$, $x \in X$, is an odd function, then its graph is symmetric about the origin.*

□ Let $M(x, f(x))$, be a point belonging to the graph of the function. Since, by hypothesis, the function is odd, we have, firstly, $(-x) \in X$ and, secondly, $f(-x) = -f(x)$. Hence, the point $M'(-x, -f(x))$ also belongs to the graph of the given function. But the points M and M' are symmetric about the origin (Fig. 34). Thus, the graph of an odd function, along with each of its points, contains a point symmetric to it with respect to the origin. Therefore the graph of an odd function is symmetric about the origin. ■

Example 2. Construct the graph of the function $y = |x|$.

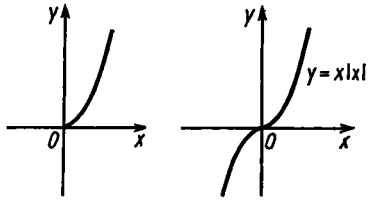
○ This function is defined throughout the number line (a symmetric set), and, in addition, $f(-x) = |-x| = |x| = f(x)$. Hence, the function is even, and therefore its graph is symmetric with respect to the axis of ordinates.

If $x \geqslant 0$, then $|x| = x$, that is, for $x \geqslant 0$ we have $y = x$. The graph of the function $y = x$, $x \geqslant 0$, is the angle bisector of Quadrant I (see Example 2 of Item 4). Adding to it its image with respect to the y-axis, we get the graph of the function $y = |x|$ (Fig. 35). ●

Example 3. Construct the graph of the function $y = x \mid x \mid$.

○ This function is defined throughout the number line (a symmetric set), and, in addition, $f(-x) = (-x) \cdot |-x| = -x \mid x \mid = -f(x)$. Hence, the given function is odd, and therefore its graph is symmetric with respect to the origin.

If $x \geqslant 0$, then $|x| = x$, and $f(x) = x \mid x \mid = x \cdot x = x^2$. Hence, for $x \geqslant 0$ we have $y = x^2$. The graph is a branch of a parabola (see Example 5 of the preceding item) which is represented in Fig. 36*a*.



(*a*) (*b*)

Fig. 36

Adding to it another branch symmetric to the obtained one with respect to the origin, we get the graph of the function $y = x \mid x \mid$ (Fig. 36*b*). ●

6. Periodic Functions. In nature and in engineering, we often come across phenomena which repeat with time. For instance, as the Earth orbits about the Sun, its distance from the Sun varies all the time, but each time, completing an orbit, the Earth is found at the same distance from the Sun as a year before. A blade of a turbine also occupies the same position on completing a revolution. Such periodically repeated processes are described by periodic functions.

Definition 1. A function f is said to have a *period* T if for any value of x for which it is defined the following equalities hold:

$$f(x - T) = f(x) = f(x + T). \tag{1}$$

This definition implies that if a function f with a period T is defined at the point x, then it is also defined at the points $x + T$ and $x - T$.

Any function has a period equal to zero (for $T = 0$ equality (1) turns into identity: $f(x - 0) \equiv f(x) \equiv f(x + 0)$).

Definition 2. A function f having a nonzero period T is called *periodic*.

For example, if time is measured in years, then the distance from the Earth to the Sun is given by a periodic function whose period is equal to 1.

Theorem 3.1.4. *If a function f has a period T, then any number multiple of T* (that is, a number of the form kT, $k \in \mathbf{Z}$) *is also its period.*

□ Let us prove that $2T$ is a period of the function f. We have

$$f(x) = f(x + T) = f((x + T) + T) = f(x + 2T),$$
$$f(x) = f(x - T) = f((x - T) - T) = f(x - 2T).$$

Similarly, we prove that $f(x) = f(x + 3T) = f(x - 3T)$. Passing from one natural number n to the next one, we make sure that for any k the following equality is valid:

$$f(x - kT) = f(x) = f(x + kT).$$

Hence, all numbers of the form kT, $k \in \mathbf{Z}$, are periods of the function f. ■

Thus, a periodic function has infinitely many periods. In most cases, among positive periods of a periodic function there is a period

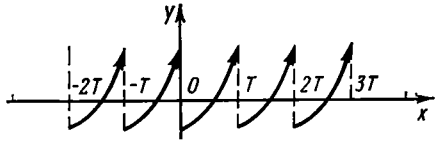


Fig. 37

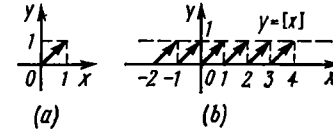


Fig. 38

which is not a multiple of any other period called the *fundamental period* of this function, all the other periods being multiple of the fundamental period.

The graphs of periodic functions possess the following property. If T is the fundamental period of a function $y = f(x)$, then, in order to construct its graph, it suffices to construct a branch of the graph on one of the subintervals of length T and then to translate this branch along the x-axis by $\pm T$, $\pm 2T$, $\pm 3T$, . . . (Fig. 37). In most cases, a subinterval with end points $(-T/2, 0)$ and $(T/2, 0)$ or $(0, 0)$ and $(T, 0)$ is chosen as such a subinterval of length T.

Example. Prove that $y = \{x\}$ ($\{x\}$ is a fractional part of the number x) is a periodic function and construct its graph.

○ It is obvious that the numbers x and $x + k$ ($k \in \mathbf{Z}$) have the same fractional part, that is, $\{x\} = \{x + k\}$. Hence, any whole number k is a period of the function $\{x\}$, $T = 1$ being the fundamental period.

First we construct the graph of the function $y = \{x\}$ on the subinterval $[0, 1)$ which is equal to the fundamental period. If $0 \leqslant x < 1$, then $\{x\} = x$, that is, on this subinterval we have $y = x$. The graph of the function $y = \{x\}$, where $0 \leqslant x < 1$, is represented in Fig. 38*a*, while the entire graph of the function $y = \{x\}$ is shown in Fig. 38*b*. ●

7. Monotonic Functions.

Definition 1. A function $y = f(x)$ is said to be *increasing* on an interval X if for any two of its points x_1 and x_2 such that $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ is satisfied; in other words, if to a greater value of the argument there corresponds a greater value of the function.

Definition 2. A function $y = f(x)$ is said to be *decreasing* on an interval X if for any two of its points x_1 and x_2 such that $x_1 < x_2$

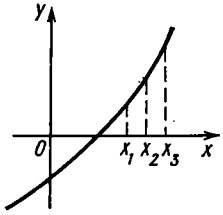


Fig. 39

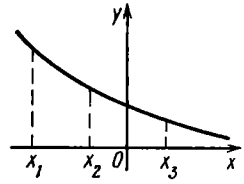


Fig. 40

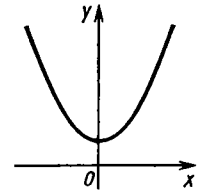


Fig. 41

the inequality $f(x_1) > f(x_2)$ is satisfied; in other words, if a smaller value of the function corresponds to a greater value of the argument.

When moving along the axis of abscissas from left to right, the ordinate of the graph of an increasing function increases (Fig. 39), while the ordinate of the graph of a decreasing function decreases (Fig. 40).

Increasing and decreasing functions are united by the generic term *monotonic* (or *monotone*) *functions*.

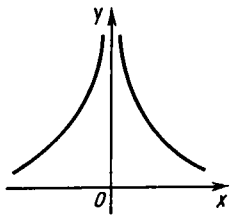


Fig. 42

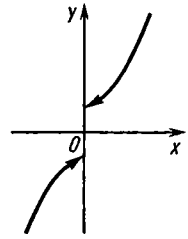


Fig. 43

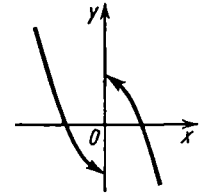


Fig. 44

Geometrical considerations enable us to make the following assertions:

1°. *If a function* $y = f(x)$ *is even and increasing for* $x > 0$, *then it is decreasing for* $x < 0$ (Fig. 41).

2°. *If a function* $y = f(x)$ *is even and decreasing for* $x > 0$, *then it is increasing for* $x < 0$ (Fig. 42).

3°. *If a function* $y = f(x)$ *is odd and increasing for* $x > 0$, *then it is also increasing for* $x < 0$ (Fig. 43).

4^0. *If a function* $y = f(x)$ *is odd and decreases for* $x > 0$, *then it also decreases for* $x < 0$ (Fig. 44).

5^0. *A monotonic function cannot be periodic.*

Example 1. Investigate the function $y = x^3$ for being monotonic.

○ This function is odd, therefore it suffices to test its interval $[0, +\infty)$. If $0 \leqslant x_1 < x_2$, then $x_1^3 < x_2^3$ (to prove this, it suffices to multiply the inequality $x_1 < x_2$ by itself three times by making use of Property 7^0 of numerical inequalities—see § 1.3.3). Hence $x_1 < x_2$ implies $f(x_1) < f(x_2)$, that is, the function is monotone increasing for $x \geqslant 0$. Then, by virtue of Property 3^0, it also increases for $x < 0$. Since, finally, it is obvious that for $x_1 < 0$ and $x_2 > 0$ $f(x_1) < f(x_2)$, we may conclude that the function is increasing not only for $x < 0$ and $x > 0$, but also throughout the number line. ●

Example 2. Test the function $y = x^2 + 2$ for being monotonic.

○ This is an even function. If $0 < x_1 < x_2$, then, using the properties of numerical inequalities (see § 1.3.3), we get $x_1^2 + 2 < x_2^2 + 2$, that is, $f(x_1) < f(x_2)$. Hence, the function is increasing for $x > 0$. Then, according to Property 1^0, it is decreasing for $x < 0$. ●

8. Graph of an Equation in Two Variables. Let there be given an equation in two variables $F(x, y) = 0$. If all of its solutions are represented by points in the coordinate plane, then we get a certain set of points in the plane. This set represents the *graph of the equation* $F(x, y) = 0$.

Example 1. Construct the graph of the equation $y - x = 0$.

○ We transform this equation to the form $y = x$. As we established above, the graph of the function $y = x$ is a straight line, namely, the angle bisector of Quadrants I and III (see Fig. 30). This straight line represents the graph of the given equation. ●

Example 2. Prove that the graph of the equation $x^2 + y^2 = 16$ is a circle with centre at the origin and of radius 4 (Fig. 45).

○ Let $A(x, y)$ be an arbitrary point of a circle belonging to neither of the coordinate axes. Then, by the Pythagorean theorem, from the right triangle OAM we get: $OM^2 + AM^2 = OA^2$. But $OM = |x|$, $AM = |y|$, $OA = 4$. Hence, $x^2 + y^2 = 16$. Obviously, the coordinates of each of the four points of intersection of the circle with the coordinate axes: $(4, 0)$, $(0, 4)$, $(-4, 0)$, and $(0, -4)$ also satisfy the equation $x^2 + y^2 = 16$. Thus, the coordinates of any point of the circle under consideration satisfy the equation $x^2 + y^2 = 16$.

Let us now show that if a point does not lie on the circle, then its coordinates do not satisfy the equation in question. Indeed, if the point $B(x, y)$ lies inside the circle, then $x^2 + y^2 < 16$, and if the point $C(x, y)$ lies outside the circle, then $x^2 + y^2 > 16$ (Fig. 45). Thus, the graph of the equation $x^2 + y^2 = 16$ is a circle with centre at the origin and radius 4. ●

In general, the *graph of the equation* $x^2 + y^2 = r^2$ *is a circle with centre at the origin and of radius* r (Fig. 46).

Note that as distinct from Example 1, the constructed graph of the equation $x^2 + y^2 = r^2$ is not the graph of a function, since to every value of x from the interval $(-r, r)$ there correspond two numbers $y_1 = \sqrt{r^2 - x^2}$ and $y_2 = -\sqrt{r^2 - x^2}$ (Fig. 46). If we confined our-

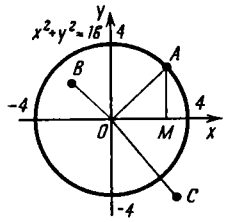


Fig. 45

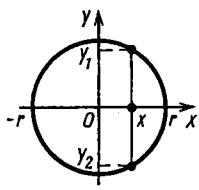


Fig. 46

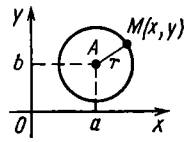


Fig. 47

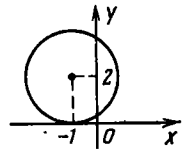


Fig. 48

selves to the upper or lower semicircle, then we would get the graphs of two functions $y = \sqrt{r^2 - x^2}$ (for the upper semicircle) and $y = -\sqrt{r^2 - x^2}$ (for the lower semicircle).

Theorem 3.1.5. *A circle of radius* r *centred at the point* $A(a, b)$ *represents the equation* $(x - a)^2 + (y - b)^2 = r^2$ (Fig. 47).

□ Let us take any point $M(x, y)$ of the circle. Then $\rho(A, M) = \sqrt{(x - a^2) + (y - b)^2}$ (see page 114). Since $\rho(A, M) = r$, we have $(x - a^2) + (y - b)^2 = r^2$. If a point does not lie on the circle, then its coordinates will not satisfy the given equation. ■

Example 3. Prove that $x^2 + y^2 + 2x - 4y + 1 = 0$ is an equation of a circle and construct this circle.

○ We have

$$(x^2 + 2x + 1) + (y^2 - 4y + 4) - 4 = 0, \quad (x + 1)^2 + (y - 2)^2 = 4.$$

This is the equation for the circle of radius 2 centred at the point $(-1, 2)$ (Fig. 48). ●

Exercises

1. A function f is given by the formula $y = x^3$. Find $f(0)$, $f(0.2)$, $f(0.5)$.
2. Given a function $y = f(x)$, where $f(x) = (2x - 3)^3/(x^2 + 1)$. Find $f(0)$, $f(a)$, $f(-x)$, and $f(x + 1)$.
3. Express by a formula the correspondence between the perimeter of the square and its side.
4. Given a function $f(x) = 3x^3 + 1$. Compute $(f(-1))^3 + f^2(1) - 2f(1)$.
5. Given a function $f(x) = 2x^2 - 3x + 1$. Determine the parameter a from the condition $f(1 - a) - f(a - 1) = 0$.
6. Given a function

$$f(x) = \begin{cases} \sqrt[3]{x} & \text{for } x \geqslant 1, \\ 1 & \text{for } x < 1. \end{cases}$$

Find $f(-8)$, $f(-1.2)$, $f(0)$, $f(1)$, and $f(4)$.

7. Give an example of a function represented analytically and defined: (a) throughout the number line; (b) throughout the number line except for the point $x = 5$, (c) only on the ray $[2, +\infty)$, (d) on the closed interval $[-2, 2]$.

8. Which of the given functions are increasing and which are decreasing: (a) $f(x) = x$, (b) $f(x) = x + 1$, (c) $f(x) = 5 - x$, (d) $f(x) = 3x$?

9. Indicate the intervals on which the given functions are decreasing (increasing): (a) $f(x) = 3x^2$, (b) $f(x) = 2x^3$.

10. Which of the functions given in Exercises 8 and 9 are even, which are odd, and which are neither even nor odd?

11. Which of the given functions are even, and which are odd: (a) $f(x) = x^6$, (b) $f(x) = x^7$, (c) $f(x) = -|x|$, (d) $f(x) = \frac{x}{|x|}$, (e) $f(x) = x^2|x|$?

12. Graph the following functions: (a) $y = 0$, (b) $y = 1$, (c) $y = -2$, (d) $y = -x$.

13. Find the domain of definition of each of the given functions:

(a) $f(x) = 2x + 1$, (b) $f(x) = \frac{1}{x}$, (c) $f(x) = \frac{2}{x-3}$,

(d) $f(x) = \frac{5x+1}{(x-1)(x+1)}$, (e) $f(x) = \sqrt{x}$, (f) $f(x) = \sqrt{x-4}$.

14. Prove that the function $y = x^2$ is nonperiodic.

15. Extend the graph of a periodic function $y = f(x)$ with period $T = 2$ (Fig. 49) along the entire number line.

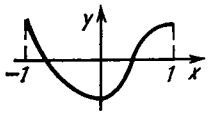


Fig. 49

16. Do the points $A(1, 2)$ and $B(-1, 3)$ belong to the graph of the equation $y + x^2 = 5$?

17. The point $A(x, -2)$ belongs to the graph of the equation $2x + y = 5$. Find its abscissa.

18. Find the distance between the points A and B if: (a) $A(2, -1)$, $B(-1, 5)$, (b) $A(0, -4)$, $B(3, 0)$.

19. Construct the graphs of the given equations: (a) $x^2 + y^2 = 25$, (b) $(x-1)^2 + (y+3)^2 = 9$, (c) $(x+2)^2 + (y-2)^2 = 0$.

20. Prove that $x^2 + y^2 - 5x + 3y + 7.5 = 0$ is an equation of a circle. Construct this circle.

§ 3.2. Linear Function

1. Linear Function and Its Properties. A function of the form $y = kx + b$, where k and b are real numbers, is called *linear*. If, in particular, $k = 0$, then a constant function $y = b$ is obtained.

Here are some examples of linear functions: $y = 2x + 3$ (here $k = 2$, $b = 3$), $y = -0.5x + 4$ (here $k = -0.5$, $b = 4$), $y = 4x$ (here $k = 4$, $b = 0$), $y = 7.5$ (here $k = 0$, $b = 7.5$), $y = 0$ (here $k = 0$, $b = 0$).

Note some properties of a linear function.

1°. *The domain of definition is the set* $\mathbf{R}$ *of real numbers.*

2°. *A linear function* $y = kx + b$ *is even if* $k = 0$, *odd if* $b = 0$, *and neither even nor odd if* $k \neq 0$ *and* $b \neq 0$.

□ Note first that the domain of the function is a symmetric set. Compare the values $f(x)$ and $f(-x)$. We have

$$f(x) = kx + b,\ f(-x) = k(-x) + b = -kx + b.$$

(1) If $k = 0$, then $f(x) = b$, $f(-x) = b$. Hence, $f(-x) = f(x)$, that is, the function is even.

(2) If $b = 0$, then $f(x) = kx$, $f(-x) = -kx$. Consequently, $f(-x) = -f(x)$, that is, the function is odd.

(3) If $k \neq 0$ and $b \neq 0$, then $-kx + b \neq kx + b$ and $-kx + b \neq -(kx + b)$. Thus, $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, that is, the function is neither even nor odd. ■

For example, $y = 4$ is an even function, $y = 5x$ and $y = -3x$ are odd functions; $y = 2x + 1$, $y = x - 5$, and $y = -4x + 3$ are neither even nor odd functions.

Note that if $k = 0$ and $b = 0$, then we get the function $y = 0$, which we may regard both as even and as odd.

3°. *A linear function $y = kx + b$ is increasing if $k > 0$, decreasing if $k < 0$; and if $k = 0$, then it is constant.*

□ Let $x_1 < x_2$. Comparing the numbers $f(x_1)$ and $f(x_2)$, where $f(x_1) = kx_1 + b$ and $f(x_2) + b$, we have

$$f(x_2) - f(x_1) = (kx_2 + b) - (kx_1 + b) = (k(x_2 - x_1).$$

Here $x_2 - x_1 > 0$, hence, the sign of the difference $f(x_2) - f(x_1)$ depends on the coefficient k. The following three cases may occur:

(1) $k > 0$. Then $f(x_2) - f(x_1) > 0$, that is, $f(x_1) < f(x_2)$. Thus, $x_1 < x_2$ implies that $f(x_1) < f(x_2)$, that is, the function is increasing.

(2) $k < 0$. Then $f(x_2) - f(x_1) < 0$, that is, $f(x_1) > f(x_2)$. Thus, $x_1 < x_2$ implies that $f(x_1) > f(x_2)$, that is, the function is decreasing.

(3) $k = 0$. In this case, the linear function has the form $y = b$, that is, it is constant. ■

For example, $y = 3x - 5$, $y = x + 2$, $y = \frac{1}{2}x + \frac{1}{3}$ are increasing functions, $y = -2x - 3$, $y = -x + 1$, $y = -\frac{1}{3}x + \frac{2}{7}$ are decreasing functions.

2. Graph of a Constant Function. In § 3.1, Item 4, we constructed the graph of the constant function $y = 2$. Reasoning in a similar way, we may conclude that the graph of any constant function $y = b$ is a straight line parallel to the axis of abscissas and passing through the point $(0, b)$ on the axis of ordinates. Figure 50 represents graphs of several constant functions. The graph of the function $y = 0$ is, in particular, the axis of abscissas.

3. Direct Variation and Its Graph. The *direct variation* is defined as a linear function of the form $y = kx$, where $k \neq 0$; the number k is called the *variation factor* (or *coefficient*).

Theorem 3.2.1. *The graph of the direct variation $y = kx$ is a straight line passing through the origin.*

□ Let us draw a straight line through the origin and the point $A(1, k)$ and prove that it is the graph of the function $y = kx$. We shall first consider the case when $k > 0$ (Fig. 51).

Let us take any point $M(x, y)$ on the straight line l. From the similarity of the triangles OAA_1 and OMM_1 we conclude that $\frac{MM_1}{AA_1} = \frac{OM_1}{OA_1}$, that is, $\frac{y}{k} = \frac{x}{1}$, whence $y = kx$. We now take a point $Q(\bar{x}, \bar{y})$ not lying on the straight line l. Then the coordinates of the point P, having the same abscissa but lying on the straight line l, satisfy the equation $y = kx$, and, consequently, the coordinates of the point Q do not satisfy this equation.

Thus, the points of the straight line l and only they satisfy the equation $y = kx$; hence, the straight line l is the graph of the function $y = kx$.

Consider now the case when $k < 0$. Let us take two functions $y = kx$ and $y = -kx$. For one and the same abscissa x the ordi-

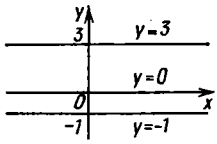


Fig. 50

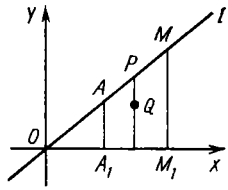


Fig. 51

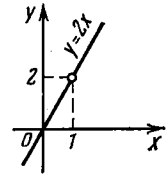


Fig. 52

nates of the graphs of these functions are equal in absolute value, but have opposite signs. Therefore the graphs of these functions are symmetric with respect to the axis of ordinates. However, $-k > 0$, and, according to the aforeproved, the graph of the function $y = -kx$ is a straight line. Since under reflection a straight line is transformed into a straight line, the graph of the function $y = kx$ is a straight line. ■

Let us, for instance, graph the function $y = 2x$. We know that its graph is a straight line passing through the origin. To construct this line, it suffices to find one point of the graph different from the origin and draw a straight line through the origin and the found point. Let us take the point (1, 2) as such a point (if $x = 1$, then $y = 2 \cdot 1 = 2$). The graph of the function $y = 2x$ is shown in Fig. 52.

Figure 53 represents the graphs of the functions $y = \frac{1}{2}x$, $y = 3x$, $y = -x$, and $y = -\frac{1}{3}x$.

Note that the size of the angle formed by the graph of the function $y = kx$ with the positive ray Ox changes with k (in particular, for $k > 0$ this angle is acute, while for $k < 0$ it is obtuse). Therefore k is called the *angular coefficient* or *slope*.

4. Graph of a Linear Function. Let us construct the graph of a linear function $y = kx + b$ for the general case, that is, when both coefficients are different from zero.

We know that the graph of a direct variation $y = kx$ is a straight line passing through the origin. The ordinate of every point of the graph of the function $y = kx + b$ differs from the ordinate of the corresponding point of the graph of the direct variation $y = kx$ by b. Hence, the graph of the function $y = kx + b$ is obtained

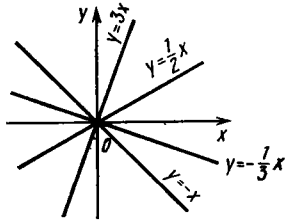


Fig. 53

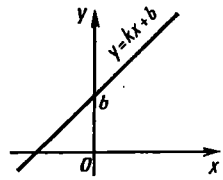


Fig. 54

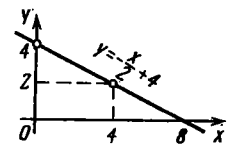


Fig. 55

from the graph of the function $y = kx$ by translating the latter upwards by $|b|$ if $b > 0$ and downwards if $b < 0$, that is, by translating by the vector $\mathbf{p}(0, b)$. But the translation carries a straight line into a straight line parallel to itself. Thereby we have proved the following theorem.

Theorem 3.2.2. *The graph of a linear function $y = kx + b$ is a straight line* (Fig. 54).

The point $(0, b)$ always belongs to the graph of the function $y = kx + b$, that is, this graph intersects the y-axis at the point whose ordinate is b (Fig. 54). Therefore the coefficient b is sometimes called the *initial ordinate*. As to the coefficient k, the same as in the case of a direct variation, it is called the *slope*.

Example 1. Construct the graph of the function $y = -\frac{1}{2}x + 4$.

○ As is known, the graph of a linear function is a straight line, and, in order to construct a straight line, it suffices to know two of its points. Let us compile a table: $\begin{array}{c|c|c} x & 0 & 4 \\ \hline y & 4 & 2 \end{array}$ (the argument x was assigned two arbitrary values, and the corresponding values of y were found by the formula $y = -\frac{1}{2}x + 4$). We now plot the points $(0, 4)$ and $(4, 2)$ in the coordinate plane and draw a straight line through them, which is the desired graph (Fig. 55). ●

Let us consider one more method of constructing the graph of a linear function. Let there be given a linear function: $y = 3x - 1$. If we take $x = 0$, then $y = -1$, and we get the point $A(0, -1)$

lying on the y-axis. Setting $y = 0$, we get the point B (1/3, 0) lying on the x-axis. These points belong to the graph of the function $y = 3x - 1$ and, consequenty, they lie on the straight line which joins them and is the graph of the given function. Drawing a straight line through the points A and B, we get the graph of the function $y = 3x - 1$ (Fig. 56).

The foregoing implies the following assertions concerning mutual positions of the graphs of two linear functions: $y = k_1x + b_1$ and $y = k_2x + b_2$:

(1) *if* $k_1 = k_2$ *and* $b_1 = b_2$, *then the straight lines representing the graphs of the functions* $y = k_1x + b_1$ *and* $y = k_2x + b_2$ *coincide*,

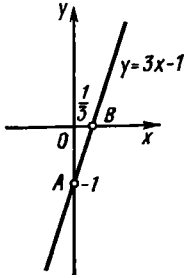


Fig. 56

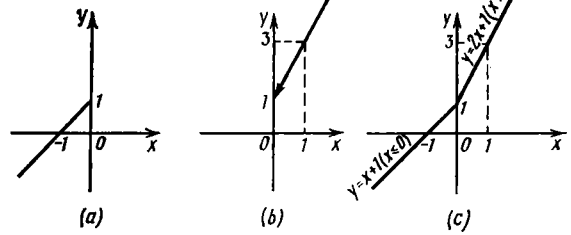


Fig. 57

(2) *if* $k_1 = k_2$ *and* $b_1 \neq b_2$, *then the straight lines are parallel and do not coincide*,

(3) *if* $k_1 \neq k_2$, *then the straight lines are nonparallel, that is, they intersect.*

Example 2. Construct the graph of the function

$$y = \begin{cases} x + 1 & \text{for } x \leqslant 0, \\ 2x + 1 & \text{for } x > 0. \end{cases}$$

○ The graph of the function $y = x + 1$ is a straight line, but we must take only the portion of this line to the left of the y-axis for $x \leqslant 0$ (Fig. 57*a*). The graph of the function $y = 2x + 1$ is a straight line, but we need take only the portion of this line to the right of the y-axis for $x > 0$ (Fig. 57*b*). The desired graph is shown in Fig. 57*c*. ●

5. Graph of a Linear Equation in Two Variables. As is known (see § 1.6, Item 1), a linear equation in two variables is defined as an equation of the form $ax + by + c = 0$, where at least one of the coefficients a or b is other than zero. In the present item, we shall consider the graph of such an equation.

Let $b \neq 0$. Then from the equation $ax + by + c = 0$ we find $y = -\frac{a}{b}x + \frac{c}{b}$. The graph of the linear function $y = -\frac{a}{b}x + \frac{c}{b}$

is a straight line (if $a = 0$, then we get a constant function $y = c/b$ whose graph is a straight line parallel to the x-axis). Hence, the graph of the linear equation $ax + by + c = 0$ is in this case a straight line.

Let $b = 0$. Then $a \neq 0$, and from the equation $ax + c = 0$ we find $x = -c/a$. The graph of this equation is the set of points in the coordinate plane having the same abscissa $-c/a$. This is the set of

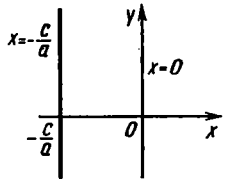


Fig. 58

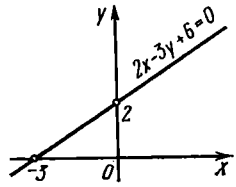


Fig. 59

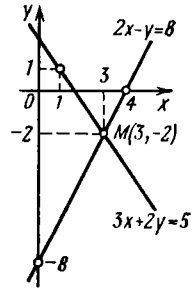


Fig. 60

points of the straight line parallel to the axis of ordinates and passing through the point $(-c/a, 0)$ on the axis of abscissas (Fig. 58). The graph of the equation $x = 0$ is, in particular, the y-axis.

Thus, *the graph of any linear equation* $ax + by + c = 0$ *is a straight line. If* $b = 0$, *then it is parallel to the y-axis, and if* $a = 0$, *then it is parallel to the x-axis.*

Example. Construct the graph of the equation $2x - 3y + 6 = 0$.

○ The graph of a linear equation is a straight line. To construct a straight line, it suffices to know two of its points. Substituting $x = 0$ into the equation $2x - 3y + 6 = 0$, we get $-3y + 6 = 0$, that is, $y = 2$. Substituting $y = 0$ into the equation $2x - 3y + 6 = 0$, we get $2x + 6 = 0$, that is, $x = -3$.

Thus, we have found two points of the desired graph: (0, 2) and (−3, 0). Drawing a straight line through them, we get the graph of the equation $2x - 3y + 6 = 0$ (Fig. 59). ●

6. Geometrical Interpretation of the Solution of Two Simultaneous Linear Equations in Two Variables. Let there be given a system of two simultaneous equations

$$\begin{cases} a_1x + b_1y + c_1 = 0, \\ a_2x + b_2y + c_2 = 0. \end{cases} \tag{1}$$

The graph of either of the equations is a straight line. Two straight lines may: (1) **intersect** at one point; then a system has *only one solution*; (2) **be parallel and not coincide**; then a system *has no solu-*

tion; and, finally, (3) **coincide** in which case a system has *infinitely many solutions.*

Example 1. Solve graphically the simultaneous equations

$$\begin{cases} 3x+2y=5, \\ 2x-y=8. \end{cases}$$

○ Let us construct the graph of the equation $3x + 2y = 5$. Having determined two solutions of this equation (1, 1) and (3, —2), we construct a straight line passing through the two points obtained (Fig. 60). Further, we construct the graph of the equation $2x - y = 8$. On having selected two solutions to this equation (0, —8) and (4, 0), we construct a straight line passing through the two points obtained (Fig. 60). The two straight lines thus constructed are non-parallel, they intersect at the point M (3, —2). Hence, (3, —2) is the solution for the given system. ●

The number of solutions satisfying system (1) can be determined without constructing the graphs of the equations forming the system; it suffices to find certain relationships between the coefficients of these equations established by the following theorems.

Theorem 3.2.3. *If* $a_1b_2 - a_2b_1 \neq 0$, *then system* (1) *has only one solution.*

□ Since $a_1b_2 \neq a_2b_1$, two cases are possible: (1) one of the coefficients b_1 or b_2 is zero, the other being nonzero, (2) both coefficients b_1 and b_2 are nonzero.

Consider the first case. Let, for instance, $b_1 = 0$ and $b_2 \neq 0$. Then $a_1 \neq 0$ (in the linear equation $a_1x + b_1y + c_1 = 0$ the coefficients of the variables cannot vanish simultaneously), and system (1) can be transformed to the form

$$\begin{cases} x=-c_1/a_1, \\ y=-(a_2/b_2)\,x-c_2/b_2. \end{cases} \tag{2}$$

The graph of the first equation of system (2) is a straight line parallel to the axis of ordinates; the graph of the second a straight line with a slope $-a_2/b_2$. These straight lines intersect at one point, therefore the system has only one solution.

In the second case, system (1) can be transformed as follows:

$$\begin{cases} y=-(a_1/b_1)\,x-c_1/b_1, \\ y=-(a_2/b_2)\,x-c_2/b_2. \end{cases} \tag{3}$$

From the condition $a_1b_2 \neq a_2b_1$ it follows that $-a_1/b_1 \neq -a_2/b_2$, that is, the slopes of the straight lines serving as graphs of the equations of system (3) are different. Consequently (see Item 4), the straight lines intersect at one point, and therefore system (3), and, hence, system (1) has only one solution. ■

Theorem 3.2.4. *If* $a_1b_2 - a_2b_1 = 0$, *and at least one of the numbers* $a_1c_2 - a_2c_1$ *or* $b_1c_2 - b_2c_1$ *is different from zero, then system* (1) *has no solution.*

□ From the condition $a_1b_2 = a_2b_1$ it follows that if $a_1 = 0$, then $a_2 = 0$ (since in this case $b_1 \neq 0$); similarly, if $b_1 \neq 0$, then $b_2 = 0$. Hence, three cases are possible: (1) $a_1 = a_2 = 0$, $b_1 \neq 0$, $b_2 \neq 0$, (2) $b_1 = b_2 = 0$, $a_1 \neq 0$, $a_2 \neq 0$, (3) all the coefficients a_1, a_2, b_1, and b_2 are nonzero.

In the first case system (1) can be transformed to the form

$$\begin{cases} y = -c_1/b_1, \\ y = -c_2/b_2. \end{cases} \tag{4}$$

This system has no solution if and only if $-c_1/b_1 \neq -c_2/b_2$, that is, $b_1c_2 - b_2c_1 \neq 0$.

In the second case system (1) can be transformed to the form

$$\begin{cases} x = -c_1/a_1, \\ x = -c_2/a_2. \end{cases} \tag{5}$$

This system has no solution if and only if $-c_1/a_1 \neq -c_2/a_2$, that is, $a_1c_2 - a_2c_1 \neq 0$.

Finally, in the third case system (1) can be transformed to the form (3). The slopes of the straight lines representing the graphs of the equations of system (3) are equal. Consequently, this system, and, hence, also system (1) has no solution if and only if $-c_1/b_1 \neq -c_2/b_2$, that is, $b_1c_2 - b_2c_1 \neq 0$.

In all the three cases, at least one of the numbers $a_1c_2 - a_2c_1$ or $b_1c_2 - b_2c_1$ is different from zero. ■

Theorem 3.2.5. *If* $a_1b_2 - a_2b_1 = 0$, $a_1c_2 - a_2c_1 = 0$, *and* $b_1c_2 - b_2c_1 = 0$, *then system* (1) *has infinitely many solutions.*

□ Here, the same three cases may occur as in proving Theorem 3.2.4. In the first case the system is transformed to form (4): it has infinitely many solutions if and only if $-c_1/b_1 = -c_2/b_2$, that is, $b_1c_2 - b_2c_1 = 0$. Since in this case $a_1 = a_2 = 0$, the equality $a_1c_2 - a_2c_1 = 0$ is also satisfied. The second case is similar to the first.

In the third case, system (1) is transformed to form (3). The slopes of the straight lines representing the graphs of the equations of system (3) are equal; hence the system has infinitely many solutions when the straight lines coincide, that is, when $-c_1/b_1 = -c_2/b_2$ or $b_1c_2 - b_2c_1 = 0$. It remains to prove that the equality $a_1c_2 - a_2c_1 = 0$ is also satisfied in this case.

From the condition $a_1b_2 = a_2b_1$ it follows that $a_2 = a_1b_2/b_1$. Substituting this expression for a_2 into the expression $a_1c_2 - a_2c_1$, we get $a_1c_2 - a_2c_1 = a_1c_2 - \frac{a_1b_2}{b_1}c_1 = \frac{a_1}{b_1}(b_1c_2 - b_2c_1) = 0$. Thus, $a_1c_2 - a_2c_1 = 0$. ■

The number $a_1b_2 - a_2b_1$ is called the *determinant of system* (1) and is denoted by Δ. Combining the results obtained in the last three theorems, we may draw the following conclusions:

(1) *if the determinant* Δ *of system* (1) *is nonzero, then the system has only one solution,*

(2) *if* $\Delta = 0$ *and at least one of the numbers* $a_1c_2 - a_2c_1$ *and* $b_1c_2 - b_2c_1$ *is nonzero, then system* (1) *has no solution,*

(3) *if* $\Delta = 0$, $a_1c_2 - a_2c_1 = 0$, *and* $b_1c_2 - b_2c_1 = 0$, *then the system has infinitely many solutions.*

Example 2. Without solving the given simultaneous equations, determine the number of its solutions:

$$(1)\ \begin{cases} 2x + 5y - 3 = 0, \\ 3x - y + 1 = 0, \end{cases} \quad (2)\ \begin{cases} 2x - 3y + 1 = 0, \\ 4x - 6y + 2 = 0, \end{cases}$$

$$(3)\ \begin{cases} 3x + y - 5 = 0, \\ -9x - 3y - 1 = 0. \end{cases}$$

○ (1) Here $a_1 = 2$, $a_2 = 3$, $b_1 = 5$, $b_2 = -1$, $\Delta = a_1b_2 - a_2b_1 = -17 \neq 0$. Hence, the system has one solution.

(2) Here $a_1 = 2$, $a_2 = 4$, $b_1 = -3$, $b_2 = -6$, $\Delta = 0$. Further, $c_1 = 1$, $c_2 = 2$, $a_1c_2 - a_2c_1 = 0$, $b_1c_2 - b_2c_1 = 0$. Therefore the system has infinitely many solutions.

(3) Here $a_1 = 3$, $a_2 = -9$, $b_1 = 1$, $b_2 = -3$, $\Delta = 0$. Further, $c_1 = -5$, $c_2 = -1$, $a_1c_2 - a_2c_1 \neq 0$. Consequently, the system has no solution. ●

In the course of algebra, the determinant Δ of system (1) is written in the form $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, that is, it is supposed that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

Similarly,

$$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1, \quad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = b_1c_2 - b_2c_1.$$

In general, the notation $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called the *determinant of order two*; this is how the number $ad - bc$ is denoted.

Let us call $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ and $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$ *auxiliary determinants* and denote them by Δ_y and Δ_x, respectively. Expressing x and y from system (1), we get (in the case $\Delta \neq 0$)

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} = \frac{\Delta_x}{\Delta}, \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\Delta_y}{\Delta}.$$

Then the above results may be formulated in the following way:

(1) *if* $\Delta \neq 0$, *then system* (1) *has only one solution* $x = \Delta_x/\Delta$, $y = \Delta_y/\Delta$ (*so-called* Cramer's formulas);

(2) *if* $\Delta = 0$, *but at least one of the auxiliary determinants* Δ_x *or* Δ_y *is non-zero, then system* (1) *has no solution*;

(3) *if* $\Delta = \Delta_x = \Delta_y = 0$, *then system* (1) *has infinitely many solutions.*

Exercises

1. Construct the graph of the given linear function: (a) $y = \frac{1}{2}x + 1$, (b) $y = -\frac{1}{3}x + 2$, (c) $y = -x - 2$, (d) $y = -2x - \frac{1}{5}$.

2. A school-boy had three copy-books. Then, during a week he bought one copy-book a day. How many copy-books (y) did he have in x days? Express y in terms of x. Is the dependence of y on x a linear function?

3. To send a telegram, one is to pay 10 kopecks plus 3 kopecks for each word in the telegram text. How much does it cost to send a telegram consisting of x words? Denote the cost of the telegram (in kopecks) by y. Write the formula for the cost of a telegram. Why is the dependence of y on x a linear function? What is the domain of definition? Construct the graph of this function.

4. Construct the graph of the function $y = 5x$ and show that to equal changes in x there correspond equal changes in y.

5. Construct the graph of the linear function $y = 3x - 5$. Using the graph, find out by how many units the value of y will increase if the value of x is increased: (a) from -2 to -1, (b) from 1 to 2, (c) from 3 to 4.

6. Without graphing the function, find out whether the point A belongs to the graph of the function $y = -3x + 5$ if: (a) A $(-15, 49)$, (b) A $(1/3, 4)$, (c) A $(1.8, -0.4)$.

7. Find the value of k and graph the function if it is known that the graph of the function $y = kx + 2$ passes through the point A $(-7, -12)$.

8. Find the value of b and graph the function if it is known that the graph of the function $y = -3x + b$ passes through the point A $(-2, -4)$.

9. A linear function is given by a formula on the closed interval $X = [1, 3]$. Find the set Y of values of the variable y if: (a) $y = 2x$, (b) $y = 2x - 1$, (c) $y = 0.5x$, (d) $y = 0.5x + 2$.

10. A linear function is given by a formula on a closed interval X. Its values form a closed interval Y. What is the formula by which the linear function is given if $X = [2, 5]$, $Y = [-5, -2]$?

11. For what values of k and b does the the graph of the function $y = kx + b$: (a) coincide with the graph of the function $y = -2x + 3$, (b) go parallel to the graph of the function $y = -2x + 3$, without coinciding with it, (c) intersect the graph of the function $y = -2x + 3$?

12. Set up the equations of the sides of a square whose diagonals are directed along the coordinate axes. The length of the side of the square is equal to a.

13. To the equation $x - y = 3$ join another equation such that the system of equations thus obtained has: (a) one solution, (b) no solution, (c) infinitely many solutions.

14. The graph of the function $y = kx + b$ passes through the points M_1 $(2, 3)$ and M_2 $(-5, -4)$. Find the values of the parameters k and b and construct the graph of the function.

15. Construct the graphs of the following equations: (a) $x + 3y = 7$, (b) $2x - y = 5$, (c) $\frac{x}{3} + \frac{y}{4} = 1$, (d) $|y| = 7$, (e) $(x - y)(x + y) = 0$.

16. Solve graphically the given simultaneous equations:

(a) $\begin{cases} 3x - 2y = 12, \\ x + 2y = -4; \end{cases}$ (b) $\begin{cases} y - 3x = 2, \\ 2y - 6x = 1; \end{cases}$ (c) $\begin{cases} x - 3y = 2, \\ 2x - 6y = 4. \end{cases}$

17. For what value of the parameter a do the simultaneous equations $\begin{cases} ax-3y=4, \\ x-y=4/3 \end{cases}$ have infinitely many solutions?

18. For what value of the parameter a do the simultaneous equations $\begin{cases} x-y=2, \\ 2x-2y=a \end{cases}$ have no solution?

19. Is it possible to indicate the value of the parameter a for which the system $\begin{cases} x+2y=a, \\ 2x+4y=5 \end{cases}$ has a solution?

20. Determine for what values of a the system

$$\begin{cases} (a-2)\,x+5y=5 \\ x+(a+2)\,y=5 \end{cases}$$

has: (a) only one solution; (b) infinitely many solutions; (c) no solution. Solve the system graphically for $a = 2$.

In Exercises 21 to 23, construct the graphs of the given functions:

21. $f(x)=\begin{cases} x & \text{for } x<0. \\ -5x & \text{for } x\geqslant 0. \end{cases}$ 22. $f(x)=\begin{cases} 2x-1 & \text{for } x<0, \\ x+1 & \text{for } x\geqslant 0. \end{cases}$

23. $f(x)=\begin{cases} 3x+2 & \text{for } x<1, \\ 5 & \text{for } x\geqslant 1. \end{cases}$

§ 3.3. Power Functions with Integral Exponent

1. Preliminaries. In the present section, we shall consider power functions with an arbitrary integral exponent, that is, power functions of the form $y = x^m$, where m is an integer. If $m = 1$, then we get $y = x$. This function has been already considered and its graph constructed.

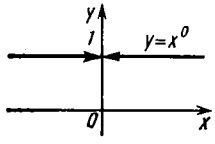


Fig. 61

If $m = 0$, then we get $y = x^0$ or, which is the same, $y = 1$, $x \neq 0$. This function is of no particular interest; its graph is shown in Fig. 61 (with a "puncture" at the point (1, 0)).

In what follows, we shall consider the other cases of a power function with an integral exponent.

2. Function $y = x^2$. Let us mention some properties of this function.

1°. *The domain of the function is the set* **R** *of real numbers.*

2°. *The function is even.*

3°. *The function increases on the interval* $[0, +\infty)$.

This follows from the fact that if $x_1 \geqslant 0$ and $x_2 \geqslant 0$, then, by virtue of the properties of numerical inequalities, $x_1 < x_2$ implies $x_1^2 < x_2^2$.

4°. *The function decreases on the interval* $(-\infty, 0]$.

This follows from the fact that the function $y = x^2$ is even and increasing on the interval $[0, +\infty)$ (see § 3.2, Item 7).

5^0. *The function is nonperiodic.*

This follows from its monotonicity on the rays $(-\infty, 0)$ and $(0, +\infty)$.

The graph of the function $y = x^2$ was constructed above (see Fig. 32); recall that it is called the parabola.

3. Function $y = x^3$. Note some properties of this function.

1^0. *The domain of the function is the set* **R** *of real numbers.*

2^0. *The function is odd.*

3^0. *The function increases throughout the set* **R**.

This assertion was proved above (see § 3.1, Item 8).

4^0. *The function is nonperiodic.*

To construct the graph of the function $y = x^3$, we first sketch a branch of the graph on the ray $[0, +\infty)$. To this end, we compile the following table:

x	0	1	2	1/2	3/2
y	0	1	8	1/8	27/8

We now plot the found points on the coordinate plane, and draw a smooth curve through them, thus obtaining a branch of the graph of the function $y = x^3$ on the ray $[0, +\infty)$ (Fig. 62).

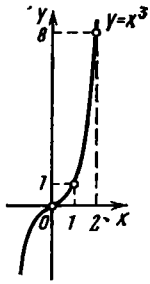


Fig. 62

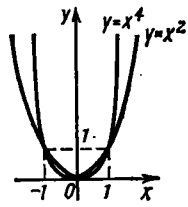


Fig. 63

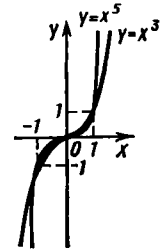


Fig. 64

Since the function $y = x^3$ is odd, we may sketch another branch symmetric to it with respect to the origin. As a result, we get the graph of the function $y = x^3$ which is called the *cubical parabola* (Fig. 62).

4. Function $y = x^n$, $n \in \mathbf{N}$. We have already considered the power function with natural exponents 1, 2, and 3. Let n be an arbitrary **even** natural number greater than 2: $n = 4, 6, 8, \ldots$. In this case, the function $y = x$ possesses the same properties as the function $y = x^2$. The graph of such a function resembles a parabola, but the greater n, the more the branches of the graph open upward (Fig. 63).

Let n be an arbitrary odd number greater than 3: $n = 5, 7, 9, \ldots$. In this case, the function $y = x^n$ possesses the same properties as the function $y = x^3$. The graph of such a function resembles a cubical parabola, and the greater n, the more the branches of the parabola open upward or downward (Fig. 64).

5. Function $y = 1/x$. Note some properties of this function.

1°. *The domain of the function is the set* **R** *of real numbers with point O removed.*

2°. *The function is odd.*

□ Indeed, firstly, the domain of definition of the function is a symmetric set. Secondly, we have $f(x) = 1/x$, $f(-x) = 1/(-x) = -1/x$, hence, $f(-x) = -f(x)$. ■

3°. *The function decreases on the interval* $(0, +\infty)$.

□ Let $x_1 > 0$, $x_2 > 0$, and $x_1 < x_2$, then $f(x_1) = 1/x_1$, $f(x_2) = 1/x_2$. We have

$$f(x_2) - f(x_1) = \frac{1}{x_2} - \frac{1}{x_1} = \frac{x_1 - x_2}{x_1 x_2} < 0.$$

Consequently, $f(x_2) < f(x_1)$. Thus, from $0 < x_1 < x_2$ it follows that $f(x_1) > f(x_2)$, and this means that the function $y = 1/x$ decreases on the interval $(0, +\infty)$. ■

4°. *The function decreases on the interval* $(-\infty, 0)$.

This follows from the fact that the function is odd and decreases on the interval $(0, +\infty)$.

5°. *The function is nonperiodic.*

To construct the graph of the function $y = 1/x$, let us first sketch its branch on the open ray $(0, +\infty)$. To this end, we compile the following table:

x	1	1/2	2	1/4	4
y	1	2	1/2	4	1/4

We now plot the found points on the coordinate plane and draw a smooth curve through them (Fig. 65), thus obtaining a branch of the graph of the function $y = 1/x$ on the interval $(0, +\infty)$. Since the function $y = 1/x$ is odd, we sketch the other branch which is symmetric to it with respect to the origin. As a result, we obtain the graph of the function $y = 1/x$ which is called the *hyperbola* (Fig. 65).

Note that the branches of the hyperbola indefinitely, or, as it is commonly said, *asymptotically* approach the coordinate axes, and the coordinate axes, that is, the straight lines $y = 0$ and $x = 0$, are called the *horizontal* and the *vertical asymptote* to the graph of the function $y = 1/x$, respectively.

6. Function $y = x^{-n}$, $n \in \mathbf{N}$. Let us first consider the case when n is odd.

If $n = 1$, then we get $y = x^{-1}$, that is, $y = 1/x$. This function was considered in the preceding subsection.

Let n be an odd number greater than 1: $n = 3, 5, 7, \ldots$. In this case, the function $y = x^{-n}$ possesses the same properties as the

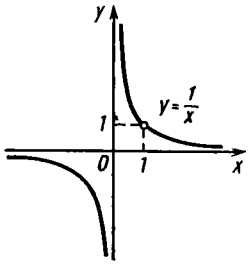


Fig. 65

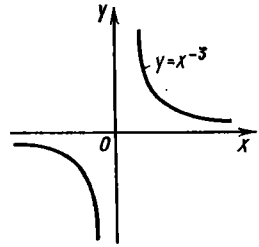


Fig. 66

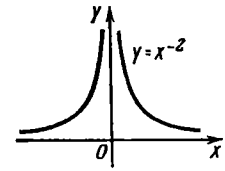


Fig. 67

function $y = 1/x$. The graph of the function $y = x^{-n}$ ($n = 3, 5, 7, \ldots$) resembles the graph of the function $y = 1/x$. Figure 66 represents the graph of the function $y = x^{-3}$.

Let now n be an even number, say, $n = 2$. Here, are some properties of the function $y = x^{-2}$, that is, the function $y = 1/x^2$.

1°. *The domain of the function is* $(-\infty, 0) \cup (0, +\infty)$.

2°. *The function is even.*

3°. *The function decreases on* $(0, +\infty)$ *and increases on* $(-\infty, 0)$.

4°. *The function is nonperiodic.*

The same properties are inherent in any functions of the form $y = x^{-2k}$, $k = 2, 3, 4, \ldots$.

The graph of the function $y = x^{-2}$ is shown in Fig. 67. The graph of the function $y = x^{-2k}$ has a similar shape if $k = 2, 3, 4, \ldots$.

Exercises

1. Construct the graphs of the following functions: (a) $y = x^5$, (b) $y = x^6$, (c) $y = x^{-4}$, (d) $y = x^{-3}$, (e) $y = 1/x$.

2. Find n if it is known that the graph of the function $y = x^n$ passes through the point: (a) A (2, 8), (b) B (1/3, 1/243).

3. Can the graph of the function $y = x^n$, where n is a natural number, pass through the point: (a) A (2, 5), (b) B ($\sqrt{3}$, 81)?

4. Construct the graphs of the functions $y = x^2$ and $y = 2x$ in the same coordinate system. Using the graphs, solve: (a) the equation $x^2 = 2x$, (b) the inequality $x^2 < 2x$.

§ 3.4. Inverse Functions

1. Notion of an Inverse Function. Let us compare two functions $y = f(x)$ and $y = g(x)$ whose graphs are represented in Fig. 68*a* and *b*, respectively. Both are defined on the closed interval $[a, b]$ and have the closed interval $[c, d]$ as their range. The function

$y = f(x)$ possesses the following property: whatever y_0 from the range of the function is taken, it is the value of the function **only at one point** x_0: $y_0 = f(x_0)$. The function $y = g(x)$ does not possess this property: thus, for the value y_0 chosen in Fig. 68*b* we have $y_0 = (x_1)$, $y_0 = f(x_2)$, and $y_0 = f(x_3)$. In other words, the range of the function $y = g(x)$ contains values which the function attains at

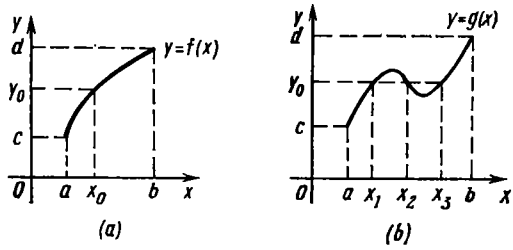


Fig. 68

more than one point of its domain. The function $y = f(x)$ is said to be reversible, while the function $y = g(x)$ irreversible.

Definition 1. A function $y = f(x)$ defined on an interval X is said to be *reversible* if it takes on any of its values only at one point of the interval X; or in other words, if different values of the function correspond to different values of the argument.

Theorem 3.4.1. *If a function $y = f(x)$ is monotonic on the interval X, then it is reversible.*

□ Let $y = f(x)$ be increasing on X. Then from $x_1 < x_2$ it follows that $f(x_1) < f(x_2)$. Thus, to the different values of the argument there correspond different values of the function, that is, the function is reversible. ■

A vivid illustration of this theorem is given in Fig. 68: the function $y = f(x)$ is monotonic and reversible, whereas the function $y = g(x)$ is nonmonotonic and irreversible.

Definition 2. Let a reversible function $y = f(x)$ be defined on an interval X and has an interval Y as its range. Let every y from Y associate with a unique value of x for which $f(x) = y$ (that is, the only root of the equation $f(x) = y$ with respect to the variable x). Then we get a function which is defined on Y and has X as its range. This function is denoted as $x = f^{-1}(y)$ and called the *inverse* of the function $y = f(x)$.

Theorem 3.4.1. implies that for any function $y = f(x)$, monotonic on X, there is an inverse function. To find it, one has to express x in terms of y from the equation $y = f(x)$.

Theorem 3.4.2. *If a function $y = f(x)$ increases (decreases) on the interval X, and Y is the set of values of the function, then the inverse function $x = f^{-1}(y)$ also increases (decreases) on Y.*

□ Let $y = f(x)$ be an increasing function, y_1 and y_2 two of its values, where $y_1 < y_2$. Since the function is reversible, each of these values is reached at one point: $y_1 = f(x_1)$ and $y_2 = f(x_2)$. The values x_1 and x_2 are related by the inequality $x_1 < x_2$. Indeed, if we suppose that $x_1 \geqslant x_2$, then from the increase of the function $y = f(x)$ it would follow that $f(x_1) \geqslant f(x_2)$, that is, $y_1 \geqslant y_2$ which contradicts the hypothesis. Hence, $y_1 < y_2$ implies that $x_1 < x_2$, that is, $f^{-1}(y_1) < f^{-1}(y_2)$, and this means that the inverse function $x = f^{-1}(y)$ increases on Y. ■

Example 1. Show that for the function $y = 5x - 3$ there is an inverse function and find its analytic expression.

○ The linear function $y = 5x - 3$ is defined on $\mathbf{R}$, increases on $\mathbf{R}$, and the set of its values (its range) is $\mathbf{R}$. Hence, the inverse function exists. In order to find its analytic expression, we solve the equation $y = 5x - 3$ with respect to x: we get $x = (y + 3)/5$. This is the desired inverse function. It is defined and increases on $\mathbf{R}$. ●

Example 2. Show that there is the inverse of the function $y = x^2$, $x \in (-\infty, 0]$ and express it analytically.

○ The function $y = x^2$ decreases on $(-\infty, 0]$, its range being represented by the ray $[0, +\infty)$ (see Fig. 32). Hence, the inverse function exists.

From the equation $y = x^2$ we find $x = \sqrt{y}$ or $x = -\sqrt{y}$. By the hypothesis, $x \in (-\infty, 0]$. The values of the function $x = -\sqrt{y}$ belong to this interval, while those of the function $x = \sqrt{y}$ do not. Thus, the desired inverse function has the form $x = -\sqrt{y}$, $y \in [0, +\infty)$. This function, the same as the original function $y = x^2$, $x \in (-\infty, 0]$, is decreasing. ●

As we saw, the monotonicity of a function is a sufficient condition for the inverse function to exist, but is not a necessary condition. Figure 69 represents the graph of a nonmonotonic but a reversible function.

2. Graph of an Inverse Function. The change from a function $y = f(x)$, $x \in X$ to an inverse function $x = f^{-1}(y)$, $y \in Y$ reduces only to a change of the roles played by the sets X and Y: in the first case X changes into Y, while in the second, Y changes into X. Therefore the graphs of the functions $y = f(x)$ and $x = f^{-1}(y)$ represent the same set of points of the coordinate plane xOy.

The argument of an inverse function is usually denoted by x, and the corresponding value of the function by y, that is, instead of $x = f^{-1}(y)$, we write $y = f^{-1}(x)$. Thus, if a pair of numbers (x, y) satisfies the equation $y = f(x)$ or the equivalent equation $x = f^{-1}(y)$, then a pair of numbers (y, x) satisfies the equation $y = f^{-1}(x)$.

Therefore the graph of the function $y = f^{-1}(x)$ is obtained from the graph of the function $y = f(x)$ by means of a transformation of

the xy-plane which carries a point (x, y) into a point (y, x). This transformation is a reflection in the straight line $y = x$ (the angle bisector of Quadrants I and III). Hence, *to obtain the graph of the function* $y = f^{-1}(x)$ *which is the inverse of the function* $y = f(x)$, *the graph of the function* $y = f(x)$ *must be reflected in the straight line* $y = x$ (Fig. 70).

Example. Given a function: $y = x^2$, $x \in [0, +\infty)$. Prove that there exists its inverse, write the analytic expression of the inverse function in the form $y = f^{-1}(x)$, and construct the graph of the inverse function.

○ The given function increases on the interval $[0, +\infty)$; hence, it has an inverse function. From the equation $y = x^2$ we find $x = \sqrt{y}$ or $x = -\sqrt{y}$. To the interval $[0, +\infty)$ belong only the values of the function $x = \sqrt{y}$. This is the inverse function which is defined on the interval $[0, +\infty)$.

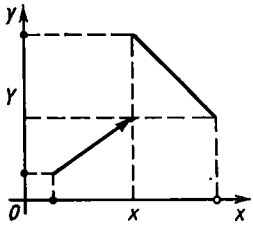


Fig. 69

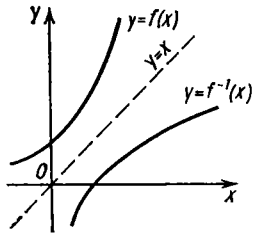


Fig. 70

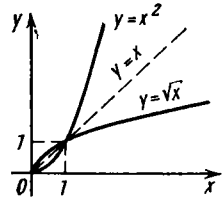


Fig. 71

Interchanging the position of x and y, we get $y = \sqrt{x}$, $x \in [0, +\infty)$. The graph of this function is obtained from the graph of the function $y = x^2$, $x \in [0, +\infty)$, by means of reflection in the straight line $y = x$ (Fig. 71). ●

Exercises

1. Prove that each of the given functions has its inverse: (a) $y = 0.5x - 3$, (b) $y = -3x + 1$, (c) $y = x^4$, where $x \geqslant 0$, (d) $y = x^3$.

2. A function is defined by the formula $y = x^2$ on the closed interval $[-3, 2]$. Prove that this function is irreversible. Single out of the interval $[-3, 2]$ the subintervals on which the function $y = x^2$ is reversible.

3. The graph of a function is represented by a semicircle with centre at the origin and of radius 3, which is situated in the upper coordinate half-plane. Is there the inverse of this function?

4. Find the inverse of each of the given functions and construct its graph in the same coordinate system: (a) $y = 2x - 3$, (b) $y = x^4$, where $x \leqslant 0$, (c) $y = x^3$.

In Exercises 5 to 11, find out which of the given functions are invertible

and find their inverses:

5. $y=2x+3$. 6. $y=\frac{3}{x}$. 7. $y=\frac{2}{x-1}$.

8. $y=5-\frac{1}{x}$. 9. $y=(x-2)^2-1$, $x\in(-\infty, 2]$.

10. $y=(x+3)^2$, $x\in[-5, 5]$. 11. $y=x^3-1$.

12. Find the inverse of the function

$$y=\begin{cases} 0.5x+1 & \text{for } x\geqslant 0, \\ 2x+1 & \text{for } x<0. \end{cases}$$

Construct the graphs for both functions.

13. Find the points of intersection of the graphs of the function $y=x^4$, $x\geqslant 0$, and its inverse.

14. Prove that the given functions are inverse: (a) $y=x^5$ and $y=\sqrt[5]{x}$, (b) $y=\frac{1}{x-2}$ and $y=\frac{2x+1}{x}$.

15. Is any linear function $y=ax+b$ reversible? Consider the cases $a\neq 0$ and $a=0$.

16. Using a formula, specify the inverse of the linear function $y=ax+b$ for $a\neq 0$.

17. A function is given by the formula $y=(-2)^n$, where $n\in\mathbf{N}$. Is this function monotonic? reversible?

18. Prove that the function $y=\frac{1-x}{1+x}$ is the inverse of itself, that is, that the inverse function is identical to the given one. What can be said about the graphs of these functions?

§ 3.5. Power Function with Any Rational Exponent

1. Function $y=\sqrt[n]{x}$. Consider the function $y=x^n$, $x\in[0, +\infty)$, where n is a natural number greater than unity. In § 3.3, we made sure that this function is increasing on the interval $[0, +\infty)$ and constructed the graph of this function (it is represented in Fig. 72*a*). The latter has the inverse which is denoted by $x=\sqrt[n]{y}$. Interchanging x and y, we get $y=\sqrt[n]{x}$. This function is defined and increases on the interval $[0, +\infty)$. Transforming the graph of the function $y=x^n$ symmetrically with respect to the straight line $y=x$, we get the graph of the function $y=\sqrt[n]{x}$ (Fig. 72*b*).

Note that if n is an odd number, then the function $y=x^n$ increases on the entire set of real numbers. Hence, in this case we may speak of the inverse of the function $y=x^n$, $x\in\mathbf{R}$. It is also denoted by $y=\sqrt[n]{x}$.

Figure 73 represents the graphs of the functions $y=x^3$ and $\sqrt[3]{x}$.

2. Function $y=x^r$, $r\in Q$. In § 3.3 we considered the power function with an integral exponent and in the preceding subsection the function $y=x^{1/n}$. We now consider the power function with any rational exponent. Three cases may happen here: $r>1$, $0<r<1$, $r<0$.

Case I: $r > 1$. Let $r = m/n$, where $m > n$. Note the properties of the function $y = x^{m/n}$.

1°. *The domain of the function is the interval* $[0, +\infty)$.

2°. *On the interval* $[0, +\infty)$ *the function increases.*

□ Indeed, let $0 < x_1 < x_2$. In the preceding subsection, we established that the function $y = \sqrt[n]{x}$ is increasing. Hence, from $x_1 < x_2$ it follows that $\sqrt[n]{x_1} < \sqrt[n]{x_2}$, that is, $x_1^{1/n} < x_2^{1/n}$.

Let us set $t_1 = x_1^{1/n}$, $t_2 = x_2^{1/n}$ and consider the function $y = t^m$. As it was noted above (see § 3.3, Item 4), it is increasing; hence from

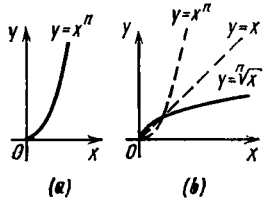


Fig. 72

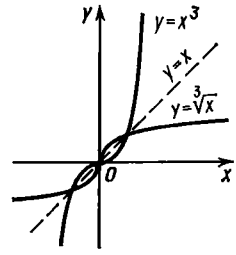


Fig. 73

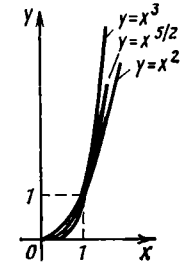


Fig. 74

$t_1 < t_2$ it follows that $t_1^m < t_2^m$. Thus, $(x_1^{1/n})^m < (x_2^{1/n})^m$, that is, $x_1^{m/n} < x_2^{m/n}$. To conclude, we see that $x_1 < x_2$ implies that $x_1^{m/n} < x_2^{m/n}$ and this means that the power function $y = x^{m/n}$ increases on the interval $[0, +\infty)$. ■

Figure 74 represents the graph of the function $y = x^{5/2}$ (it is enclosed between the graphs of the functions $y = x^2$ and $y = x^3$).

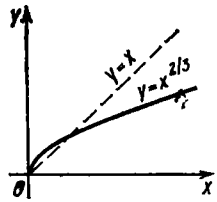


Fig. 75

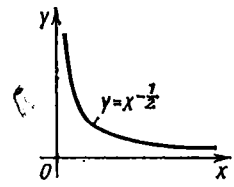


Fig. 76

The graph of any function of the form $y = x^r$, $x \in [0, +\infty)$, where $r > 1$ has a similar shape. The greater r, the steeper upward opens the branch of the graph.

Case II: $0 < r < 1$. As in the preceding case, the function $y = x^r$ is defined on the ray $[0, +\infty)$ and increases on it. Figure 75 shows the graph of the function $y = x^{2/3}$. The graph of any power function $y = x^r$, $x \in [0, +\infty)$, where $0 < r < 1$, has a similar shape.

Case III: $r < 0$. Let $r = -m/n$, where m and n are relatively prime natural numbers and $n \neq 1$. Note the properties of the function $y = x^{-m/n}$.

1^0. *The domain of the function is the interval* $(0, +\infty)$.

2^0. *The function decreases on the interval* $(0, +\infty)$.

□ Indeed, let $0 < x_1 < x_2$. Then, as it was established above, $x_1^{m/n} < x_2^{m/n}$. By virtue of the property of numerical inequalities, we have $1/x_1^{m/n} > 1/x_2^{m/n}$, that is, $x_1^r > x_2^r$. Thus, from $x_1 < x_2$ it follows that $x_1^r > x_2^r$, and this means that the function on the interval $(0, +\infty)$ decreases. ■

Let us construct the graph of the function $y = x^{-1/2}$. To this end, we compile the following table:

x	1	4	9	1/4	1/9
y	1	1/2	1/3	2	3

We then plot the obtained points on the coordinate plane and join them with a smooth curve (Fig. 76). The graph of any function $y = x^r$, $x \in [0, +\infty)$, where $r < 0$, has a similar shape. Note that the branches of the graph asymptotically approach the coordinate axes.

Exercises

1. Construct the graphs of the following functions: (a) $y = x^{5/3}$, (b) $y = x^{2/5}$, (c) $y = x^{-1/3}$, (d) $y = x^{-4/3}$.

2. Prove that the functions $y = x^{4/3}$ and $y = x^{3/4}$ are inverse. Graph both functions.

3. Given a function $y = x^{1/2}$: (a) find the domain of definition and the range of the function, (b) prove that the given function increases on its entire domain, (c) construct the graph of the function, (d) construct the graphs of the functions $y = -x^{1/2}$, $y = |x|^{1/2}$. What is the difference between the graph of each of these functions and the function $y = x^{1/2}$?

4. Find the domain of each of the given functions: (a) $y = \sqrt{x-5}$, (b) $y = \sqrt[3]{x-1}$, (c) $y = \sqrt[4]{2-x}$, (d) $y = \sqrt[10]{-x}$.

5. Construct the graph of the function $y = \sqrt[3]{x}$ and with its aid solve the given equation and inequalities: (a) $\sqrt[3]{x} = 2$, (b) $\sqrt[3]{x} > 2$, (c) $\sqrt[3]{x} \leq 2$.

6. Construct the graphs of the following functions: (a) $y = \sqrt[3]{x}$, (b) $y = \sqrt[3]{-x}$, (c) $y = -\sqrt[3]{x}$, (d) $y = \sqrt[3]{|x|}$.

§ 3.6. Transformations of Graphs

1. Compression and Extension. Consider in general form the problems on constructing graphs of functions by means of transformations of known graphs.

Problem I. Given the graph of a function $y = f(x)$. Construct the graph of the function $y = -f(x)$.

○ The graphs of the functions $y = f(x)$ and $y = -f(x)$ are symmetric with respect to the axis of abscissas. Hence, the graph of the function $y = -f(x)$ can be obtained from the graph of the function $y = f(x)$ by means of a reflection in the x-axis (Fig. 77). ●

Problem II. Given the graph of a function $y = f(x)$. Construct the graph of the function $y = mf(x)$.

○ Consider the case when $m > 0$. All the ordinates of the graph of the function $y = mf(x)$ are obtained by multiplying the corresponding ordinates of the graph of the function $y = f(x)$ by m. Such

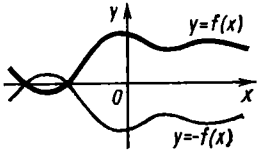


Fig. 77

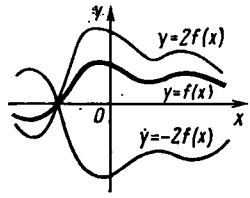


Fig. 78

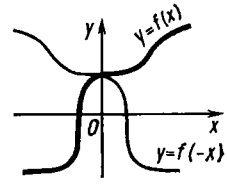


Fig. 79

a transformation of the graph of the function $y = f(x)$ is called the *extension* along the x-axis with coefficient m if $m > 1$ or *compression* if $0 < m < 1$.

Let now $m < 0$. Since in this case $mf(x) = -|m|f(x)$, the graph of the function $y = mf(x)$ can be obtained by extending (or compressing) the graph $y = f(x)$ along the x-axis with coefficient $|m|$ and successive reflection in the x-axis. ●

Figure 78 represents the graphs of the functions $y = f(x)$, $y = 2f(x)$ and $y = -2f(x)$.

Problem III. Given the graph of a function $y = f(x)$. Construct the graph of the function $y = f(-x)$.

○ It is easy to see that the ordinate of the graph of the function $y = f(-x)$ at the point x is equal to the ordinate of the graph of the function $y = f(x)$ at the point $-x$. Thus, $f(x) = f(1)$ for $x = 1$, and $f(-x) = f(1)$ for $x = -1$; $f(x) = f(2)$ for $x = 2$, and $f(-x) = f(2)$ for $x = -2$, etc. This means that the graph of the function $y = f(-x)$ can be obtained from the graph of the function $y = f(x)$ by means of a reflection in the y-axis. ●

The graphs of the functions $y = f(x)$ and $y = f(-x)$ are shown in Fig. 79.

Problem IV. Given the graph of a function $y = f(x)$. Construct the graph of the function $y = f(kx)$.

○ Consider the case when $k > 0$. It is clear that the ordinate of the graph of the function $y = f(kx)$ at the point x is equal to the ordinate of the graph of the function $y = f(x)$ at the point kx. Thus, $f(x) = f(k)$ for $x = k$, and $f(kx) = f(k)$ for $x = 1$, $f(x) = f(2k)$ for $x = 2k$; and $f(kx) = f(2k)$ for $x = 2$, etc. This means that the

graph of the function $y = f(kx)$ is obtained from the graph of the function $y = f(x)$ by compressing with coefficient k along the y-axis if $k > 1$, and by extending if $0 < k < 1$.

Let now $k < 0$. Then $f(kx) = f(-|k|x)$. Therefore the graph of the function $y = f(kx)$ can be obtained by compressing (or extending) the graph of the function $y = f(x)$ with coefficient $|k|$ along the y-axis followed by the reflection of the obtained graph of the function $y = f(|k|x)$ in the y-axis. ●

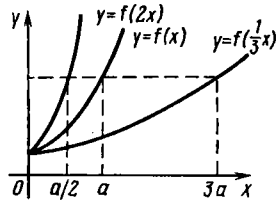


Fig. 80

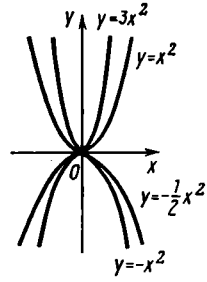


Fig. 81

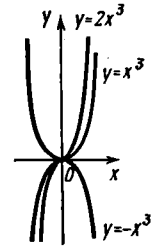


Fig. 82

The graphs of the functions $y = f(x)$, $y = f(2x)$, and $y = f\left(\frac{1}{3}x\right)$ are shown in Fig. 80.

2. Graphs of the Functions $y = ax^2$, $y = ax^3$, $y = k/x$. Consider the function $y = ax^2$. As is known, the graph of the function $y = x^2$ is a parabola. According to the result of Problem II, in order to construct the graph of the function $y = ax^2$, one has to extend the parabola $y = x^2$ along the x-axis with coefficient $|a|$. In this case, if $a < 0$, then the graph of the function $y = |a|x^2$ must be reflected in the x-axis.

Figure 81 represents the graphs of the functions of the form $y = ax^2$ for $a = 1, -1, 3, -1/2$. All these graphs are also *parabolas*. For $a > 0$ the branches of the parabola $y = ax^2$ are open upward, and for $a < 0$ downward.

The graphs of functions of the form $y = ax^3$ can be constructed in a similar way. Figure 82 shows these graphs for the cases $a = 1, -1, 2$. All of them are *cubical parabolas*.

Finally, consider the function $y = k/x$. This function is known as the *inverse variation* with coefficient k. As is known, the graph of the function $y = 1/x$ is a hyperbola. The graph of the inverse variation is also a *hyperbola*. Figure 83 represents the graphs of the functions $y = 1/x$ and $y = 2/x$ (where, $k > 0$), while Fig. 84 the graphs of the functions $y = -1/x$ and $y = -2/x$ (where $k < 0$).

3. Translation. Here we shall consider three general problems on constructing the graphs of functions by transforming the known graphs.

Problem V. Given the graph of a function $y = f(x)$. Construct the graph of the function $y = f(x) + b$.

○ In § 3.2, Item 4, we established that the graph of the function $y = kx + b$ is obtained from the graph of the function $y = kx$

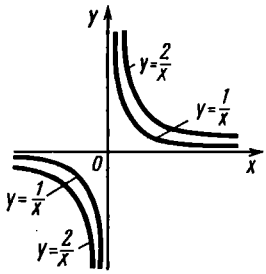


Fig. 83

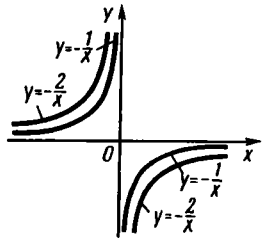


Fig. 84

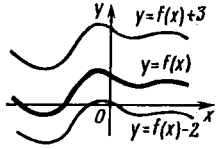


Fig. 85

by translating the latter along the vector $\mathbf{p}(0, b)$. The graph of the function $y = f(x) + b$ is obtained from the graph of the function $y = f(x)$ in exactly the same manner. ●

Figure 85 represents the graphs of the functions $y = f(x)$, $y = f(x) + 3$, and $y = f(x) - 2$.

Problem VI. Given the graph of a function $y = f(x)$. Construct the graph of the function $y = f(x + a)$.

○ It is easy to see that the ordinate of the graph of the function $y = f(x)$ at the point x is equal to the ordinate of the graph of the function $y = f(x + a)$ at the point $x - a$. Thus, $f(x) = f(1)$ for $x = 1$, and $f(x + a) = f(1)$ for $x = 1 - a$; $f(x) = f(2)$ for $x = 2$, and $f(x + a) = f(2)$ for $x = 2 - a$, etc. Consequently, if (x, y) is a point belonging to the graph of the function $y = f(x)$, then $(x - a, y)$ is a point belonging to the graph of the function $y = f(x + a)$. But the point $(x - a, y)$ is obtained from the given point (x, y) by translating it over $|a|$. The direction of the translation is parallel to the x-axis; if $a < 0$, then it coincides with the positive direction of the x-axis, and if $a > 0$, with the negative direction of the x-axis. This translation is specified by the vector $\mathbf{p}(-a, 0)$. ●

Figure 86 represents the graphs of the functions $y = f(x)$, $y = f(x - 2)$, and $y = f(x + 3)$.

Problem VII. Given the graph of a function $y = f(x)$. Construct the graph of the function $y = f(x + a) + b$.

○ Using the results of Problems V and VI, we conclude that, in order to construct the graph of the function $y = f(x + a) + b$, we

have to translate the graph of the function $y = f(x)$ along the vector $\mathbf{p}(-a, b)$.

Note that if the coordinate axes are translated with the new origin placed at the point $O'(-a, b)$, then the graph of the function $y =$

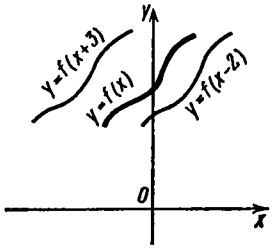


Fig. 86

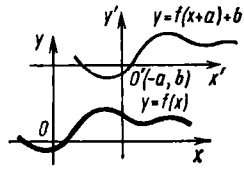


Fig. 87

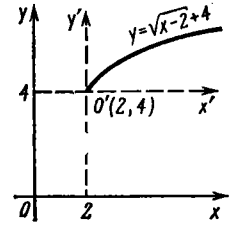


Fig. 88

$f(x + a) + b$ will occupy the same position in the new coordinate system $x'O'y'$ as the graph of the function $y = f(x)$ in the old coordinate system xOy (Fig. 87).

Thus, in order to construct the graph of the function $y = f(x + a) + b$, we have to translate the coordinate axes by placing the new origin at the point $O'(-a, b)$ and to construct the graph of the function $y' = f(x')$ in the new coordinate system $x'O'y'$. ●

Example 1. Construct the graph of the function $y = \sqrt{x - 2} + 4$.

○ Let us translate the coordinate axes by placing the origin of the new system at the point $O'(2, 4)$ and construct the graph of the function $y' = \sqrt{x'}$ in the system $x'O'y'$ (Fig. 88) which is just the desired graph. ●

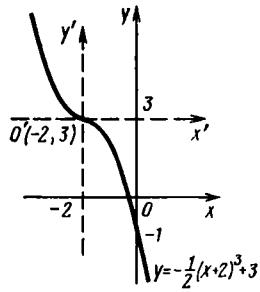


Fig. 89

Example 2. Construct the graph of the function $y = \frac{1}{2}(x + 2)^3 + 3$.

○ Let us translate the coordinate axes by placing the new origin at the point $O'(-2, 3)$. We then construct the graph of the function $y' = -\frac{1}{2}x'^3$ in the system $x'O'y'$. This is the desired graph (Fig. 89).

4. Graph of a Quadratic Function. An expression of the form $ax^2 + bx + c$ (where $a \neq 0$) is called the *quadratic trinomial*, and the function $y = ax^2 + bx + c$ the *quadratic function*. Prior to constructing the graph of the quadratic function, we perform the following trans-

formations of the quadratic trinomial:

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x\right)+c=a\left(x^2+2\frac{b}{2a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}\right)+c$$
$$=a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}\right)+c$$
$$=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c$$
$$=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}.$$

Using the result of Problem VII, we conclude that to graph the function

$$y=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$$

we have to translate the coordinate system by placing the origin of the new system at the point $O'\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$ and to construct the parabola $y'=ax'^2$ in the new coordinate system.

Hence, the graph of the quadratic function $y=ax^2+bx+c$ is a parabola. The latter can be obtained from the graph of the function $y=ax^2$ by translating the coordinate system along the vector $\mathbf{p}=\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$. The axis of symmetry of this parabola is perpendicular to the x-axis and passes through the point (x_0, y_0), where $x_0=-\frac{b}{2a}$, $y_0=\frac{4ac-b^2}{4a}$. The vertex of the parabola lies at the same point, and the parabola opens upward for $a>0$ and downward for $a<0$.

Example 1. Graph the function $y=x^2-x+1$.

○ Let us carry out the following transformations:

$$y=x^2-x+1=x^2-x+\frac{1}{4}-\frac{1}{4}+1=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}.$$

Consequently, the graph of the function $y=x^2-x+1$ is obtained from the graph of the function $y=x^2$ by translating along the vector $\mathbf{p}$ (1/2, 3/4) (Fig. 90). ●

We proved above that the graph of the function $y=ax^2+bx+c$ is a parabola which is obtained from the parabola $y=ax^2$ by means of a translation. This makes it possible to construct the graph of the function $y=ax^2+bx+c$ without applying the method of completing the square for realizing a translation. Let us explain the method of constructing the graph considering the following examples.

Example 2. Construct the graph of the function $y=x^2-2x-3$.

○ Let us find the point of intersection of the desired graph and the y-axis. Since the abscissa of this point is equal to zero, its ordi-

nate is equal to -3. Consequently, A $(0, -3)$ is the point of intersection of the required parabola and the y-axis.

Let us find the point at which the graph intersects the x-axis. For this purpose, we solve the equation $x^2 - 2x - 3 = 0$ (for solving quadratic equations see § 4.1):

$$x_{1,2} = 1 \pm \sqrt{1+3} = 1 \pm 2, \quad x_1 = -1, \quad x_2 = 3.$$

Consequently, B $(-1, 0)$ and C $(3, 0)$ are the points of intersection of the graph with the x-axis. We then plot all the obtained points on the coordinate plane (Fig. 91). The points B and C lie on the required

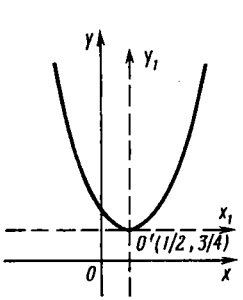


Fig. 90

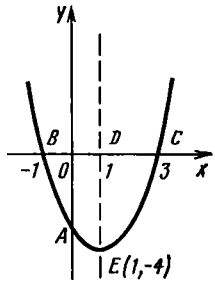


Fig. 91

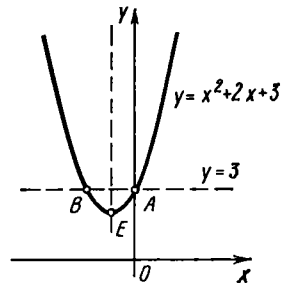


Fig. 92

parabola and on the straight line perpendicular to the axis of symmetry of the parabola; consequently, they are symmetric with respect to the parabola axis. Hence, the axis of symmetry of the parabola intersects the x-axis at the point D equidistant from the points B and C, that is, its abscissa $(-1 + 3)/2 = 1$.

We now find the coordinates of the vertex of the parabola. Since the vertex of the parabola lies on the axis of symmetry, its abscissa $x = 1$. Hence, $y = 1 - 2 - 3 = -4$. Thus, the coordinates of the vertex of the parabola are: E $(1, -4)$. By drawing a smooth curve through the points A, B, E, and C, we get the desired graph of the given function (Fig. 91). ●

Example 3. Construct the graph of the function $y = x^2 + 2x + 3$.

○ We first find the point of intersection with the y-axis. Setting $x = 0$, we get $y = 3$, that is, the desired graph passes through the point A $(0, 3)$. Since the equation $x^2 + 2x + 3 = 0$ has no root, the graph of the given function does not intersect the x-axis. We then find the points of intersection of the graph and a certain straight line parallel to the x-axis. It is suitable to take the straight line $y = 3$ $(y = c)$ for such a line. For this purpose, it is necessary to solve the equation $3 = x^2 + 2x + 3$. Hence, $x^2 + 2x = 0$, and $x_1 = 0$, $x_2 = -2$. Consequently, A $(0, 3)$ and B $(-2, 3)$ are the points of intersection of the straight line $y = 3$ and the parabola.

Reasoning as in the preceding example, we find the coordinates of the vertex E of the parabola: $x = (0 - 2)/2 = -1$ and $y = 1-2 + 3 = 2$. Joining the points A, B, and E $(-1, 2)$ with a smooth curve, we get the required graph (Fig. 92). ●

5. Graph of a Linear Fractional Function. The *linear fractional function* is defined as a function of the form $y = \frac{ax+b}{cx+d}$. Note that if $b/a = d/c$, then

$$y = \frac{ax+b}{cx+d} = \frac{a(x+b/a)}{c(x+d/c)} = \frac{a}{c},$$

that is, the linear fractional function turns into a constant function defined everywhere except for the point $x = -d/c$.

If $c = 0$, then we have a function of the form $y = \frac{ax+b}{d}$, that is, a linear function.

Consider a linear fractional function of general form $y = \frac{ax+b}{cx+d}$, where $a \neq 0$, $c \neq 0$, and $b/a \neq d/c$. We divide $ax+b$ by $cx+d$:

$$\begin{array}{r|l} & \frac{a}{c} \\ \hline cx+d & ax+b \\ & ax+\frac{ad}{c} \\ \hline & b-\frac{ad}{c} \end{array}.$$

Thus,

$$\frac{ax+b}{cx+d} = \frac{a}{c} + \frac{b-ad/c}{cx+d} = \frac{a}{c} + \frac{(bc-ad)/c^2}{x+d/c}.$$

Let us set $k = (bc - ad)/c^2$; from the condition $b/a \neq d/c$ it follows that $k \neq 0$. Then the function takes the form

$$y = \frac{a}{c} + \frac{k}{x+d/c}.$$

If we set $f(x) = k/x$, then

$$f\left(x+\frac{d}{c}\right) + \frac{a}{c} = \frac{k}{x+d/c} + \frac{a}{c}.$$

Thus, we have to construct the graph of the function $y = f(x + d/c) + a/c$. As is known (see Problem VII), to this end, we have to change to a new coordinate system, by placing its origin at the point O' $(-d/c, a/c)$, and to construct the graph of the function $y' = f(x') = k/x'$ in the new coordinate system. We know that the graph of the function $y' = k/x'$ is a hyperbola (see p. 147); hence, the graph

of the function $y = \frac{ax+b}{cx+d}$ is also a hyperbola with asymptotes $x = -d/c$ and $y = a/c$ (Fig. 93).

Note that the number $-d/c$ is a root of the denominator of the expression $(ax + b)/(cx + d)$, and the number a/c is equal to the ratio of the senior coefficients of the numerator and the denominator.

Example 1. Construct the graph of the function $y = \frac{x-2}{x+2}$.

○ Here, the function is not defined at the point $x = -2$ (the root of the denominator), and the ratio of the senior coefficients is

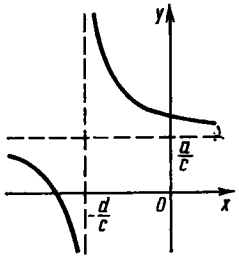


Fig. 93

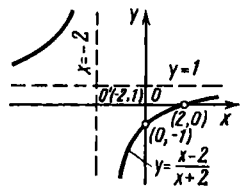


Fig. 94

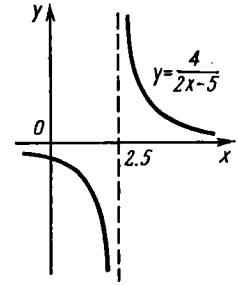


Fig. 95

equal to 1. Hence, $x = -2$ is a vertical asymptote, and $y = 1$ is a horizontal asymptote of the sought-for hyperbola (dashed lines in Fig. 94). Further, if $x = 2$, then $y = 0$, and if $x = 0$, then $y = -1$.

With the aid of the check points (2, 0) and (0, −1), we construct one branch of the hyperbola. The second branch is symmetric to the first one with respect to the point O' (−2, 1). The graph of the given function is represented in Fig. 94. ●

Example 2. Construct the graph of the function $y = \frac{4}{2x-5}$.

○ We have

$$\frac{4}{2x-5} = \frac{4}{2(x-2.5)} = \frac{2}{x-2.5}.$$

According to the result of Problem VI, we have to translate the graph $y = 2/x$ along the vector $\mathbf{p}$ (2.5, 0). The graph of the given function is represented in Fig. 95. ●

6. Graph of the Function $y = |f(x)|$. Consider the following problem.

Problem VIII. Given the graph of a function $y = f(x)$. Construct the graph of the function $y = |f(x)|$.

○ As is known, $|f(x)| = f(x)$ if $f(x) \geqslant 0$, and $|f(x)| = -f(x)$ if $f(x) < 0$. Therefore the graph of the function $y = |f(x)|$ coincides with the graph of the function $y = f(x)$ on the subintervals where $f(x) \geqslant 0$. Wherever $f(x) < 0$, the graph of the function

$y = |f(x)|$ can be obtained by the reflection of the graph of the function $y = f(x)$ in the x-axis. ●

The graphs of the functions $y = f(x)$ and $y = |f(x)|$ are represented in Fig. 96.

Example. Construct the graph of the function $y = |4x - x^2|$.

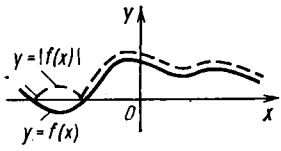


Fig. 96

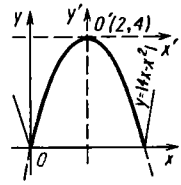


Fig. 97

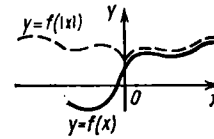


Fig. 98

○ We have $|4x - x^2| = |-(x-2)^2 + 4|$. We construct the required graph in several steps:

(1) translate the coordinate system, placing the origin of the new coordinate system at the point $O'(2, 4)$,

(2) construct the parabola $y' = -x'^2$ in the new coordinate system; thus, the graph of the function $y = 4x - x^2$ has been completed,

(3) we now construct the graph of the function $y = |4x - x^2|$ (see Problem VIII). The successive steps of constructing this graph are shown in Fig. 97. ●

7. Graph of the Function $y = f(|x|)$. Consider the following problem.

Problem IX. Given the graph of a function $y = f(x)$. Construct the graph of the function $y = f(|x|)$.

○ If $x \geqslant 0$, then $|x| = x$, and, hence $f(|x|) = f(x)$. Therefore, in the right-hand half-plane, the graph of the function $y = f(|x|)$ coincides with the graph of the function $y = f(x)$.

Since, further, $|-x| = |x|$, then $y = f(|x|)$ is an even function, and its graph is symmetric with respect to the y-axis. Therefore, adjoining to the branch of the graph of the function $y = f(x)$, $x \geqslant 0$ another branch symmetric to it with respect to the y-axis, we get the graph of the function $y = f(|x|)$. ●

The graphs of the functions $y = f(x)$ and $y = f(|x|)$ are shown in Fig. 98.

Example 1. Construct the graph of the function $y = |x| - 5$.

○ (1) We construct the graph of the function $y = x - 5$ (this is a straight line).

(2) We retain the branch of this graph corresponding to the values $x \geqslant 0$ (a ray) and replace the branch corresponding to the values $x < 0$ by a branch symmetric to the remained one with respect to the

y-axis. The two constructed branches form the graph of the function $y = |x| - 5$ (Fig. 99). ●

Example 2. Construct the graph of the function $y = \sqrt{|x-2|} + 3$.

○ We translate the coordinate axes, placing the origin of the new coordinate system at the point O' (2, 3). Using, further, the result

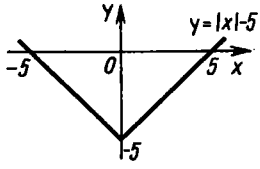


Fig. 99

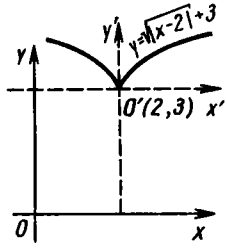


Fig. 100

of Problem IX, we construct the graph of the function $y' = \sqrt{|x'|}$ in the system $x'O'y'$. This is the required graph (Fig. 100). ●

Exercises

1. Graph the following functions: (a) $y = 2x^2$, (b) $y = -3x^2$, (c) $y = 0.2x^2$.
2. Find the value of a for which the graph of the function $y = ax^3$ passes through the point K (0.1, 25).
3. Graph the following functions: (a) $y = x^3$, (b) $y = 0.5x^3$, (c) $y = 2x^3$, (d) $y = -x^3$.
4. Construct the graph of the function $y = x^3/27$, where $x \in [-3, 6]$. From the graph obtained, find the sets of values of the variable x for which: (a) $y = 0$, (b) $y > 0$, (c) $y < 0$, (d) $y = 1$, (e) $y > 1$, (f) $y < 1$.
5. The area of a right triangle is 7.5 cm^2. One of its legs equals x cm, the other y cm. Express y in terms of x. Construct the graph of y versus x.
6. It is known that the point P (−9, 18) belongs to the graph of the inverse variation. Find the inverse variation factor.
7. Graph the following functions: (a) $y = -5/x$, (b) $y = 2/x$, (c) $y = 2/(3x)$.
8. In the same coordinate system, construct the graphs of the functions $y = 0.5x^2$ and $y = 4/x$. For what values of x: (a) $0.5x^2 = 4/x$, (b) $0.5x^2 < 4/x$, (c) $0.5x^2 > 4/x$, (d) $4/x < 1$?
9. Graph the following functions: (a) $y = x^2 - 9$, (b) $y = -x^2 - 5$, (c) $y = -(x + 2)^2$, (d) $y = (x + 3)^2 - 4$.
10. Graph the function $y = x^2 + 2x - 8$. From the constructed graph find: (a) the points of intersection of the graph with the x- and y-axes, (b) the intervals on which the function takes on negative (positive) values, (c) the intervals on which the function is increasing (decreasing), (d) the greatest or the least value of the function, (e) the range of the function.
11. Construct the graphs of the given functions and do the same as in the preceding exercise: (a) $y = x^2 + 4x$, (b) $y = x^2 + 2x - 3$, (c) $y = -x^2 + 6x - 10$, (d) $y = -x^2 + 5x$, (e) $y = (x - 1)(x + 5)$.
12. Prove that the graph of the function $y = x^2 + 6x + 9$ is situated in the upper coordinate half-plane. Construct the graph.

13. Construct the graph of the function $y = (x - 2)(x + 4)$ and, using the obtained graph, solve the following inequalities: (a) $(x - 2)(x + 4) > 0$; (b) $(x - 2)(x + 4) < 0$.

14. Find the coordinates of the vertex of each of the given parabolas: (a) $y = x^2 + 4x - 5$, (b) $y = 5x - x^2$, (c) $y = (x - 2)(x + 3)$, (d) $y = 2(x - 1)(x + 2)$.

15. Prove with the aid of the graph that for all values of x the following inequalities are satisfied: (a) $x^2 - 3x + 8 > 0$, (b) $-x^2 + 6x - 10 < 0$.

16. Find out whether or not the graph of the function $y = x^2 - 7x - 31$ passes through the points: (a) A (3,—43), (b) B (—8, 89), (c) C (—5, —29).

17. (a) For what value of c does the graph of the function $y = 2x^2 + 7x + c$ pass through the point A (—10, 150)?

(b) For what value of b does the graph of the function $y = x^2 + bx - 19$ pass through the point D (—11, —30)?

18. It is known that the graph of the function $y = ax^2 + bx - 48$ passes through the points A (1, 2) and B (2, 10). Find the values of the coefficients a and b. Construct the graph of the function.

In Exercises 19 to 27, construct the graph for each of the given functions:

19. (a) $y = -\frac{3}{x+1}$, (b) $y = \frac{1}{x} + 2$, (c) $y = \frac{1}{x+5} - 1$,
(d) $y = \frac{1}{x-2} + 3$, (e) $y = \frac{x+3}{x+2}$, (f) $y = \frac{2x-1}{x-2}$.

20. (a) $y = (x+1)^3$, (b) $y = -3(x-2)^3$, (c) $y = x^3 - 5$, (d) $y = (x-1)^3 = 8$.

21. (a) $y = \frac{4}{x^2} + 2$, (b) $y = \frac{1}{(x-5)^2}$, (c) $y = \frac{1}{(x+1)^2} - 4$.

22. (a) $y = \frac{1}{(x+7)^3}$, (b) $y = -\frac{1}{x^3} + 3$, (c) $y = \frac{1}{(x-4)^3} + 5$.

23. (a) $y = (x-2)^4$, (b) $y = x^4 + 3$, (c) $y = -(x-6)^4 + 1$.

24. (a) $y = -\frac{1}{x^4} - 2$, (b) $y = \frac{1}{(x-1)^4}$, (c) $y = 4 - \frac{1}{(x+3)^4}$.

25. (a) $y = |x^2 - x - 12|$, (b) $y = |x^2 + 2x + 3|$, (c) $y = x^2 - |x| - 12$,
(d) $y = |x - 2|(x + 3)$.

26. (a) $y = \left|\frac{3x-1}{x+2}\right|$, (b) $y = \frac{|x-2|}{x-1}$, (c) $y = \frac{|x|-2}{|x|-1}$.

27. (a) $y = \sqrt{x+2} - 3$, (b) $y = \frac{1}{(x-1)^3} - 2$,
(c) $y = |x - 2|^{1/3}$, (d) $y = |\sqrt[4]{(x-4)^5} - 5|$.

28. Graph the following equations:

(a) $(|x| - y)\left(y + \frac{1}{x}\right) = 0$, (b) $(y - |x^2 - 4|)\left(y + \frac{1}{|x-2|}\right) = 0$.

§ 3.7. Trigonometric Functions

1. Number Circle. In § 1.3, Item 2, we spoke of one-to-one correspondence between the set of real numbers and the set of points of the number line. Here, we shall establish the correspondence between the set of real numbers and the set of points of the circle of unit radius.

We take a circle of unit radius (Fig. 101) with point A (the right-hand end point of the horizontal diameter) as a reference point. We assume an anticlockwise motion round the circle to be a positive direction, and a clockwise motion to be a negative direction.

To every point M of the circle there correspond infinitely many arcs beginning at the point A and terminating at the point M. One of them is the shortest arc connecting these two points, all the other arcs being obtained from the shortest arc by adding or subtracting an integral number of complete revolutions.

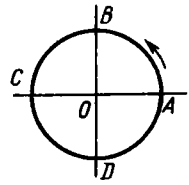


Fig. 101

Thereby every point M of the circle is associated with an infinite set of numbers which consists of the values of all the arcs beginning at the point A and terminating at the point M (the lengths of the arcs being taken with the plus or minus sign according as the motion from the point A to the point M is anticlockwise or clockwise). Since the circumference of the circle of unit radius is equal to 2π, the lengths of all the arcs terminating at the given point M differ from one another by a whole number multiple of 2π. In other words, the general form of these quantities is $x + 2\pi k$, where k is any whole number, that is, $k \in \mathbf{Z}$, while x is the length of the shortest arc joining A and M.

A circle of unit radius with a reference point and the direction of tracing specified is called the *number circle*. To every real number x there corresponds a point $M(x)$ of the number circle such that the length of the arc AM is x, and every point M of the circle corresponds to an infinite set of numbers of the form $x + 2\pi k$, $k \in \mathbf{Z}$, where x is the length of one of the arcs connecting the points A and M.

The number circle is divided by its vertical and horizontal diameters into four equal parts—the arcs AB, BC, CD, and DA in Fig. 101. These arcs are called *Quadrants* I, II, III, and IV, respectively.

Example 1. Find on the number circle the points corresponding to the numbers 0, $\pi/2$, π, $3\pi/2$, 2π, $-\pi/2$.

○ The point of reference (point A), that is, $A = A\ (0)$ corresponds to the number 0. Further, since the length of the circumference is 2π, then $\pi/2$ is the length of an arc equal to one-fourth of the circumference. Hence, if we lay off an arc of length equal to a quarter of the circumference from the point A in the positive direction, then we get the required point B. Thus, $B = B\ (\pi/2)$.

Similarly, we get that $C = C\ (\pi)$, $D = D\ (3\pi/2)$, $A = A\ (2\pi)$.

To find the point corresponding to the number $-\pi/2$, we have to start from the point A in the negative direction and cover the path of length $\pi/2$. We then arrive at the point D. Thus, $D = D\ (-\pi/2)$. ●

Example 2. Find on the number circle, the points correspoding to the numbers $k\pi/4$, where $k = 1, 3, 5, 7$.

○ To find the desired points, we bisect each of the quadrants of the number circle. The points of interest are marked in Fig. 102 by K, L, M, and N. ●

Example 3. Find the points on the number circle that correspond to the numbers $k\pi/6$, where $k = 1, 2, 4, 5, 6, 8, 10, 11$.

○ To find the desired points, we divide each of the quadrants of the number circle into three equal parts. Then the arc length of one such part $(1/3)\cdot(\pi/2) = \pi/6$. The points of interest are marked in Fig. 103 by E, F, P, Q, R, S, T, H. ●

Example 4. Find the points on the number circle that correspond to the numbers 1, 2, 4, and -6.

○ We have $\overset{\frown}{AB} = \pi/2 \approx 3.14/2 \approx 1.57$. Hence, to the number 1 there corresponds a point of the first quadrant lying somewhat high-

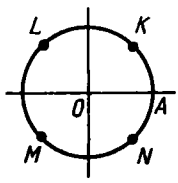


Fig. 102

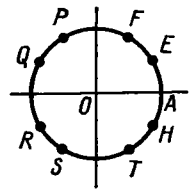


Fig. 103

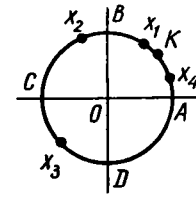


Fig. 104

er than the midpoint K of the arc AB (the point x_1 in Fig. 104). The number 2 is greater than the number $\pi/2$ by approximately 0.43 which is about a quarter of the length of the arc BC. Thus, we get the point x_2 (Fig. 104) corresponding to the number 2. To the number 4, there corresponds the point x_3 lying in Quadrant III.

It remains to find the point corresponding to the number -6. The number 6 differs from the length of the circumference 2π by approximately 0.28. Starting from the point A, we cover the path of length 6 in the negative direction round the circle, that is, the path by 0.28 less than a complete revolution. The end point of this path is denoted by x_4 (Fig. 104): this is the point corresponding to the number -6. ●

2. Trigonometric Functions Defined. Let us take a number circle and arrange it so that the centre coincides with the origin O of the rectangular coordinate system XOY (Fig. 105). Let there be given some number x. We find the point $M(x)$ on the number circle that corresponds to this number. The ordinate of the point $M(x)$ is called the *sine* of the number x and is denoted by $\sin x$, and the abscissa of this point is called the *cosine* of the number x and is denoted by $\cos x$ (Fig. 105). The ratio $\dfrac{\sin x}{\cos x}$ is called the *tangent* of the number x

and is denoted by tan x; the ratio $\frac{\cos x}{\sin x}$ is called the *cotangent* of the number x and is denoted by cot x. We have associated every number x with the values of sin x, cos x, tan x, and cot x. Thus, we have defined the four numerical functions $y = \sin x$, $y = \cos x$, $y = \tan x$, and $y = \cot x$ which are called *trigonometric functions.*

The functions $y = \sin x$ and $y = \cos x$ are defined for any values of x. The function $y = \tan x$ is defined for values of x such that $\cos x \neq 0$. The function $y = \cot x$ is defined for the values of x for which $\sin x \neq 0$.

Let us compute the values of trigonometric functions for some values of the argument. Recall that the point A (the reference point on

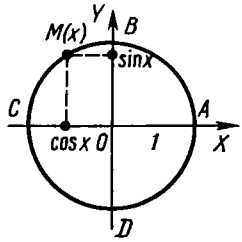


Fig. 105

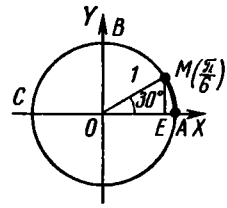


Fig. 106

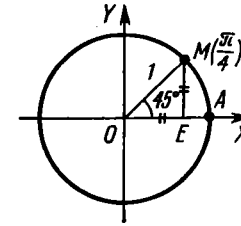


Fig. 107

the circle) corresponds to the number 0, that is, $A = A$ (0). Since, the abscissa and the ordinate of this point is 1 and 0, respectively, we get

$$\cos 0 = 1, \ \sin 0 = 0, \ \tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

(cot 0 is not defined since sin 0 = 0). The point B of intersection of the circle and the positive ray of the axis OY corresponds to the number $\pi/2$. Since the abscissa and the ordinate of the point B is 0 and 1, respectively, we have

$$\cos(\pi/2) = 0, \ \sin(\pi/2) = 1, \ \cot(\pi/2) = \frac{\cos(\pi/2)}{\sin(\pi/2)} = \frac{0}{1} = 0$$

(tan $(\pi/2)$ is not defined since cos $(\pi/2) = 0$). In the same manner, knowing the coordinates of the points C and D, we get

$$\cos \pi = -1, \ \sin \pi = 0, \ \tan \pi = 0 \ (\cot \pi \text{ is not defined}),$$

$$\cos(3\pi/2) = 0, \ \sin(3\pi/2) = -1, \ \cot(3\pi/2) = 0$$

(tan $(3\pi/2)$ is not defined).

Let us now compute sin $(\pi/6)$. Let M be the point of the number circle corresponding to the number $\pi/6$ (Fig. 106). Then $\overset{\frown}{AOM} = 30°$ and from the triangle EOM we get $|ME| = 1/2$. But, according to the definition of sine, $|ME| = \sin(\pi/6)$. Hence, sin $(\pi/6) =$

1/2. Further, we have $\cos(\pi/6) = |OE| = \sqrt{|OM|^2 - |ME|^2} = \sqrt{1 - 1/4} = \sqrt{3}/2$. Then $\tan(\pi/6) = (1/2) \div (\sqrt{3}/2) = 1/\sqrt{3} = \sqrt{3}/3$, $\cot(\pi/6) = \sqrt{3}$. Exactly in the same way, we get

$$\sin(\pi/3) = \sqrt{3}/2,\ \cos(\pi/3) = 1/2,\ \tan(\pi/3) = \sqrt{3},\ \cot(\pi/3) = \sqrt{3}/3,$$
$$\sin(\pi/4) = \sqrt{2}/2,\ \cos(\pi/4) = \sqrt{2}/2,\ \tan(\pi/4) = 1,\ \cot(\pi/4) = 1$$

(Fig. 107). Finally we tabulate the obtained results:

Function \ Argument x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$
$\sin x$	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0	−1
$\cos x$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0	−1	0
$\tan x$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	—	0	—
$\cot x$	—	$\sqrt{3}$	1	$\sqrt{3}/3$	0	—	0

In practice, we sometimes consider trigonometric functions whose argument is not a number but an angle; for example, sin 30°, tan 45°, and so on. By $T(\alpha^\circ)$, where T is a trigonometric function, one should understand $T(\alpha\pi/180)$, where $\alpha\pi/180$ is a number representing the radian measure of the angle α°. Thus, we have

$$\sin 30^\circ = \sin(30\pi/180) = \sin(\pi/6) = 1/2,$$
$$\tan 45^\circ = \tan(45\pi/180) = \tan(\pi/4) = 1.$$

3. Signs of Trigonometric Functions. The coordinate axes divide the number circle into four quadrants. As is seen from Fig. 105, in each of the quadrants, the coordinates of points of the circle have the same sign. Namely, in the first quadrant both coordinates are positive, that is, $X > 0$, $Y > 0$; in the second quadrant $X < 0$, $Y > 0$; in the third quadrant $X < 0$, $Y < 0$; and in the fourth quadrant $X > 0$, $Y < 0$. Therefore, knowing in which quadrant a point $M(x)$ lies, we are able to determine the signs of the trigonometric functions.

For example, if $0 < x < \pi/2$, then the point $M(x)$ lies in the first quadrant, where $X > 0$ and $Y > 0$. Therefore $\sin x > 0$, $\cos x > 0$, $\tan x > 0$, and $\cot x > 0$. If $\pi/2 < x < \pi$, then the point $M(x)$ is in the second quadrant, where $X < 0$, $Y > 0$. Therefore $\sin x > 0$, $\cos x < 0$, $\tan x < 0$, and $\cot x < 0$. If, further, $\pi < x < 3\pi/2$, then the point $M(x)$ is in the third quadrant and therefore $\sin x$

< 0, $\cos x < 0$, $\tan x > 0$, and $\cot x > 0$. If, finally $3\pi/2 < x < 2\pi$, then the point $M(x)$ is in the fourth quadrant and therefore $\sin x < 0$, $\cos x > 0$, $\tan x < 0$, and $\cot x < 0$.

We tabulate the results obtained:

x	Quadrant	$\sin x$	$\cos x$	$\tan x$	$\cot x$
$0 < x < \pi/2$	I	+	+	+	+
$\pi/2 < x < \pi$	II	+	−	−	−
$\pi < x < 3\pi/2$	III	−	−	+	+
$3\pi/2 < x < 2\pi$	IV	−	+	−	−

4. Solving Simple Trigonometric Equations. In what follows, we shall consider a few examples which show how, with the aid of the number circle, equations of the form $\sin x = m$, $\cos x = m$, $|m| \leqslant 1$, are solved.

Let us solve the equation $\sin x = 0$. This means that we have to find the set of all such numbers x to which there correspond on the number circle points with ordinate equal to zero. There are two points on the number circle whose ordinates are equal to zero, viz. A and C (see Fig. 105). The point A corresponds to the numbers 0, $\pm 2\pi$, $\pm 4\pi$, ..., and the point C to the numbers $\pm\pi$, $\pm 3\pi$, $\pm 5\pi$, ... Hence, the points A and C correspond to numbers of the form πn, where n is any integer*, that is, $n \in \mathbf{Z}$. Thus, $x = \pi n$ are the roots of the equation $\sin x = 0$.

Reasoning in a similar way, we find that the roots of the equation $\sin x = 1$ are numbers of the form $x = \pi/2 + 2\pi n$, and the roots of the equation $\sin x = -1$ numbers of the form $x = -\pi/2 + 2\pi n$.

Let us now solve the equation $\cos x = 0$. There are two points on the number circle whose abscissas are equal to zero, viz. B and D (see Fig. 105). The point B corresponds to numbers of the form $\pi/2 + 2\pi k$, while the point D to numbers of the form $3\pi/2 + 2\pi k$ or $\pi/2 + \pi(2k + 1)$. Hence, the points B and D correspond to numbers of the form $\pi/2 + \pi n$. Thus, $x = \pi/2 + \pi n$, are the roots of the equation $\cos x = 0$.

Similarly, we get that the roots of the equation $\cos x = 1$ are numbers of the form $x = 2\pi n$, and those of the equation $\cos x = -1$ are numbers of the form $x = \pi + 2\pi n$.

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* From this point on, everywhere, when writing an answer to a trigonometric equation or inequality, we imply that the parameter (n, k, m, ...) takes on any integral values.

We tabulate the results obtained:

Equation	$\sin x = 0$	$\sin x = 1$	$\sin x = -1$	$\cos x = 0$	$\cos x = 1$	$\cos x = -1$
Solution	$x = \pi n$	$x = \frac{\pi}{2} + 2\pi n$	$x = -\frac{\pi}{2} + 2\pi n$	$x = \frac{\pi}{2} + \pi n$	$x = 2\pi n$	$x = \pi + 2\pi n$

Let us now solve the equation $\sin x = 1/2$. We mark two points on the number circle whose ordinate is 1/2 (the points M and N in Fig. 108). Since $\overset{\frown}{AM} = \pi/6$, the point M corresponds to numbers of the form $\pi/6 + 2\pi k$. Further, $\overset{\frown}{AN} = \overset{\frown}{AC} - \overset{\frown}{NC} = \pi - \pi/6 = 5\pi/6$; hence, the point N corresponds to numbers of the form $5\pi/6 +$

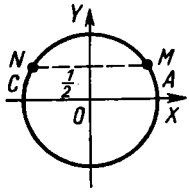


Fig. 108

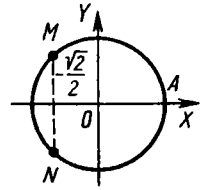


Fig. 109

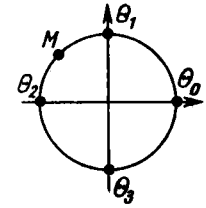


Fig. 110

$2\pi k$. As a result, we get the following solutions to the given equation: $x = \pi/6 + 2\pi k$, $x = 5\pi/6 + 2\pi k$.

Let us also solve another equation $\cos x = -\sqrt{2}/2$. We mark two points on the number circle whose abscissa is equal to $-\sqrt{2}/2$ (the points M and N in Fig. 109). Since $\overset{\frown}{AM} = 3\pi/4$, the point M corresponds to numbers of the form $3\pi/4 + 2\pi k$. The point N, which is symmetric to the point M with respect to the horizontal diameter, corresponds to numbers of the form $-3\pi/4 + 2\pi k$. Finally, we get the following solutions to the given equation: $x = 3\pi/4 + 2\pi k$, $x = -3\pi/4 + 2\pi k$, or, which is the same, $x = \pm 3\pi/4 + 2\pi k$.

5. Solving Simple Trigonometric Inequalities. Consider the solution of inequalities of the form $\sin x > a$ ($\sin x < a$), $\cos x > a$ ($\cos x < a$) with the help of the number circle.

Making use of inequalities, we can specify intervals on the number line. Similarly, with the aid of inequalities, we can specify arcs on the number circle. Note that in this subsection it is more convenient to change somewhat the notation: the points of the number circle corresponding to the numberes $0, \pi/2, \pi, 3\pi/2$ will be respectively denoted by Θ_0, Θ_1, Θ_2, Θ_3 (Fig. 110).

Let us agree to denote by the symbol $\smile M_1M_2$ an arc for which M_1 is the initial point (in the arc notation it occupies the first place) and M_2 is the terminal point of the path described by a running point on the circle in the positive direction (that is, anticlockwise).

Example 1. With the aid of inequalities, write the following closed arcs of the number circle (Fig. 110): (1) $\smile \Theta_0\Theta_1$, (2) $\smile \Theta_1\Theta_3$, (3) $\smile \Theta_1\Theta_0$, (4) $\smile \Theta_2\Theta_1$, (5) $\smile \Theta_0M$, (6) $\smile M\Theta_0$, (7) $\smile \Theta_3M$, where M is the midpoint of the arc $\Theta_1\Theta_2$.

○ (1) The point Θ_0 corresponds to the number 0, and the point Θ_1 to the number $\pi/2$, therefore the running point of the arc $\Theta_0\Theta_1$ corresponds to a number x such that $0 \leqslant x \leqslant \pi/2$.

However, taking into account that if a point of the circle corresponds to the number x, then it also corresponds to all numbers of the form $x + 2\pi k$ (k an integer), we get that the points of the arc $\Theta_0\Theta_1$ correspond to the numbers x that satisfy the following system of inequalities:

$$0 + 2\pi k \leqslant x \leqslant \pi/2 + 2\pi k \text{ or } 2\pi k \leqslant x \leqslant \pi/2 + 2\pi k.$$

This is an analytic notation of the arc $\Theta_0\Theta_1$.

2) For the arc $\Theta_1\Theta_3$ we get $\pi/2 + 2\pi k \leqslant x \leqslant 3\pi/2 + 2\pi k$.

3) As it was noted above, in this case by the notation $\smile \Theta_1\Theta_0$ the arc $\Theta_1\Theta_2\Theta_3\Theta_0$ is understood. During the first tracing of the circle the point Θ_1 corresponds to the number $\pi/2$, and the point Θ_0 to the number 2π (but not to the number 0, since from Θ_1 to Θ_0 the circle is traced in the positive direction). Hence, the arc $\Theta_1\Theta_0$ can be written analytically in the following way: $\pi/2 + 2\pi k \leqslant x \leqslant 2\pi + 2\pi k$.

(4) The arc $\Theta_2\Theta_1$ can be written in two ways:

$$-\pi + 2\pi k \leqslant x \leqslant \pi/2 + 2\pi k \text{ or } \pi + 2\pi n \leqslant x \leqslant 5\pi/2 + 2\pi n.$$

(5) $\smile \Theta_0M$: $2\pi k \leqslant x \leqslant 3\pi/4 + 2\pi k$.

(6) $\smile M\Theta_0$: $3\pi/4 + 2\pi k \leqslant x \leqslant 2\pi + 2\pi k$.

(7) $\smile \Theta_3M$: $-\pi/2 + 2\pi k \leqslant x \leqslant 3\pi/4 + 2\pi k$.

Remark. Writing an arc in the form $\alpha + 2\pi k \leqslant x \leqslant \beta + 2\pi k$, one should make sure that the inequality $\alpha < \beta$ is satisfied, otherwise this double inequality may turn out to be contradictory.

Example 2. Each quadrant of the number circle is divided into three equal parts (Fig. 111). Find the analytical notation for each of the following closed arcs: (1) $\smile B_1B_2$, (2) $\smile \Theta_1B_4$, (3) $\smile B_3A_1$, (4) $\smile A_2B_1$, (5) $\smile A_4\Theta_2$, (6) $\smile A_3B_2$.

○ (1) Consider the arc B_1B_2. Since each of the arcs Θ_0A_1, A_1B_1, $B_1\Theta_1$, Θ_1A_2, . . ., $B_4\Theta_0$ is of length $\pi/6$, during the first positive tracing of the circle the point B_1 corresponds to the number $\pi/3$, and the point B_2 to the number $5\pi/6$. Consequently, the arc B_1B_2 is respresented analytically as follows: $\pi/3 + 2\pi k \leqslant x \leqslant 5\pi/6 + 2\pi k$.

(2) $\smile \Theta_1 B_4$: $\pi/2 + 2\pi k \leqslant x \leqslant 11\pi/6 + 2\pi k$.
(3) $\smile B_3 A_1$: $-2\pi/3 + 2\pi k \leqslant x \leqslant \pi/6 + 2\pi k$ (or $4\pi/3 + 2\pi n \leqslant x \leqslant 13\pi/6 + 2\pi n$).

We draw reader's attention to the necessity of being careful when writing the end points of arcs. Thus, during the first tracing of the number circle, the point B_3 corresponds to the number $4\pi/3$; continuing moving in the direction from the point B_3 to A_1, when passing

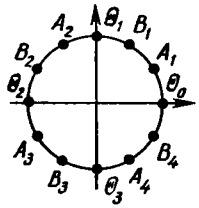


Fig. 111

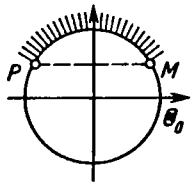


Fig. 112

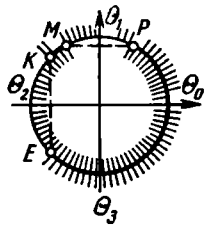


Fig. 113

through the point Θ_0, we begin tracing the circle for the second time, that is, the point A_1 now corresponds to the number $13\pi/6$. Hence, a second notation for the arc B_3A_1 is obtained.

(4) $\smile A_2 B_1$: $-4\pi/3 + 2\pi k \leqslant x \leqslant \pi/3 + 2\pi k$ (or $2\pi/3 + 2\pi n \leqslant x \leqslant 7\pi/3 + 2\pi n$).
(5) $\smile A_4 \Theta_2$: $-\pi/3 + 2\pi k \leqslant x \leqslant \pi + 2\pi k$ (or $5\pi/3 + 2\pi n \leqslant x \leqslant 3\pi + 2\pi n$).
(6) $\smile A_3 B_2$: $-5\pi/6 + 2\pi k \leqslant x \leqslant 5\pi/6 + 2\pi k$ (or $7\pi/6 + 2\pi n \leqslant x \leqslant 17\pi/6 + 2\pi n$).

Example 3. Solve the inequality $\sin x > 1/2$.

○ By definition, $\sin x$ is the ordinate of the point on the number circle corresponding to the number x. Let us mark on the number circle two points having the ordinate equal to 1/2 (points M and P in Fig. 112). Then the points, whose ordinate is greater than 1/2 fill in the open arc MP. It is natural to call this arc the geometrical solution of the given inequality.

Let us represent analytically the open arc MP: $\pi/6 + 2\pi k < x < 5\pi/6 + 2\pi k$. This is the solution of the given inequality. ●

Example 4. Solve the system of inequalities

$$\begin{cases} \sin x < \sqrt{3}/2, \\ \cos x > -\sqrt{2}/2. \end{cases}$$

○ Let us find the geometrical solution of the inequality $\sin x < \sqrt{3}/2$ (the arc MP of the number circle is marked in Fig. 113 by an inner hatching). On the same circle we find the geometrical solution of the inequality $\cos x > -\sqrt{2}/2$ (the corresponding arc

EK is marked in Fig. 113 by an outer hatching). Then the geometrical solution of the given system is the intersection of the arcs MP and EK, that is, the union of the arcs MK and EP. It remains only to represent these arcs analytically. For the arc MK we have $2\pi/3 + 2\pi k < x < 3\pi/4 + 2\pi k$; for the arc EP we have $-3\pi/4 + 2\pi k < x < \pi/3 + 2\pi k$. ●

6. Properties and the Graph of the Function $y = \sin x$. Note the properties of the function $y = \sin x$.

1^0. *The domain of definition is the set* **R** *of real numbers.*

2^0. *The function is periodic with fundamental period* 2π.

□ We know that if two numbers differ by an integer multiple of 2π, then they are associated with the same point M (x) of the number cir-

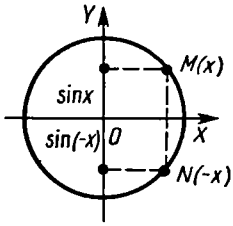


Fig. 114

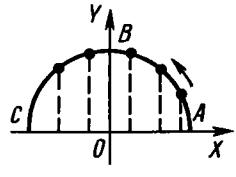


Fig. 115

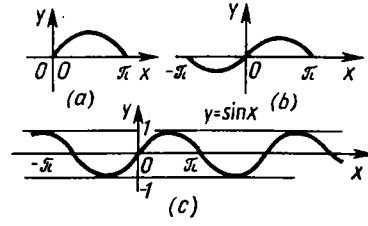


Fig. 116

cle, and $\sin x$ is the ordinate of the point M. Hence it follows that if two numbers differ by 2π, then the same values of the sine, $\sin(x + 2\pi) = \sin x$, correspond to them.

In other words, the function $y = \sin x$ is periodic, and 2π is one of its periods. It is easy to see that this is the fundamental period. Indeed, let there exist a positive period T less than 2π, that is, $0 < T < 2\pi$. We show that then the equality $\sin(x + T) = \sin x$ is not satisfied for $x = \pi/2$. Suppose the contrary, that is, $\sin(\pi/2 + T) = \sin(\pi/2)$. Then $\sin(\pi/2 + T) = 1$. Hence it follows that $\pi/2 + T = \pi/2 + 2\pi k$ (see Item 4), consequently, $T = 2\pi k$. But none of the values $T = 2\pi k$ satisfy the inequality $0 < T < 2\pi$. The obtained contradiction shows that 2π is the fundamental period of the function $y = \sin x$. ■

Since 2π is the fundamental period of the function $y = \sin x$, in order to construct the graph of the function $y = \sin x$ throughout its domain, it suffices to construct its branch on the closed interval $[-\pi, \pi]$.

3^0. *The function is odd.*

□ Indeed, the points M (x) and N $(-x)$ of the number circle are symmetric with respect to the axis OX. Therefore they have ordinates equal in absolute value, but opposite in sign. Thus, $\sin(-x) = -\sin x$ (Fig. 114). ■

The graph of an odd function is symmetric about the origin, therefore, in order to construct the graph of the function $y = \sin x$ on the closed interval $[-\pi, \pi]$, it suffices to construct a branch of the graph on the interval $[0, \pi]$.

4^0. *The function is increasing on* $[0, \pi/2]$ *and decreasing on* $[\pi/2, \pi]$.

□ Indeed, when moving round the number circle from A to B anticlockwise, the ordinate increases from 0 to 1, and when moving from B to C, the ordinate decreases from 1 to 0 (Fig. 115). Hence, the function $y = \sin x$ increases on $[0, \pi/2]$ and decreases on $[\pi/2, \pi]$. ■

5^0. *The range of the function is the closed interval* $[-1, 1]$. This follows from the definition of the function $y = \sin x$.

Using the results of the investigation performed and taking the check points $(0, 0)$, $(\pi/6, 1/2)$, $(\pi/2, 1)$, and $(\pi, 0)$, we construct the graph of the function $y = \sin x$ on the interval $[0, \pi]$ (Fig. 116*a*). Since the function $y = \sin x$ is odd, reflecting the constructed graph in the origin, we get the graph of the function on the interval $[-\pi, \pi]$ (Fig. 116*b*]. Finally, taking advantage of the periodicity of the function $y = \sin x$, we extend the graph to the entire domain of definition (Fig. 116*c*). Note that the graph lies within the band $-1 \leqslant y \leqslant 1$.

Example 1. Compute $\sin(25\pi/6) + \sin(-13\pi/6)$.

○ Taking into consideration the oddness and periodicity of the function $y = \sin x$, we get

$$\sin(25\pi/6) + \sin(-13\pi/6) = \sin(\pi/6 + 4\pi) - \sin(\pi/6 + 2\pi)$$
$$= \sin(\pi/6) - \sin(\pi/6) = 0. \bullet$$

Example 2. Determine the sign of the number

$$a = \frac{(\sin 0.6\pi - \sin 0.62\pi)(\sin 1.3 - \sin 1.2)}{\sin 4 - \sin 3.5}.$$

○ The values 0.6π and 0.62π belong to the second quadrant. But $\sin x$ decreases in the second quadrant, hence $\sin 0.6\pi > \sin 0.62\pi$, and therefore $\sin 0.6\pi - \sin 0.62\pi > 0$. The values 1.3 and 1.2 belong to the first quadrant; $\sin x$ increases in the first quadrant, consequently $\sin 1.3 > \sin 1.2$, and therefore $\sin 1.3 - \sin 1.2 > 0$. Finally, the values 4 and 3.5 belong to the third quadrant. Since the function $y = \sin x$ decreases in the third quadrant (see Fig. 116*c*), we get $\sin 4 - \sin 3.5 < 0$. Thus, $a < 0$. ●

7. Properties and the Graph of the Function $y = \cos x$. The investigation of the function $y = \cos x$ is carried out similarly to that of the function $y = \sin x$ (the relevant proofs are left to the reader).

1^0. *The domain of the function is the set* **R** *of real numbers.*

2^0. *The function is periodic with fundamental period* 2π.

3^0. *The function is even.*

4°. *The function is decreasing on the closed interval* $[0, \pi]$.

5°. *The range of the function is the closed interval* $[-1, 1]$.

The graph of the function is represented in Fig. 117.

Example. Construct the graph of the function $y = -3 \cos 2x$.

○ We begin with constructing the graph of the function $y = \cos x$ (confining ourselves to one half-wave of the graph). Compressing it towards the y-axis with coefficient 2, we get the graph of the function $y = \cos 2x$. We then stretch the obtained graph along the x-axis

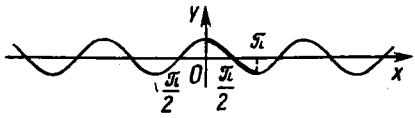


Fig. 117

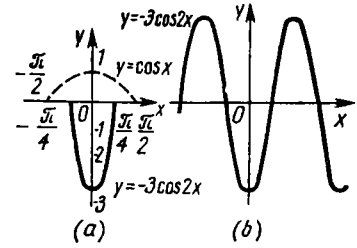


Fig. 118

with coefficient 3, and, finally, reflect it in the x-axis. As a result, we get the graph of the function $y = -3 \cos 2x$. Figure 118*a* represents one half-wave of the required graph, while Fig. 118*b* the entire graph.

8. Harmonic Oscillations. Trigonometric functions are used to describe oscillatory processes. One of the most important processes is described by the formula

$$y = A \sin (\omega x + \alpha) \tag{1}$$

which is called the **formula of harmonic** (or **sinusoidal**) **oscillations,** where A is the *amplitude* of oscillation which characterizes the swing of the oscillation and ω is the *frequency* of oscillation. The greater ω, the greater the number of oscillations per unit time (the number of oscillations per unit time is $\omega/(2\pi)$), and, finally, α is the *initial phase* of oscillation.

If, for instance, a load suspended by a spring is disturbed from the equilibrium position, then it starts to oscillate vertically. It turns out that the law of oscillatory motion is expressed by formula (1), where y is the displacement of the load from its equilibrium position, and x is the time. The same law is encountered in the theory of alternating electric current. If a rectangular frame made from a current-conducting material rotates in a magnetic field, then an alternating current is induced in it. If the frame rotates uniformly, then the current varies according to the law of harmonic oscillations (1).

Let us find the period of the function $y = A \sin(\omega x + \alpha)$. Let T be the fundamental period of this function. Then

$$A \sin(\omega(x + T) + \alpha) = A \sin(\omega x + \alpha),$$

hence we get

$$\sin(\omega x + \alpha + \omega T) = \sin(\omega x + \alpha). \quad (2)$$

Since the fundamental period of the function $y = \sin x$ is 2π, the least positive value of T satisfying relationship (2) is found from the equation $\omega T = 2\pi$, that is, $T = 2\pi/\omega$. Thus, the fundamental period of the function $y = A \sin(\omega x + \alpha)$ equals $2\pi/\omega$.

Example. Construct the graph of the function $y = 2 \sin\left(\frac{x}{3} - \frac{\pi}{6}\right)$.

○ Write the given function as follows: $y = 2 \sin\left(\frac{1}{3}\left(x - \frac{\pi}{2}\right)\right)$. Construct the desired graph in several steps:

(1) Translate the coordinate system, choosing the point O' $(\pi/2, 0)$ as the new origin. Graph the function $y' = 2 \sin(x'/3)$ in the new system $x'O'y'$.

(2) Graph the function $y' = \sin x'$.

(3) Compressing the graph along the axis $O'y'$ with coefficient $1/3$ (that is, extending with coefficient 3), we get the graph of the function $y' = \sin(x'/3)$.

(4) Extend the last graph along the axis $O'x'$ with coefficient 2. The obtained graph is exactly the one of the function $y = 2 \sin\left(\frac{x}{3} - \frac{\pi}{6}\right)$ (Fig. 119). ●

In the general case, the graph of the function $y = A \sin(\omega x + \alpha)$ is constructed in a similar way (as it was done in the above example). First we transform the function to the form $y = A \sin \omega\left(x + \frac{\alpha}{\omega}\right)$ and then construct the graph in several steps:

(1) Translate the coordinate system with the point O' $(-\alpha/\omega, 0)$ taken as the origin of the new coordinate system $x'O'y'$.

(2) Sketch the graph of the function $y' = \sin x'$ in the system $x'O'y'$ (first its one half-wave).

(3) Compressing the constructed graph along the axis $O'y'$ with ratio ω, we get the graph of the function $y' = \sin \omega x'$.

(4) Extending the last graph along the axis $O'x'$ with coefficient A, we get the required graph.

9. Properties and the Graph of the Function $y = \tan x$. Note the properties of the function $y = \tan x$.

1°. *The domain*: $x \neq \pi/2 + \pi k$ (since $\tan x = \sin x/\cos x$, in the domain of the function $\tan x$ we have $\cos x \neq 0$, that is, $x \neq \pi/2 + \pi k$).

2°. *The function is periodic with fundamental period* π.

□ Note first that the points $M(x)$ and $N(x+\pi)$ of the number circle are diametrically opposite since the length of the unit semicircumference is equal to π. Consequently, the coordinates of these points differ only in sign (Fig. 120): $\sin(x+\pi) = -\sin x$, $\cos(x+\pi) = -\cos x$. Then

$$\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin x}{-\cos x} = \tan x.$$

Thus, $y = \tan x$ is a periodic function and π is one of its periods.

It is easy to see that this period is fundamental. Indeed, let there exist a period T which satisfies the inequality $0 < T < \pi$. We

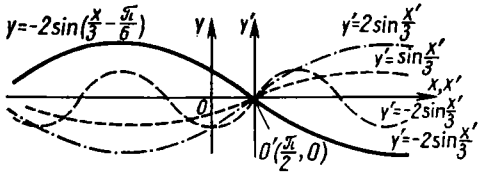


Fig. 119

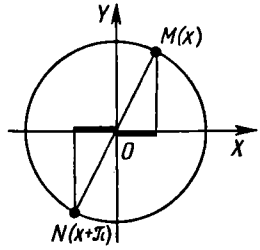


Fig. 120

show that the equality $\tan(x+T) = \tan x$ is then not satisfied for $x = 0$. Suppose the contrary, that is, $\tan(0+T) = \tan 0$. Then $\tan T = 0$, whence $\sin T = 0$, consequently, $T = \pi k$. But none of the values $T = \pi k$ satisfies the inequality $0 < T < \pi$. The obtained contradiction shows that $T = \pi$ is the fundamental period of the function $y = \tan x$. ■

Since π is the fundamental period of the function $y = \tan x$, to graph the function $y = \tan x$ on the entire domain, it suffices first to construct one branch of the graph on the interval $(-\pi/2, \pi/2)$.

3°. *The function is odd.*

□ Indeed,

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x. \quad ■$$

The graph of an odd function is symmetric with respect to the origin, therefore, to construct the graph of the function $y = \tan x$ on the interval $(-\pi/2, \pi/2)$, it suffices first to sketch its branch on the interval $[0, \pi/2)$.

4°. *The function is increasing on the interval* $[0, \pi/2)$.

□ Let $0 \leqslant x_1 < x_2 < \pi/2$. Then $\sin x_1 < \sin x_2$ ($y = \sin x$ increases on the interval under consideration), and $\cos x_1 > \cos x_2$

($y = \cos x$ decreases on $[0, \pi/2)$). Hence

$$\tan x_1 = \frac{\sin x_1}{\cos x_1} < \frac{\sin x_2}{\cos x_2} = \tan x_2.$$

Thus, from $x_1 < x_2$ it follows: $\tan x_1 < \tan x_2$, and this means that $y = \tan x$ is an increasing function on $[0, \pi/2)$. ■

Note that if x approaches the right-hand end point of the interval $(0, \pi/2)$, that is, $\pi/2$, then $\sin x$ approaches unity, while $\cos x$ ap-

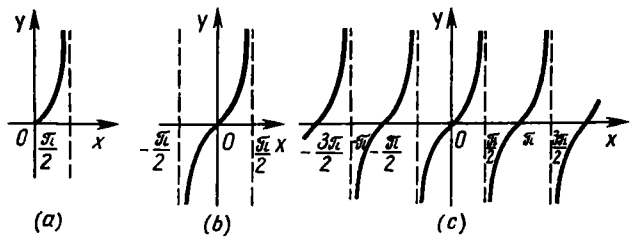


Fig. 121

proaches zero and, hence, the fraction $\sin x/\cos x$ increases without bound. This means that the graph of the function $y = \tan x$ (or, as we put it above, asymptotically) approaches the straight line $x = \pi/2$ indefinitely. This straight line is a vertical asymptote to the graph of the function $y = \tan x$.

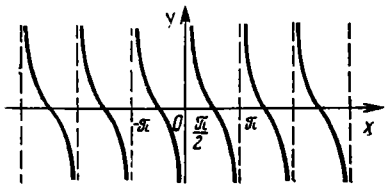


Fig. 122

Using the investigation results and choosing several check points: $(0, 0)$, $(\pi/4, 1)$, $(\pi/3, \sqrt{3})$, we get the graph of the function $y = \tan x$ on the interval $[0, \pi/2)$ (Fig. 121*a*). Since the function $y = \tan x$ is odd, we construct the graph on the interval $(-\pi/2, \pi/2)$ (Fig. 121*b*). Finally, taking advantage of the periodicity of the function $y = \tan x$, we continue the graph to the entire domain (Fig. 121*c*).

10. Function $y = \cot x$. The function $y = \cot x$ is investigated similarly to the function $y = \tan x$. Therefore we only mention the properties of the function $y = \cot x$, leaving their proof for the student.

1°. *The domain is* $x \neq \pi k$.

2°. *The function is periodic with fundamental period* π.

3°. *The function is odd.*

Thus, to graph the function $y = \cot x$ on the entire domain, it suffices first to construct the graph on the interval $(0, \pi/2]$.

4°. *The function is decreasing on the interval* $(0, \pi/2]$.

The graph of the function $y = \cot x$ is represented in Fig 1.22.

Exercises

1. Using the number circle, compute the values of the trigonometric functions $\sin x$, $\cos x$, $\tan x$, and $\cot x$ if: (a) $x = 2\pi/3$, (b) $x = 7\pi/6$, (c) $x = 7\pi/4$.

2. Which is greater: (a) $\sin 2$ or $\sin 2.1$, (b) $\cos 3$ or $\cos 3.1$, (c) $\tan 5$ or $\tan 5.2$?

3. Solve the following inequalities: (a) $\dfrac{\sin 5 - \sin 5.2}{-2x+1} > 0$,

(b) $\dfrac{\cos(-3.2)}{x-3} < 0$.

4. Determine whether the following equalities are possible: (a) $\sin\alpha = -\sqrt[4]{1.75}$, (b) $\cos x = \pi/4$?

5. Determine the signs of the given products: (a) $-\cot(5\pi/3)\cdot\sin 6$, (b) $\cos(-2.8)\cdot\tan(-5)$.

6. Simplify the following expressions:

(a) $x^2\sin^2(-\pi/2) + y^2\cos^2 0 + \tan\pi + 2xy\sin(3\pi/2)$,

(b) $(a\cos(-\pi)^2 + ab\sin(3\pi/2)\cos\pi + (b\sin(-\pi))^2$.

7. Find the greatest and the least values of the function $2 - \sin x$.

8. Find which of the given functions are even, which are odd, and which are neither even nor odd:

(a) $f(x) = 1 - \cos x$, (b) $f(x) = x - \sin x$, (c) $f(x) = x^2\cos x$, (d) $f(x) = \tan x\sin^2 x$, (e) $f(x) = \cos x\sin x$, (f) $f(x) = \tan x\cot^3 x$. (g) $f(x) = x^2 + \sin x + \tan x$.

9. Compute: (a) $\cos(-91\pi/6)$, (b) $\cos(216\pi - 1.4) - \cos(312\pi + 1.4)$.

10. Compare the numbers a and b if: (a) $a = \sin(0.3 - 2\pi)$, $b = \sin(0.3 + 302\pi)$, (b) $a = \cos(11\pi/18)$, $b = \cos(121\pi/180)$.

11. Determine the sign of the number a if:

(a) $a = (\sin(2\pi/5) - \sin(2\pi/7))(\sin 4 - \sin 3.9)$,

(b) $a = \dfrac{\cos(-\pi/5) - \cos(-\pi/10)}{\sin(-3\pi/5) - \sin(-4\pi/7)}$.

12. Graph the following functions:

(a) $y = 2\sin x$, (b) $y = \sin 2x$, (c) $y = -2\sin\dfrac{x}{2}$,

(d) $y = \dfrac{1}{2}\sin\left(2x - \dfrac{\pi}{2}\right)$, (e) $y = -2\cos\left(3x - \dfrac{\pi}{2}\right)$, (f) $y = |\sin x|$,

(g) $y = \sin|x|$, (h) $y = \left|4\cos\dfrac{x}{2}\right|$, (i) $y = \left|3\sin\left(2x - \dfrac{\pi}{3}\right) + 1\right|$.

13. Graph the following functions:

(a) $y = \dfrac{1}{2}\tan x$, (b) $y = \tan(-2x)$, (c) $y = -2\tan 3x$,

(d) $y = \tan\left(x - \dfrac{\pi}{4}\right)$, (e) $y = \cot|x|$, (f) $y = \left|\cot\left(x - \dfrac{\pi}{4}\right)\right|$.

14. Compute:

(a) $\cot(29\pi/6)$, (b) $\tan(-11\pi/6)$, (c) $\dfrac{\sin(-13\pi/4) + \cos(31\pi/4)}{\tan(-11\pi/6) + \cot(-11\pi/3)}$.

15. Determine the sign of the number a if:

(a) $a = \dfrac{\tan(5\pi/3) - \tan(11\pi/7)}{\tan(\pi/9) - \tan(\pi/11)}$, (b) $a = \left(\cos\dfrac{31\pi}{6} - \cos\dfrac{36\pi}{7}\right)(\tan 6 - \tan 5)$.

16. Using the number circle, solve the following equations: (a) $\sin x = \sqrt{2}/2$, (b) $\sin x = \sqrt{3}/2$, (c) $\sin x = -1/2$, (d) $\sin x = -\sqrt{2}/2$, (e) $\sin x = -\sqrt{3}/2$, (f) $\cos x = 1/2$, (g) $\cos x = \sqrt{2}/2$, (h) $\cos x = \sqrt{3}/2$, (i) $\cos x = -1/2$, (j) $\cos x = -\sqrt{3}/2$.

In Exercises **17** to **19**, solve the given inequalities and systems of inequalities:

17. (a) $\sin x > 0$, (b) $\sin x < 0$, (c) $\sin x > \sqrt{2}/2$, (d) $\sin x < \sqrt{3}/2$, (e) $\sin x > -1/2$, (f) $\sin x < -\sqrt{3}/2$.

18. (a) $\cos x > 0$, (b) $\cos x < 0$, (c) $\cos x > 1/2$, (d) $\cos x < \sqrt{2}/2$, (e) $\cos x > -\sqrt{3}/2$, (f) $\cos x < -1/2$.

19. (a) $\begin{cases} \sin x > 0, \\ \cos x > 0, \end{cases}$ (b) $\begin{cases} \sin x < 0, \\ \cos x < 1/2, \end{cases}$ (c) $\begin{cases} \cos x < 0, \\ \sin x < \sqrt{2}/2, \end{cases}$

(d) $\begin{cases} \sin x > -1/2, \\ \cos x < \sqrt{2}/2, \end{cases}$ (e) $\begin{cases} \sin x > -\sqrt{3}/2, \\ \sin x < 1/2. \end{cases}$

§ 3.8. Inverse Trigonometric Functions

1. Function $y = \arcsin x$ (or $\sin^{-1} x$). Consider the function $y = \sin x$. It is monotonic on each of the following closed intervals: . . ., $[-\pi/2, \pi/2]$, $[\pi/2, 3\pi/2]$, $[3\pi/2, 5\pi/2]$, . . . and takes on each of them all the values from -1 to 1 (see Fig. 116). Hence, on each

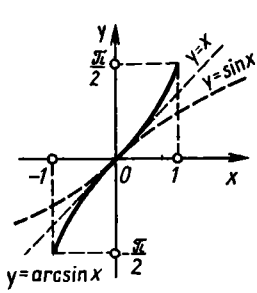


Fig. 123

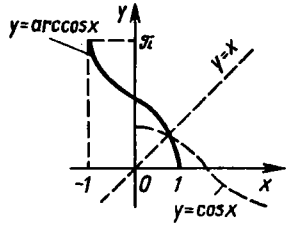


Fig. 124

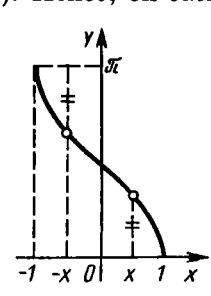


Fig. 125

of these intervals there is an inverse function for the function $y = \sin x$ (according to the theorem on the inverse of a function; see § 3.4 Item 2). It should be noted that these functions are different. The function inverse of the function $y = \sin x$ on the closed interval $[-\pi/2, \pi/2]$ is of a particular interest. This function is denoted by $x = \arcsin y$. Interchanging, as usual, x and y, we write $y = \arcsin x$ (or $\sin^{-1} x$).

Thus, $y = \arcsin x$ *is the inverse of the function* $y = \sin x$, where $-\pi/2 \leqslant x \leqslant \pi/2$.

The graph of the function $y = \arcsin x$ can be obtained from the graph of the function $y = \sin x$, $-\pi/2 \leqslant x \leqslant \pi/2$ by means of a reflection in the straight line $y = x$ (Fig. 123).

Note the properties of the function $y = \arcsin x$.

1°. *The domain*: $D(f) = [-1, 1]$.

2°. *The function is odd*: $\arcsin(-x) = -\arcsin x$.

3°. *The function is increasing.*

4°. *The range of the function*: $E(f) = [-\pi/2, \pi/2]$.

It follows from the aforesaid that the notations $y = \arcsin x$ and $x = \sin y$, $-\pi/2 \leqslant y \leqslant \pi/2$ are equivalent. Substituting the expression of y, that is, $\arcsin x$ into the equality $x = \sin y$, we get $x = \sin(\arcsin x)$. Consequently, for any $x \in [-1, 1]$ we have

$$\sin(\arcsin x) = x, \quad -\pi/2 \leqslant \arcsin x \leqslant \pi/2.$$

2. Function $y = \arccos x$ (or $\cos^{-1} x$). The function $y = \cos x$ is monotonic on the closed interval $[0, \pi]$ (decreases on this interval) and takes on all the values from -1 to 1 (see Fig. 117). Hence, there is an inverse of the function $y = \cos x$ considered on the closed interval $[0, \pi]$. It is denoted by $y = \arccos x$ (or $\cos^{-1} x$). The graph of this function is obtained from the graph of the function $y = \cos x$, $0 \leqslant x \leqslant \pi$ by means of a reflection in the straight line $y = x$ (Fig. 124).

Note the properties of the function $y = \arccos x$.

1°. *The domain*: $D(f) = [-1, 1]$.

2°. *The function is neither even nor odd.* This follows, say, from the fact that the graph of the function is symmetric neither with respect to the axis of ordinates nor about the origin.

Note, however, that the graph of the function $y = \arccos x$ is symmetric about the point $(0, \pi/2)$. Hence we get $\arccos(-x) - \pi/2 = \pi/2 - \arccos x$, that is, $\arccos(-x) = \pi - \arccos x$ (Fig. 125).

3°. *The function is decreasing.*

4°. *The range of the function*: $E(f) = [0, \pi]$.

It follows from the aforesaid that the notations $y = \arccos x$ and $x = \cos y$, $0 \leqslant y \leqslant \pi$ are equivalent. Substituting the expression $\arccos x$ for y into the equality $x = \cos y$, we get $x = \cos(\arccos x)$. Consequently, for any $x \in [-1, 1]$ we have

$$\cos(\arccos x) = x, \quad 0 \leqslant \arccos x \leqslant \pi.$$

3. Function $y = \arctan$ (or $\tan^{-1} x$). The function $y = \tan x$ is monotonic on the open interval $(-\pi/2, \pi/2)$ (increasing on this interval) and takes on all the values (see Fig. 121). Therefore there is an inverse of the function $y = \tan x$ on the indicated interval. It is denoted by $y = \arctan x$ (or $\tan^{-1} x$). The graph of this function is obtained from the graph of the function $y = \tan x$, $-\pi/2 < x < \pi/2$ by means of a reflection in the straight line $y = x$ (Fig. 126).

Note the properties of the function $y = \arctan x$.

1°. *The domain*: $D(f) = \mathbf{R}$.

2°. *The function is odd*: $\arctan(-x) = -\arctan x$.

3^0. *The function is increasing.*

4^0. *The range of the function*: $E(f) = (-\pi/2, \pi/2)$.

It follows from the aforesaid that $y = \arctan x$ and $y = \tan x$, $-\pi/2 < y < \pi/2$ are equivalent. For any x, $-\infty < x < +\infty$, we have

$$\tan(\arctan x) = x, \quad -\pi/2 < \arctan x < \pi/2.$$

4. Function $y = \operatorname{arccot} x$ (or $\cot^{-1} x$). The function $y = \cot x$ is monotonic on the open interval $(0, \pi)$ (decreasing on this interval) and takes on all the values (see Fig. 122). Consequently, there is an inverse of the function $y = \cot x$ on this interval. It is denoted by

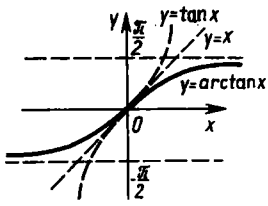


Fig. 126

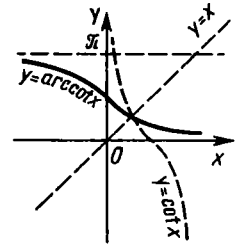


Fig. 127

$y = \operatorname{arccot} x$ (or $\cot^{-1} x$). The graph of the function $y = \operatorname{arccot} x$ is obtained from the graph of the function $y = \cot x$, $0 < x < \pi$ by means of a reflection in the straight line $y = x$ (Fig. 127).

Note the properties of the function $y = \operatorname{arccot} x$.

1^0. *The domain*: $D(f) = \mathbf{R}$.

2^0. *The function is neither even nor odd.* The graph of this function like the graph of the function $y = \arccos x$, is symmetric with respect to the point $(0, \pi/2)$. Hence we get $\operatorname{arccot}(-x) = \pi - \operatorname{arccot} x$.

3^0. *The function is decreasing.*

4^0. *The range of the function*: $E(f) = (0, \pi)$.

It follows from the aforesaid that the notations $y = \operatorname{arccot} x$ and $x = \cot y$, $0 < y < \pi$ are equivalent. For any x, $-\infty < x < +\infty$, we have

$$\cot(\operatorname{arccot} x) = x, \quad 0 < \operatorname{arccot} x < \pi.$$

The functions $y = \arcsin x$, $y = \arccos x$, $y = \arctan x$, and $y = \operatorname{arccot} x$ are called *inverse trigonometric functions.*

Prior to considering examples, let us write out the most important relationships for inverse trigonometric functions widely used in practice (all of them were obtained above):

(1) $-\pi/2 \leqslant \arcsin x \leqslant \pi/2$,
(2) $\arcsin(-x) = -\arcsin(x)$,
(3) $0 \leqslant \arccos x \leqslant \pi$,
(4) $\arccos(-x) = \pi - \arccos x$,
(5) $-\pi/2 < \arctan x < \pi/2$,
(6) $\arctan(-x) = -\arctan x$,
(7) $0 < \operatorname{arccot} x < \pi$,
(8) $\operatorname{arccot}(-x) = \pi - \operatorname{arccot} x$.

Example 1. Compute:

(1) $\arcsin(\sqrt{3}/2)$, (2) $\arcsin(-1/2)$, (3) $\arccos(\sqrt{2}/2)$, (4) $\arccos(-\sqrt{2}/2)$, (5) $\arctan 1$, (6) $\arctan(-\sqrt{3})$, (7) $\operatorname{arccot}(-\sqrt{3})$.

○ (1) By definition, $y = \arcsin(\sqrt{3}/2)$ is a number such that $\sin y = \sqrt{3}/2$ and $-\pi/2 \leqslant y \leqslant \pi/2$. Hence it follows that $y = \pi/3$. Thus, $\arcsin(\sqrt{3}/2) = \pi/3$.

(2) Reasoning in a similar way, we get $\arcsin(1/2) = \pi/6$. But $\arcsin(-1/2) = -\arcsin(1/2)$, consequently, $\arcsin(-1/2) = -\pi/6$.

(3) By definition, $y = \arccos(\sqrt{2}/2)$ is a number such that $\cos y = \sqrt{2}/2$ and $0 \leqslant y \leqslant \pi$. Hence it follows that $y = \pi/4$. Thus, $\arccos(\sqrt{2}/2) = \pi/4$.

(4) We noted that $\arccos(-x) = \pi - \arccos x$. Hence $\arccos(-\sqrt{2}/2) = \pi - \arccos(\sqrt{2}/2) = \pi - \pi/4 = 3\pi/4$.

(5) By definition, $y = \arctan 1$ is a number such that $\tan y = 1$ and $-\pi/2 < y < \pi/2$. Hence it follows that $y = \pi/4$. Thus, $\arctan 1 = \pi/4$.

(6) Reasoning in a similar way, we get $\arctan\sqrt{3} = \pi/3$. Since $\arctan(-x) = -\arctan x$, we have $\arctan(-\sqrt{3}) = -\arctan\sqrt{3} = -\pi/3$.

(7) We first compute $y = \operatorname{arccot}\sqrt{3}$. This is a number such that $\cot y = \sqrt{3}$ and $0 < y < \pi$. Hence, $y = \pi/6$. We noted that $\operatorname{arccot}(-x) = \pi - \operatorname{arccot} x$. Therefore $\operatorname{arccot}(-\sqrt{3}) = \pi - \operatorname{arccot}\sqrt{3} = \pi - \pi/6 = 5\pi/6$. ●

Example 2. Determine the sign of the number a if:

(1) $a = \sin(\arccos x)$, (2) $a = \tan\left(\frac{\pi}{2} + \operatorname{arccot}\left(-\frac{2}{3}\right)\right)$.

○ (1) By definition, $0 \leqslant \arccos x \leqslant \pi$. But on the interval $[0, \pi]$ the sine attains only nonnegative values. Thus, $\sin(\arccos x) \geqslant 0$.

(2) By definition, $0 < \operatorname{arccot} x < \pi$. In the present case, $x = -2/3$, consequently, $\pi/2 < \operatorname{arccot}(-2/3) < \pi$. Then $\pi < \pi/2 + \operatorname{arccot}(-2/3) < 3\pi/2$, and in the interval $(\pi, 3\pi/2)$ tangent attains only positive values. Thus, $\tan\left(\frac{\pi}{2} + \operatorname{arccot}\left(-\frac{2}{3}\right)\right) > 0$. ●

Example 3. Compute:
(1) $\arcsin(\sin(\pi/5))$, (2) $\arcsin(\sin 4\pi/5))$, (3) $\arccos(\cos(-17\pi/5))$.

○ (1) Let $\arcsin(\sin \pi/5)) = y$. Then $\sin y = \sin(\pi/5)$ and $-\pi/2 \leqslant y \leqslant \pi/2$. Hence it follows that $y = \pi/5$.

(2) Let $\arcsin(\sin(4\pi/5)) = y$. Then $\sin y = \sin(4\pi/5)$, $-\pi/2 \leqslant y \leqslant \pi/2$. However, it is impossible here to set $y = 4\pi/5$ as in the preceding case since $4\pi/5 \notin [-\pi/2, \pi/2]$. We have $\sin(4\pi/5) = \sin(\pi - \pi/5) = \sin(\pi/5)$. Thus, $\sin y = \sin(\pi/5)$. Since $\pi/5 \in [-\pi/2, \pi/2]$, we have $y = \pi/5$.

(3) Let $\arccos(\cos(-17\pi/5)) = y$. Then $\cos y = \cos(-17\pi/5)$ and $0 \leqslant y \leqslant \pi$. Here, as in the preceding case, it is impossible to

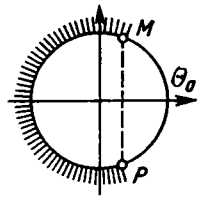


Fig. 128

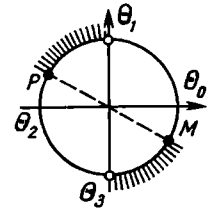


Fig. 129

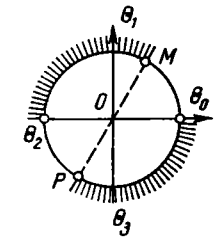


Fig. 130

set $y = -17\pi/5$, since $-17\pi/5 \notin [0, \pi]$. We have $\cos(-17\pi/5) = \cos(17\pi/5) = \cos(4\pi - 3\pi/5) = \cos(3\pi/5)$. Thus, $\cos y = \cos(3\pi/5)$, But $0 < 3\pi/5 < \pi$, consequently, $y = 3\pi/5$. ●

Example 4. Solve the given inequalities: (1) $\cos x < 1/3$, (2) $\tan x \leqslant -1/2$, (3) $\cot x < \sqrt{3}/3$.

○ (1) By definition, $\cos x$ is the abscissa of the point on the number circle associated with the number x. Let us mark on the number circle the points having the abscissa equal to 1/3 (the points M and P in Fig. 128). Then the geometrical solution of the inequality $\cos x < 1/3$ is represented by the open arc MP (the points of this arc have the abscissa less than 1/3). We represent the arc MP analytically

$$\arccos(1/3) + 2\pi k < x < 2\pi - \arccos(1/3) + 2\pi k.$$

(2) Note that $\tan x$ is not defined for $x = \pi/2 + \pi k$. These numbers are associated with the points Θ_1 and Θ_3 of the number circle (Fig. 129). Let us mark on the semicircle $\Theta_3\Theta_1$ a point $M = M(x)$ such that $\tan x = -1/2$. Since on the arc $\Theta_3\Theta_1$ (or more exactly, on each of the intervals of the number line $\mathbf{R}$ reflected on the arc $\Theta_3\Theta_1$) the function $y = \tan x$ increases, the inequality $\tan x \leqslant -1/2$ will be satisfied for all the points of the arc $\Theta_3\Theta_1$, lying in the negative direction from the point M, that is, on the half-open arc $\Theta_3 M$.

Further, since the fundamental period of the tangent is equal to π, the inequality will be also fulfilled for all the points of the arc $\Theta_1 P$, which differs from the arc $\Theta_3 M$ by a semicircle.

Thus, the geometrical solution of the inequality is the union of two half-open arcs $\Theta_3 M$ and $\Theta_1 P$. Let us represent these arcs analytically. For the arc $\Theta_3 M$ we have

$$-\pi/2 + 2\pi k < x \leqslant -\arctan(1/2) + 2\pi k,$$

and for the arc $\Theta_1 P$ we have

$$\pi/2 + 2\pi k < x \leqslant \pi - \arctan(1/2) + 2\pi k.$$

However, the solution of the given inequality can be written shorter:

$$-\pi/2 + \pi n < x \leqslant -\arctan(1/2) + \pi n.$$

(3) Note that $\cot x$ is not defined for $x = \pi k$. These numbers are associated with the points Θ_0 and Θ_2 on the number circle (Fig. 130). We mark a point $M = M(x)$ on the semicircle $\Theta_0\Theta_2$ such that $\cot x = \sqrt{3}/3$; for this purpose, we lay off an arc $\Theta_0 M$, whose length is equal to $\operatorname{arccot}(\sqrt{3}/3) = \pi/3$. Since on the arc $\Theta_0\Theta_2$ the function $y = \cot x$ is decreasing, the inequality $\cot x < \sqrt{3}/3$ will be satisfied for all the points of the arc $\Theta_0\Theta_2$, lying in the positive direction from the point M, that is, on the open arc $M\Theta_2$ (when solving the inequality $\cot x > \sqrt{3}/3$ we would have to take the arc $\Theta_0 M$). Taking into account that the fundamental period of cotangent is equal to π, we mark another arc $P\Theta_0$ on which the given inequality is satisfied (this arc is obtained from the arc $M\Theta_2$ by rotating the latter about the point O through an angle of 180°).

Thus, the geometrical solution of the given inequality is the union of two open arcs $M\Theta_2$ and $P\Theta_0$. Analytically the arc $M\Theta_2$ is represented as $\pi/3 + 2\pi k < x < \pi + 2\pi k$, and the arc $P\Theta_0$ as $4\pi/3 + 2\pi k < x < 2\pi + 2\pi k$.

The solution of the given inequality can be written shorter: $\pi/3 + \pi k < x < \pi + \pi k$. ●

Exercises

1. Compute: (a) $2\arcsin(\sqrt{3}/2) + \arctan(-1) + \arccos(1/\sqrt{2}) + 0.5\arccos(-1)$, (b) $\tan(5\arctan(\sqrt{3}/3) - 0.25\arcsin(\sqrt{3}/2))$, (c) $\sin(3\arctan\sqrt{3} + 2\arccos(1/2))$, (d) $\arcsin(\sin(\pi/3))$, (e) $\arccos(-\cos(3\pi/4))$, (f) $\operatorname{arccot}(\cot(-\pi/5))$.

2. Determine the signs of the given expressions:

(a) $a = \cos\left(\pi + \arcsin\frac{1}{3}\right)$, (b) $a = \tan\left(\arccos\left(-\frac{2}{7}\right) - \frac{\pi}{2}\right)$,

(c) $a = \cot(\pi + \arctan(-2))$, (d) $a = \sin(\pi + \operatorname{arccot} x)$.

3. Determine on which of the following closed intervals there exists the inverse of the function $y = \sin x$: (a) $\pi/2 \leqslant x \leqslant 3\pi/2$, (b) $-5\pi/2 \leqslant x \leqslant -3\pi/2$, (c) $0 \leqslant x \leqslant \pi$, (d) $1 \leqslant x \leqslant 2$?

4. On which of the following intervals is it possible to define the inverse of the function $y = \cot x$: (a) $\pi < x < 2\pi$, (b) $-\pi/2 < x < \pi/2$?

5. Determine the sign of the number a if:

(a) $a = \arccos 0.5 - \arccos 0.7$,

(b) $a = \dfrac{\arcsin 0.85 - \arcsin 0.8}{\arccos 0.85 - \arccos 0.8}$,

(c) $a = \dfrac{\arctan(-0.6) + \pi/6}{\arctan 0.6 - \pi/3}$.

6. Solve the given equations:

(a) $\arccos(x/2) = -\pi/2$, (b) $\arcsin x - \pi/4 = 0$, (c) $\arctan 2x = 1$, (d) $2 \operatorname{arccot} 2x - \pi/2 = 0$.

7. Find the domain of definition for each of the given functions: (a) $y = \arcsin 2x$, (b) $y = \arccos \dfrac{3x-1}{3}$, (c) $y = \arcsin(x^2+1)$, (d) $y = \arctan \dfrac{x}{x^2-9}$, (e) $y = \operatorname{arccot} 2x$.

8. Graph the following functions: (a) $y = \arcsin 2x$, (b) $y = -3 \arccos x$, (c) $y = 0.5 \arctan 4x$, (d) $y = -\operatorname{arccot}(-2x)$, (e) $y = \arcsin(x-1)$, (f) $y = \arccos(x+2)$, (g) $y = |\arcsin x|$, (h) $y = \arccos|x|$.

9. Using the number circle, solve the following equations: (a) $\sin x = 1/3$, (b) $\sin x = -1/4$, (c) $\cos x = 2/3$, (d) $\cos x = -3/5$.

10. Using the number circle, solve the following inequalities: (a) $\sin x > 2/7$, (b) $\sin x < 0.3$, (c) $\sin x < -0.2$, (d) $\sin x > -0.4$, (e) $\cos x > 0.1$, (f) $\cos x < 2/7$, (g) $\cos x > -0.7$, (h) $\cos x < -0.2$, (i) $\tan x < 1$, (j) $\tan x > -\sqrt{3}/3$, (k) $\cot x > -1$.

§ 3.9. Graphical Solution of Equations, Inequalities and Systems of Equations

1. Solving Equations. In practice, it frequently turns out that the solution of an equation $f(x) = 0$ involves cumbersome calculations. In such cases, we resort to one of the methods of approximate solution of the equation. In this section, we shall consider the graphical method which is one of the simplest methods, although it does not ensure high accuracy. It consists in the following: we construct the graph of the function $y = f(x)$ and find the abscissas of the points of intersection of the graph with the x-axis. Thus, to solve the equation $ax^2 + bx + c = 0$, it suffices to construct the graph of the quadratic trinomial $y = ax^2 + bx + c$ and find the abscissas of the points of intersection of this graph and the x-axis.

But in most cases the above method of graphical solution of an equation is not very suitable. Thus, to find the roots of the equation $x^3 - 3x + 1 = 0$, it would be required to construct the graph of the function, and this is not an easy task. In such cases, the equation $f(x) = 0$ is transformed to the form $g(x) = h(x)$, and then the graphs of the functions $y = g(x)$ and $y = h(x)$ are sketched (if, of

course, it is simpler than to construct the graph of the function $y = f(x)$) and the abscissas of the points of intersection of the constructed graphs are found.

Thus, to solve the equation $x^3 - 3x + 1 = 0$, we can transform it to the form $x^3 = 3x - 1$, then construct the graphs of the functions $y = x^3$ and $y = 3x - 1$, and find the abscissas of the points of intersection of these graphs.

It is clear that the equation $f(x) = 0$ can be transformed to the form $g(x) = h(x)$ in different ways. For instance, the equation $x^2 - 5x - 3 = 0$ can be transformed to one of the following equations: $x^2 = 5x + 3$, $x^2 - 3 = 5x$, $x^2 - 5x = 3$. In the first case, we

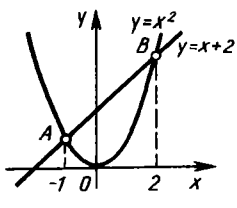


Fig. 131

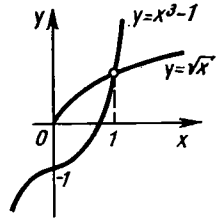


Fig. 132

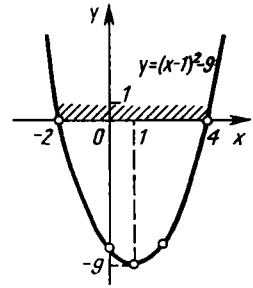


Fig. 133

have to construct the graphs of the functions $y = x^2$ and $y = 5x + 3$, in the second $y = x^2 - 3$ and $y = 5x$, and in the third, $y = x^2 - 5x$ and $y = 3$.

Example 1. Solve graphically the equation $x^2 - x - 2 = 0$.

○ It is advisable to rewrite the given equation in the form $x^2 = x + 2$. Now, the solution of the equation can be reduced to finding the abscissa of the points of intersection of the graphs of the functions $y = x^2$ and $y = x + 2$.

Figure 131 shows the graphs of the functions $y = x^2$ and $y = x + 2$ constructed in the same coordinate system. From the figure, we determine the abscissas of the points A and B of intersection of these graphs: $x_A = -1$, $x_B = 2$. Thus, the given equation has two roots: —1 and 2. ●

Example 2. Solve graphically the equation $\sqrt{x} = x^3 - 1$.

○ Construct the graphs of the functions $y = \sqrt{x}$ and $y = x^3 - 1$ in the same coordinate system (Fig. 132). They intersect at the point whose abscissa is approximately equal to 1.3. Hence, the given equation has only one solution $x \approx 1.3$. ●

2. Solving Inequalities. The graphical method of solving inequalities of the form $f(x) > 0$ consists in the following: we construct the graph of the function $y = f(x)$ and choose on the x-axis the inter-

vals in which the graph of the function lies above the axis of abscissas. If we have to solve an inequality of the form $f(x) < 0$, then we choose on the axis of abscissas the intervals on which the graph of the function lies below the axis of abscissas.

Example 1. Solve the inequality $-\frac{x}{2} + 4 < 0$ graphically.

○ The graph of the function $y = -\frac{x}{2} + 4$ (see Fig. 55) lies below the axis of abscissas if $x > 8$. Hence, the set of solutions of the inequality is the open interval $(8, +\infty)$. ●

Example 2. Solve the inequality $x^2 - 2x - 8 \leqslant 0$ graphically.

○ **First method.** Let us construct the graph of the function $y = x^2 - 2x - 8$. We have: $y = x^2 - 2x - 8 = (x^2 - 2x + 1) - 9 = (x - 1)^2 - 9$. Consequently, the vertex of the parabola is the point $(1, -9)$. The graph is represented in Fig. 133.

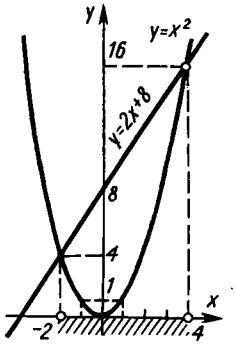


Fig. 134

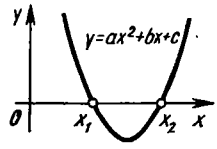


Fig. 135

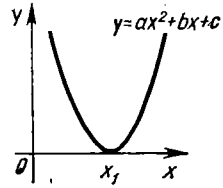


Fig. 136

We see that the graph intersects the x-axis at the points -2 and 4 and is below the x-axis for the values of x belonging to the interval $(-2, 4)$. As a result, we get that the solution of the given inequality is the closed interval $[-2, 4]$.

Second method. Transform the given inequality to the form $x^2 \leqslant 2x + 8$. Construct the graphs of the functions $y = x^2$ and $y = 2x + 8$ in the same coordinate system (Fig. 134). The inequality $x^2 \leqslant 2x + 8$ is fulfilled for all values of x for which the graph of the function $y = x^2$ lies not higher than the graph of the function $y = 2x + 8$. As a result, we get the answer $[-2, 4]$. ●

Let us dwell in more detail on solving *quadratic inequalities*, that is, inequalities of the form $ax^2 + bx + c > 0$ (or $ax^2 + bx + c < 0$), where $a \neq 0$.

Let $a > 0$. Then the graph of the function $y = ax^2 + bx + c$ is a parabola opened upward (Fir. 135). Suppose that the parabola intersects the x-axis at two points $(x_1, 0)$ and $(x_2, 0)$, where $x_1 < x_2$.

As is seen from the figure, the inequality $ax^2 + bx + c > 0$ is satisfied in two cases: when $x < x_1$ or when $x > x_2$. The solution of the inequality $ax^2 + bx + c < 0$ is the interval (x_1, x_2).

Let now the parabola have the vertex $(x_1, 0)$ lying on the axis of abscissas (Fig. 136). As is seen from the figure, in this case the inequality $ax^2 + bx + c \geqslant 0$ is fulfilled for any values of x. For the inequality $ax^2 + bx + c > 0$ the solution is $x < x_1$, $x > x_1$. In the case under consideration, the inequality $ax^2 + bx + c < 0$ has no solution.

Consider the third possible type of the parabola, when it opens upward and lies entirely above the axis of abscissas (Fig. 137). In

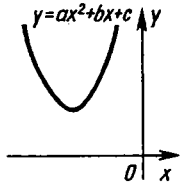


Fig. 137

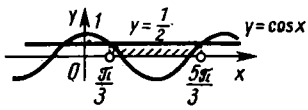


Fig. 138

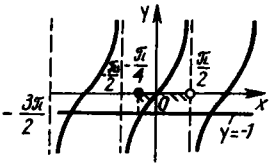


Fig. 139

this case, the inequality $ax^2 + bx + c > 0$ is satisfied for any values of x, the inequality $ax^2 + bx + c \leqslant 0$ has no solution.

If $a < 0$, then multiplying both sides of the inequality $ax^2 + bx + c > 0$ by -1, we get an equivalent inequality $-ax^2 - bx - c < 0$, where the coefficient of x^2 is positive, and this case was considered above.

Trigonometric inequalities solved earlier with the aid of the number circle (see § 3.7, Item 5) can also be solved graphically.

Example 3. Solve the inequality $\cos x < 1/2$.

○ We construct the graph of the function $y = \cos x$ and draw the straight line $y = 1/2$. Only the intervals of the x-axis on which the graph lies below the straight line $y = 1/2$ are of interest. One of such intervals is the open interval $(\pi/3, 5\pi/3)$ (it is hatched in Fig. 138). Taking advantage of periodicity of the function $y = \cos x$, we write the answer $\pi/3 + 2\pi k < x < 5\pi/3 + 2\pi k$. ●

Example 4. Solve the inequality $\tan x \geqslant -1$.

○ We construct the graph of the function $y = \tan x$ and draw the straight line $y = -1$. We are interested in the intervals of the x-axis on which the graph lies not lower than the straight line $y = -1$. One of such intervals is the interval $[-\pi/4, \pi/2)$ (it is hatched in Fig. 139). There are infinitely many such intervals, each of them, by virtue of periodicity of the function $y = \tan x$, being obtained from the interval $[-\pi/4, \pi/2)$ by shifting along the x-axis by πk. This enables us to write the solution in the following way: $-\pi/4 + \pi k \leqslant x < \pi/2 + \pi k$. ●

3. Solving Simultaneous Equations. In order to solve graphically two simultaneous equations in two variables, we have to construct the graphs of the given equations in the same coordinate system and find the coordinates of the points of intersection of these graphs.

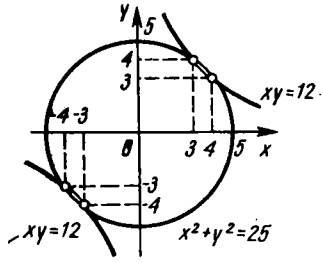


Fig. 140

Example. Solve graphically the simultaneous equations

$$\begin{cases} x^2+y^2=25, \\ xy=12. \end{cases}$$

○The graph of the equation $x^2 + y^2 = 25$ is a circle of radius 5 with centre at the origin. The graph of the equation $xy = 12$ is the hyperbola $y = 12/x$. Constructing these graphs in the same coordinate system (Fig. 140), we find the coordinates of the points A, B, C, D of intersection of the circle and the hyperbola A (4, 3), B (3, 4), C (— 4, —3), D (—3, —4). Hence the solutions of the given system are (4, 3), (3, 4) (—4, —3), and (—3, —4). ●

Exercises

1. Solve the following equations graphically: (a) $2x - 1 = -x + 2$, (b) $-3x + 1 = -3x + 2$, (c) $|x - 2| = 3$, (d) $|x - 2| = |x + 1|$, (e) $|x| = -x$, (f) $x^2 + 3x - 4 = 0$, (g) $x^2 = 5x - 8$, (h) $0.5x^2 = 7 - x$, (i) $\frac{2}{x} = x + 1$, (j) $\frac{x+1}{x-1} = 3x - 2$, (k) $\frac{1}{1-2x} = x^2$, (l) $\cos 2x = x^2 + 1$.

2. Solve graphically the given inequalities: (a) $x - 2 < 0$, (b) $7 - 2x > 0$.

3. Construct (in the same coordinate system) the graphs of the functions $y = 12/x$ and $y = x + 1$. Find with the aid of the constructed graphs for what values of x: (a) $12/x = x + 1$, (b) $12/x > x + 1$, (c) $12/x < x + 1$.

4. Prove graphically that the inequality $x^2 > x - 1$ is true for any value of x.

5. Prove graphically that the inequality $x^2 < x - 2$ has no solution.

6. Solve the following inequalities graphically: (a) $x^2 > -x + 12$, (b) $0.5x^2 < 2x$, (c) $x^2 - x + 4 < 0$.

7. Solve the following inequalities graphically: (a) $\sin x < 0$, (b) $\sin x > 1/2$, (c) $\sin x < \sqrt{2}/2$, (d) $\sin x > -1/2$, (e) $\cos x > 0$, (f) $\cos x > \sqrt{2}/2$ (g) $\cos x \leqslant -1/2$, (h) $\tan x > 0$, (i) $\tan x \geqslant \sqrt{3}$, (j) $\tan x < -1$, (k) $\cot x \leqslant 0$, (l) $\cot x > -\sqrt{3}/3$, (m) $|\cot x| < 1$.

8. Find the domain of definition for each of the given functions: (a) $y = \sqrt{-\cos x}$, (b) $y = \sqrt{\cot x - 1}$.

9. Solve the following systems of equations graphically:

(a) $\begin{cases} 3x-2y=-12, \\ x+2y=-4, \end{cases}$ (b) $\begin{cases} 2x+5y=10, \\ 4x+10y=15, \end{cases}$ (c) $\begin{cases} x+y=1, \\ 2x+2y=2, \end{cases}$

(d) $\begin{cases} x^2+y^2=25, \\ x-y=5, \end{cases}$ (e) $\begin{cases} x^2+y^2=36, \\ y-0.5x=0, \end{cases}$ (f) $\begin{cases} x^2+y^2=100, \\ y=0.5x^2, \end{cases}$ (g) $\begin{cases} y=x^2, \\ xy=8. \end{cases}$

10. Using the graphs, determine the number of roots of the following equations: (a) $x^3 = -2x + 1$, (b) $x^{-2} = 5$, (c) $\sqrt{x} = 5x - 1$, (d) $4x^{-1} = 7x$, (e) $\sqrt{x} = -5x^2 - 1$.

Chapter 4

ALGEBRAIC EQUATIONS AND SYSTEMS OF EQUATIONS

§ 4.1. Quadratic Equations

1. Quadratic Equation Defined. An equation of the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers, $a \neq 0$, and x is a variable, is called the *quadratic equation*. If $a = 1$, the quadratic equation is said to be *reduced*, if $a \neq 1$, then it is said to be *non-reduced*.

If at least one of the coefficients b or c is equal to zero, then the quadratic equation is said to be *incomplete*. Incomplete quadratic equations are solved by factorizing the left member of the equation into linear factors; if such a factorization is impossible, then the equation has no solution.

Example 1. Solve the equation $2x^2 - 5x = 0$.

○ We factor the left-hand member of the equation: $x(2x - 5) = 0$. Since the product is equal to zero, either the first factor or the second is equal to zero, that is, either $x = 0$ or $2x - 5 = 0$, whence we find $x = 2.5$. Thus, the given equation has two roots: 0 and 2.5. ●

Example 2. Solve the equation $x^2 - 7 = 0$.

○ We have $x^2 - (\sqrt{7})^2 = 0$, that is, $(x - \sqrt{7})(x + \sqrt{7}) = 0$. The equation has two roots: $\sqrt{7}$ and $-\sqrt{7}$. ●

2. Quadratic Formula. In the preceding subsection, we were busy solving incomplete quadratic equations. Now we shall consider a quadratic equation in the following *standard form*:

$$ax^2 + bx + c = 0, \tag{1}$$

where $a \neq 0$, $b \neq 0$, $c \neq 0$.

In the preceding chapter, when constructing the graph of a quadratic trinomial (see § 3.6, Item 4), we, by completing a square, transformed the quadratic trinomial $ax^2 + bx + c$ to the form

$$a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right).$$

Bearing this in mind, we rewrite equation (1) as follows:

$$a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) = 0. \tag{2}$$

Setting $b^2 - 4ac = D$ and dividing both sides of equation (2) by a number $a \neq 0$, we get the equation

$$\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2} = 0, \tag{3}$$

which is equivalent to equation (2) and, consequently, also to equation (1). Therefore, seeking for the roots of equation (3), we thereby find the roots of equation (1).

If $D < 0$, then for any x the left-hand side of equation (3) is positive, and, hence, this equation has no real roots. If $D = 0$, then it takes the form

$$\left(x + \frac{b}{2a}\right)^2 = 0,$$

whence we find $x = -\frac{b}{2a}$ which is the unique root of equation (3).

Let, finally, $D > 0$. Then the fraction $\frac{D}{4a^2}$ can be represented in the form $\left(\frac{\sqrt{D}}{2a}\right)^2$, and equation (3) rewritten as follows:

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{D}}{2a}\right)^2 = 0.$$

Factoring the difference of squares, we get

$$\left(x + \frac{b}{2a} - \frac{\sqrt{D}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{D}}{2a}\right) = 0,$$

which, in turn, leads to the collection of equations

$$x + \frac{b}{2a} - \frac{\sqrt{D}}{2a} = 0, \quad x + \frac{b}{2a} + \frac{\sqrt{D}}{2a} = 0.$$

From the first equation of the collection we find $x = \frac{-b + \sqrt{D}}{2a}$, from the second we get $x = \frac{-b - \sqrt{D}}{2a}$.

We usually use the following notation:

$$x = \frac{-b \pm \sqrt{D}}{2a} \text{ or } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{4}$$

This is the **quadratic formula.**

Note that, setting $D = 0$ in formula (4), we get $x = -\frac{b}{2a}$. This means that formula (4) can be used not only when $D > 0$, but also when $D = 0$.

The expression $D = b^2 - 4ac$ is called the *discriminant* of the quadratic equation (1). Thus, we have seen that if $D < 0$, then an equation has no roots; if $D = 0$, then an equation has one root; and if $D > 0$, an equation has two roots.

Sometimes, formula (4) is written in a somewhat different form:

$$x=\frac{-\frac{b}{2}\pm\frac{\sqrt{D}}{2}}{a} \quad \text{or} \quad x=\frac{-\frac{b}{2}\pm\sqrt{\frac{D}{4}}}{a}. \tag{5}$$

Here

$$\frac{D}{4}=\frac{b^2-4ac}{4}=\frac{b^2}{4}-ac.$$

If we set $b = 2k$, this formula takes the form

$$x=\frac{-k\pm\sqrt{k^2-ac}}{a}.$$

Formula (5) is especially useful when $-b/2$ is a whole number, that is, when the coefficient b is an even number.

Example 1. Solve the equation $2x^2 - 5x + 2 = 0$.

○ Here $a = 2$, $b = -5$, $c = 2$. We have $D = b^2 - 4ac = (-5)^2 - 4\cdot 2\cdot 2 = 9$. Since $D > 0$, the equation has two roots which are found by formula (4):

$$x=\frac{-b\pm\sqrt{D}}{2a}=\frac{5\pm\sqrt{9}}{4}=\frac{5\pm 3}{4}.$$

Thus, $x_1 = (5 + 3)/4 = 2$, $x_2 = (5 - 3)/4 = 1/2$ are roots of the given equation. ●

Example 2. Solve the equation $x^2 - 6x + 9 = 0$.

○ Here $a = 1$, $b = -6$, $c = 9$. We have $D = (-6)^2 - 4\cdot 1\cdot 9 = 0$. By formula (4), we find

$$x=\frac{6\pm\sqrt{0}}{2}=\frac{6}{2}=3.$$

Thus, the equation has only one root $x = 3$. ●

Note that the last equation can be solved without using formula (4). Since $x^2 - 6x + 9 = (x - 3)^2$, the given equation can be rewritten as $(x - 3)^2 = 0$, whence we find $x = 3$.

Example 3. Solve the equation $2x^2 + 8x + 19 = 0$.

○ Here, $a = 2$, $b = 8$, and $c = 19$. Since b is an even number, it is more convenient to find $\frac{D}{4}$ rather than D. We have $\frac{D}{4}=\left(\frac{b}{2}\right)^2-ac=\left(\frac{8}{2}\right)^2-2\cdot 19=-22$. Since $D<0$, the equation has no real roots. ●

Note that incomplete quadratic equations can also be solved by formulas (4) and (5). As an example, consider the equation $2x^2 - 5x = 0$. Here, $a = 2$, $b = -5$, $c = 0$, and $D = 25$. Hence, $x=\frac{5\pm\sqrt{25}}{4}=\frac{5\pm 5}{4}$, that is, $x_1 = 2.5$ and $x_2 = 0$.

3. Viète's Theorem. Let us derive the formulas relating the roots of a quadratic equation with its coefficients; these formulas were originally obtained by the French mathematician F. Viète.

Let the discriminant of the quadratic equation $ax^2 + bx + c = 0$ be positive. Then, as we know, the equation has two roots:

$$x_1 = \frac{-b+\sqrt{D}}{2a}, \quad x_2 = \frac{-b-\sqrt{D}}{2a}.$$

We find the sum and the product of the roots:

$$x_1 + x_2 = \frac{-b+\sqrt{D}}{2k} + \frac{-b-\sqrt{D}}{2a} = \frac{-b+\sqrt{D}-b-\sqrt{D}}{2a} = -\frac{b}{a},$$

$$x_1 \cdot x_2 = \frac{-b+\sqrt{D}}{2a} \cdot \frac{-b-\sqrt{D}}{2a}$$

$$= \frac{(-b)^2-(\sqrt{D})^2}{4a^2} = \frac{b^2-D}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2} = \frac{c}{a}.$$

Thus,

$$x_1 + x_2 = -\frac{b}{a}, \quad x_1 x_2 = \frac{c}{a}. \tag{6}$$

For $D=0$ the quadratic equation has one root: $x = -\frac{b}{2a}$. It is common to assume that for $D=0$ the equation has two equal roots: $x_1 = x_2 = -\frac{b}{2a}$. Then

$$x_1 + x_2 = -\frac{b}{a},$$

$$x_1 x_2 = \frac{b^2}{4a^2} = \frac{b^2-D}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2} = \frac{c}{a}.$$

Consequently, the formulas for the sum and the product of the roots of a quadratic equation hold true for any equation having roots. Thereby the following theorem has been proved.

Theorem 4.1.1 (Viète's Theorem). *If a quadratic equation* $ax^2 + bx + c = 0$ *has real roots, then their sum is equal to* $-b/a$, *while the product of the roots to* c/a.

Corollary. *If a reduced quadratic equation* $x^2 + px + q = 0$ *has real roots* x_1 *and* x_2, *then*

$$x_1 + x_2 = -p, \quad x_1 x_2 = q, \tag{7}$$

that is, the sum of the roots of a reduced quadratic equation is equal to the second coefficient taken with the opposite sign, and the product of the roots is equal to the constant term.

We derive some more relationships between the roots and the coefficients of the reduced quadratic equation $x^2 + px + q = 0$. We

have

$$x_1^2 + x_2^2 = (x_1^2 + 2x_1x_2 + x_2^2) - 2x_1x_2 = (x_1 + x_2)^2 - 2x_1x_2.$$

Using formulas (7), we get

$$x_1^2 + x_2^2 = p^2 - 2q. \tag{8}$$

Now we consider the sum of the cubes of the roots. We have

$$x_1^3 + x_2^3 = (x_1 + x_2)(x_1^2 - x_1x_2 + x_2^2) = (x_1 + x_2)((x_1^2 + x_2^2) - x_1x_2).$$

Making use of equalities (7) and (8), we get

$$x_1^3 + x_2^3 = -p(p^2 - 3q). \tag{9}$$

The converse of Viète's theorem is also true.

Theorem 4.1.2. *If the numbers x_1 and x_2 are such that $x_1 + x_2 = -b/a$ and $x_1x_2 = c/a$, then x_1 and x_2 are roots of the quadratic equation $ax^2 + bx + c = 0$.*

□ We have $x_2 = -\frac{b}{a} - x_1$. Substituting this expression into the equality $x_1x_2 = \frac{c}{a}$, we get $x_1\left(-\frac{b}{a} - x_1\right) = \frac{c}{a}$ and, further, $-\frac{bx_1}{a} - x_1^2 = \frac{c}{a}$, whence $ax_1^2 + bx_1 + c = 0$. The last equality means that x_1 is a root of the equation $ax^2 + bx + c = 0$. It is proved in a similar manner that x_2 is also a root of equation $ax^2 + bx + c = 0$. ■

Theorems 4.1.1 and 4.1.2 are widely used. For instance, in many cases, with their aid, it is possible to find orally the roots of a reduced quadratic equation.

Example 1. Solve the following equations:

(1) $x^2 - 7x + 12 = 0$, (2) $x^2 + 11x + 30 = 0$, (3) $x^2 + 2x - 15 = 0$.

○(1) Let x_1 and x_2 be roots of the given equation (suppose that they exist). Then $x_1x_2 = 12$, that is, x_1 and x_2 are numbers having the same sign. Further, $x_1 + x_2 = 7$, and, hence, x_1 and x_2 are positive numbers. Thus, we have to find two positive numbers whose sum is equal to 7, and the product to 12. It is easy to guess that these numbers are 3 and 4. Thus, the roots of the given equations are $x_1 = 3$ and $x_2 = 4$.

(2) We have $x_1x_2 = 30$, and, hence, x_1 and x_2 are numbers having the same sign. Further, $x_1 + x_2 = -11$, therefore x_1 and x_2 are negative numbers. The sum of what two negative numbers is -11, their product being equal to 30? These numbers are -5 and -6. Thus, the roots of the given equation are $x_1 = -5$ and $x_2 = -6$.

(3) Here $x_1x_2 = -15$, that is, x_1 and x_2 are numbers having opposite signs. Further, $x_1 + x_2 = -2$, and, hence, the absolute

value of the negative root exceeds the positive root by 2. It is easy to figure out that the roots of the given equation are $x_1 = -5$ and $x_2 = 3$. ●

Example 2. Set up a quadratic equation with the given numbers as its roots: (1) 7 and -3, (2) $1\frac{2}{5}$ and $-\frac{5}{7}$.

○(1) We shall seek the desired equation in the form $x^2 + px + q = 0$. We have $p = -(7 - 3) = -4$, $q = 7 \cdot (-3) = -21$, then $x^2 - 4x - 21 = 0$.

(2) We shall look for the desired equation in the form $x^2 + px + q = 0$. We have $p = -\left(1\frac{2}{5} - \frac{5}{7}\right) = -\left(\frac{7}{5} - \frac{5}{7}\right) = -\frac{24}{35}$, $q = 1\frac{2}{5} \cdot \left(-\frac{5}{7}\right) = -\frac{7}{5} \cdot \frac{5}{7} = -1$. Then we get $x^2 - \frac{24}{35}x - 1 = 0$ or $35x^2 - 24x - 35 = 0$. ●

Example 3. Let α and β be roots of the equation $2x^2 - 3x - 7 = 0$. Without computing α and β, set up a quadratic equation with α/β and β/α as its roots.

○We shall seek the required equation in the form $x^2 + px + q = 0$, then

$$p = -\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) = -\frac{\alpha^2 + \beta^2}{\alpha\beta}, \quad q = \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 1.$$

By hypothesis, α and β are roots of the equation $2x^2 - 3x - 7 = 0$ or $x^2 - \frac{3}{2}x - \frac{7}{2} = 0$; then $\alpha\beta = -\frac{7}{2}$, $\alpha + \beta = \frac{3}{2}$. Applying formula (8), we get

$$\alpha^2 + \beta^2 = \left(\frac{3}{2}\right)^2 - 2\left(-\frac{7}{2}\right) = \frac{9}{4} + 7 = \frac{37}{4},$$

hence,

$$p = -\frac{\alpha^2 + \beta^2}{\alpha\beta} = -\frac{37/4}{-7/2} = \frac{37 \cdot 2}{4 \cdot 7} = \frac{37}{14}.$$

Thus, the coefficients p and q of the desired quadratic equation have been found: $p = 37/14$ and $q = 1$. The equation has the form

$$x^2 + \frac{37}{14}x + 1 = 0 \text{ or } 14x^2 + 37x + 14 = 0. \quad ●$$

4. Factoring Quadratic Trinomials. Let there be given a quadratic trinomial $ax^2 + bx + c$, and x_1 and x_2 be roots of the equation $ax^2 + bx + c = 0$. Then, by Viète's theorem, $x_1 + x_2 = -b/a$, $x_1x_2 = c/a$. Bearing this in mind, we perform the following transformations of the quadratic trinomial:

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 - (x_1 + x_2)x + x_1x_2\right) \\ &= a\left((x^2 - xx_1) - (xx_2 - x_1x_2)\right) = a\left(x(x - x_1) - x_2(x - x_1)\right) \\ &= a(x - x_1)(x - x_2). \end{aligned}$$

Thus,

$$ax^2 + bx + c = a(x - x_1)(x - x_2), \tag{10}$$

where x_1 and x_2 are roots of the trinomial $ax^2 + bx + c$.

The obtained identity (10) is called the **formula for factoring quadratic trinomials.**

Example. Factor the given quadratic trinomials: (1) $2x^2 - 5x + 2$, (2) $4x^2 + 4x + 1$, (3) $x^2 - 8x - 33$, (4) $3x^2 + 3x + 8$.

○(1) From the equation $2x^2 - 5x + 2 = 0$ we find $x_1 = 2$ and $x_2 = 0.5$. Consequently,

$$2x^2 - 5x + 2 = 2(x - 2)(x - 0.5) = (x - 2)(2x - 1).$$

(2) From the equation $4x^2 + 4x + 1 = 0$ we find $x_1 = x_2 = -0.5$. Therefore

$$4x^2 + 4x + 1 = 4(x + 0.5)(x + 0.5) = (2x + 1)^2.$$

(3) From the equation $x^2 - 8x - 33 = 0$ we find $x_1 = 11$ and $x_2 = -3$. Thus,

$$x^2 - 8x - 33 = 1 \cdot (x - 11)(x + 3) = (x - 11)(x + 3).$$

(4) The equation $3x^2 + 3x + 8 = 0$ has no real roots ($D = 3^2 - 4 \cdot 3 \cdot 8 < 0$). Hence, the trinomial $3x^2 + 3x + 8$ is not factorable into linear factors. ●

5. Equations Containing a Variable in the Denominator. Consider the equation

$$\frac{P(x)}{Q(x)} = 0, \tag{11}$$

where $P(x)$ and $Q(x)$ are polynomials of degree one or two, and also equations reducing to the indicated form.

The solution of an equation of form (11) is based on the following assertion: *a fraction m/n is equal to zero if and only if its numerator is zero and the denominator is a numeral other than zero* (it is impossible to divide by zero!). This is written in the following manner: $\begin{cases} m = 0, \\ n \neq 0. \end{cases}$

In accordance with the aforesaid, the solution of equation (11) is reduced to solving the following mixed system:

$$\begin{cases} P(x) = 0, \\ Q(x) \neq 0. \end{cases}$$

For this purpose, we have first to solve the equation $P(x) = 0$ and then to see whether the statement $Q(x) \neq 0$ is true for the found value of the variable x (for each root). If this statement is true, then the found root of the equation $P(x) = 0$ is also a root of equation (11). If this statement is false (that is, the statement $Q(x) = 0$

is true), then the obtained root of the equation $P(x) = 0$ is not a root of equation (11).

Example 1. Solve the equation $\frac{2x-3}{3x+5} = 0$.

○Taking advantage of the above condition that the fraction is equal to zero, we get the system $\begin{cases} 2x-3=0, \\ 3x+5 \neq 0. \end{cases}$ From the equation $2x - 3 = 0$ we find $x = 1.5$. Since the statement $3 \cdot 1.5 + 5 \neq 0$ is true, $x = 1.5$ is a root of the given equation.

Example 2. Solve the equation $\frac{3x-6}{x^2-x-2} = 0$.

○We have $\begin{cases} 3x-6=0, \\ x^2-x-2 \neq 0. \end{cases}$ From the equation $3x - 6 = 0$ we find $x = 2$. The statement $2^2 - 2 - 2 \neq 0$ is false (for the found value of the variable $x = 2$ the denominator vanishes); hence, the given equation has no solution. ●

Example 3. Solve the equation $\frac{2}{2-x} + \frac{1}{2} = \frac{4}{2x-x^2}$.

○We transpose all the terms of the equation to the left-hand side and reduce the obtained expression to a fraction:

$$\frac{2}{2-x} + \frac{1}{2} - \frac{4}{x(2-x)} = 0, \quad \frac{4x+x(2-x)-8}{2x(2-x)} = 0, \quad \frac{x^2-6x+8}{2x(2-x)} = 0.$$

From the equation $x^2 - 6x + 8 = 0$ we find $x = 2$, $x = 4$. For $x = 2$ the denominator vanishes; hence 2 is not a root of the equation. Thus, we get the answer: 4. ●

6. Equations with a Parameter. Let there be given an equality with variables x and a:

$$f(x, a) = 0. \tag{12}$$

It can be regarded as a problem of finding all such pairs (x, a) which satisfy equality (12) (that is, turn an equality with variables into a true numerical equality). In this case, we have an equation in two variables x and a. If a fixed value is assigned to the variable a, then equality (12) will turn into an equation in one variable x.

If the problem is to solve equation (12) for each real value of a for x, then equation (12) is said to be an equation with a variable x and a *parameter* a.

To solve an equation with a parameter a means to find all the values of x, satisfying equation (12), for every real value of a.

Example 1. Solve the equation

$$2a(a-2)x = a - 2. \tag{13}$$

○Consider first the values of the parameter which turn the coefficient of x into zero. For these values of the parameter the division of both sides of the equation by the coefficient of x is impossible, while for other values of the parameter such a division is possible.

Such values are $a = 0$ and $a = 2$. For $a = 0$ equation (13) takes the form $0 \cdot x = -2$. This equation has no solution. For $a = 2$ equation (13) takes the form $0 \cdot x = 0$; any real number serves as its solution. For $a \neq 0$ and $a \neq 2$ equation (13) can be transformed to the form $x = \frac{a-2}{2a(a-2)}$, whence we find $x = \frac{1}{2a}$.

Thus, if $a = 0$, then the equation has no solution; if $a = 2$, then its solution is any real number; if $a \neq 0$ and $a \neq 2$, then $x = 1/(2a)$. ●

Example 2. Solve the equation

$$\frac{3x^2-2}{a^2+3a} + \frac{x-1}{a+3} + \frac{2}{a} = 0. \tag{14}$$

○Note that for $a = 0$ and $a = -3$ the given equation has no sense. If $a \neq 0$ and $a \neq -3$, then on clearing the fractions and collecting like terms, equation (14) takes the form

$$3x^2 + ax + a + 4 = 0. \tag{15}$$

To get the solutions of the above quadratic equation, it is reasonable to pick out those values of the parameter for which the discriminant D vanishes. We have $D = a^2 - 12(a + 4) = a^2 - 12a - 48$. Equating the discriminant to zero, we find $a_1 = 6 - 2\sqrt{21}$; $a^3 = 6 + 2\sqrt{21}$. The quadratic trinomial $a^2 - 12a - 48$ takes on negative values in the interval (a_1, a_2) (see § 3.9, Item 2), that is, $D < 0$ if $6 - 2\sqrt{21} < a < 6 + 2\sqrt{21}$. Thus, for $a \neq 0$ and $a \neq -3$ equation (15) must be considered in each particular case forming the following collection:

$$6 - 2\sqrt{21} < a < 6 + 2\sqrt{21};\quad a \leqslant 6 - 2\sqrt{21}:\quad a \geqslant 6 + 2\sqrt{21}.$$

In the first case equation (15), and, hence, equation (14) have no real solutions. In the second and third cases, we find from equation (15)

$$x = \frac{-a \pm \sqrt{a^2 - 12a - 48}}{6}.$$

Note that the values $a = 0$ and $a = -3$ satisfy the inequality $6 - 2\sqrt{21} < a < 6 + 2\sqrt{21}$. This enables us to write the answer in a more compact form.

Thus, if $6 - 2\sqrt{21} < a < 6 + 2\sqrt{21}$, then there is no solution; if $a \leqslant 6 - 2\sqrt{21}$, $a \geqslant 6 + 2\sqrt{21}$, then $x = \frac{-a \pm \sqrt{a^2 - 12a - 48}}{6}$. ●

Example 3. Solve the equation

$$(a - 1)x^2 + 2(2a + 1)x + 4a + 3 = 0. \tag{16}$$

○Let us pick out the value of the parameter $a = 1$. The thing is that for $a = 1$ the given equation is not quadratic, while for $a \neq 1$ it will be quadratic. Hence, in each of these cases it must be

solved in a different way. For $a = 1$ equation (16) takes the form $6x + 7 = 0$, whence we find $x = -7/6$. In the case $a \neq 1$, we select those values of the parameter for equation (16) for which the discriminant of the equation vanishes. We have $D = 5a + 4$. Hence, $a = -4/5$ is the value of the parameter worthy of attention.

If $a < -4/5$, then $D < 0$, and, consequently, the equation has no solution, if $a \geqslant -4/5$ and $a \neq 1$, then $D \geqslant 0$, and we get $x = \dfrac{-(2a+1) \pm \sqrt{5a+4}}{a-1}$.

Thus, if $a < -4/5$, then there is no solution; if $a = 1$, then $x = -7/6$; if $a \geqslant -4/5$ and $a \neq 1$, then $x = \dfrac{-(2a+1) \pm \sqrt{5a+4}}{a-1}$. ●

The examples considered show that there is no unified method for solving equations with a parameter. Here, much depends on the student's keenness of observation and quickness of wit. The solution of an equation with a parameter differs actually from that of an equation without a parameter only in that at each step of the solution the student must carefully follow each operation from the point of view of its feasibility for different values of the parameter.

7. Problems on Setting Up Quadratic Equations. In the process of solution, many practical problems reduce to quadratic equations or to equations reducible to quadratic.

Example 1. To transport 60 tons of load from one place to another, a certain number of trucks were required. Since the road was in bad repair, each truck's load was reduced by 0.5 ton. Therefore, another four trucks were necessary to carry out this job. How many trucks were required originally?

○ Let x denote the number of trucks required originally. Then $x + 4$ trucks actually took part in transporting the load. Since a total of 60 tons were to be transported, each truck was supposed to carry $60/x$ tons whereas, in fact, they carried 0.5 ton less than it had been supposed, that is, $60/(x + 4)$ tons. As a result, we arrive at the equation

$$\frac{60}{x} - \frac{60}{x+4} = 0.5.$$

This equation has two roots: -24 and 20. It is clear that -24 does not suit here. Thus, 20 trucks were required originally. ●

Example 2. A motor-boat moving at a speed of 20 km/h covered the distance between two points in 6 hours and 15 minutes without stopping. The distance between the points is 60 km. Determine the flow rate of the river.

○ Let x km/h be the flow rate of the river. The motor-boat moving at a speed of 20 km/h goes with the current at a speed of $(20 + x)$ km/h and against the current at a speed of $(20 - x)$ km/h. The time required to cover the path with the current is then $60/(20 + x)$ hours,

while the time required to cover the path against the current is $60/(20-x)$ hours. Since it took the motor-boat 6 hours and 15 minutes to go there and back we arrive at the following equation

$$\frac{60}{20+x}+\frac{60}{20-x}=\frac{25}{4}.$$

Solving this equation, we find two roots: $x = 4$ and $x = -4$. It is clear that the value $x = -4$ does not suit here. Thus, the flow rate of the river is 4 km/h. ●

Example 3. A 54-litre vessel was filled with acid. Several litres of the acid were poured out of the vessel and then the vessel was topped up with water. Then the same amount of the solution was poured out of the vessel again. As a result, the vessel turned out to contain a solution corresponding to 24 litres of acid at the initial concentration. How much acid was originally poured out of the vessel?

○ Let x litres of acid be poured out of the vessel for the first time. Then $(54 - x)$ litres of acid remained. The vessel filled up with water contained 54 litres of solution with $(54 - x)$ litres of acid solved in it. Hence, 1 litre of solution contains $\frac{54-x}{54}$ litres of acid (concentration of the solution). For the second time, x litres of solution were poured out of the vessel; this amount of solution contained $\left(\frac{54-x}{54}\cdot x\right)$ litres of acid. Thus, for the first time, x litres of acid were poured out of the vessel, and for the second time $\frac{54-x}{54}\cdot x$ litres, and the total amount of acid poured out of the vessel is $54-24 = 30$ litres. As a result, we get the following equation:

$$x+\frac{54-x}{54}\cdot x=30.$$

Having solved this equation, we find two roots $x = 90$ and $x = 18$. It is clear that the value $x = 90$ does not satisfy the conditions of the problem. Consequently, 18 litres of acid were poured out of the vessel for the first time. ●

Example 4. Two workers, working together, performed a job in 6 hours. The first of them, working alone, could have completed the whole job 5 hours earlier than the second worker could have done working alone. How much time would each worker need to complete the whole job alone?

○ Prior to solving this problem (or other similar problems "on joint work"), we should like to note the following: productivity of labour A (that is, the job done by a worker per unit time) and time t necessary to complete the whole job are mutually inverse quantities, that is, $At = 1$. Therefore, if we denote the time necessary for the first worker to complete the whole job by x hours and by $x + 5$ hours for the second worker, then the portion of the job carried out

by the first worker per hour is $1/x$, and by the second worker $1/(x+5)$. By hypothesis, working together, they completed the whole job in 6 hours. The job carried out by the first worker amounts to $6/x$, and by the second worker to $6/(x+5)$.

Thus, we get the equation $\frac{6}{x}+\frac{6}{x+5}=1$, whose solution is $x=10$.

Hence, it takes the first worker 10 hours and the second worker 15 hours to complete the whole job. ●

Exercises

In Exercises 1 to 11, solve the given equations:

1. (a) $9x^2-361=0$, (b) $0.0009x^2=0.6084$.

2. (a) $7\frac{1}{4}-\frac{3}{5}x^2=3\frac{1}{2}$, (b) $4x^2+5x=9x^2-15x$,

(c) $\frac{5x^2+9}{6}-\frac{4x^2-9}{5}=3$.

3. (a) $\frac{x+3}{x-3}+\frac{x-3}{x+3}=3\frac{1}{3}$, (b) $x^2-|x|=0$.

4. (a) $x^2+12x-64=0$, (b) $x^2+14x+24=0$,
(c) $x^2-x-12=0$, (d) $4x^2+12x+9=0$,
(e) $10x^2-x+1=0$.

5. $1-\frac{3-2x}{5-x}=\frac{3}{3-x}-\frac{x+3}{x+1}$.

6. $\frac{x+1}{4x}-\frac{5x-1}{2x-4}=\frac{8-x}{3x^2-6x}-\frac{x-5}{x-2}$.

7. $\frac{30}{x^2-1}-\frac{13}{x^2+x+1}=\frac{7+18x}{x^3-1}$.

8. $\frac{1}{4x+8}=\frac{20x+1}{4x^2-16}-\frac{7-5x}{x^2-4x+4}$.

9. $\frac{5}{x^2-4}-\frac{8}{x^2-1}=\frac{2}{x^2-3x+2}-\frac{20}{x^2+3x+2}$.

10. $\frac{1}{x-8}-\frac{1}{x-2}=\frac{1}{x-11}-\frac{1}{x-10}$.

11. (a) $x^4-13x^2+36=0$, (b) $3x^4=28x^2-9$, (c) $7x^2=3x^4+2$.

12. Without solving the given equation, determine the number of its roots: (a) $x^2-4x+3=0$, (b) $x^2+7x+15=0$, (c) $x^2-2x+5=0$, (d) $x^2-10x+25=0$, (e) $3x^2+x-1=0$, (f) $4x^2+6x+9=0$, (g) $4x^2-12x+9=0$.

13. Determine for what values of a the given equation has two equal roots: (a) $9x^2+6x+a=0$, (b) $x^2+2(a-4)x+a^2+6a+3=0$, (c) $(2+a)x^2+6ax+4a+1=0$.

14. Determine for what value of k the equation: (a) $x^2+kx+15=0$ has a root equal to 5, (b) $x^2+kx-24=0$ has a root equal to -3, (c) $kx^2+12x-3=0$ has a root equal to 1/5, find the second root of the given equation.

15. Set up a quadratic equation having two roots x_1 and x_2 if: (a) $x_1=-2$, $x_2=8$, (b) $x_1=3/4$, $x_2=1/2$, (c) $x_1=\sqrt{2}$, $x_2=\sqrt{3}$, (d) $x_1=(3+2\sqrt{5})/3$, $x_2=(3-2\sqrt{5})/3$, (e) $x_1=(m+n)/2$, $x_2=(m-n)/2$.

16. Factorize the following quadratic trinomials: (a) $x^2 - 2x - 35$, (b) $a^2 - 29a + 198$, (c) $m^2 - m - 12$, (d) $5x^2 + 17x - 126$, (e) $2y^2 - y - 3$.

17. Reduce the given fractions:

(a) $\dfrac{x^2+6x-91}{x^2+8x-105}$, (b) $\dfrac{2a^2+8a-90}{3a^2-36a+105}$, (c) $\dfrac{b^2-9bx+14x^2}{b^2-bx-2x^2}$.

18. Without solving the equation $x^2 + 3x - 15 = 0$, find: (a) the sum of the squares of its roots, (b) the sum of the cubes of its roots.

19. Set up a quadratic equation whose roots would exceed those of the equation $2x^2 - 5x + 2 = 0$ by unity.

20. Set up a quadratic equation whose roots would be equal to the squares of the roots of the equation $5x^2 - x - 2 = 0$.

21. Set up a quadratic equation whose roots would be inverse of the roots of the equation $2x^2 - 8x + 3 = 0$.

22. Solve the equation $x^2 - 8x + c = 0$ knowing that the sum of the squares of its roots is equal to 34.

23. Find the domains of the given functions:

(a) $y = \sqrt{x^2+20x+100}$, (b) $y = \dfrac{1}{\sqrt{45-4x-x^2}}$.

24. Construct the graph of the function $y = \dfrac{-2x^2+7x-3}{2x-1}$.

In Exercises 25 to 31, solve the given equations with parameter a:

25. $(a^2-1)\,x-(2a^2+a-3)=0$. **26.** $a=\dfrac{1}{a}+\dfrac{a-1}{a\,(x-1)}$.

27. $\dfrac{x}{2a+x}-\dfrac{2a+x}{x-2a}=\dfrac{16a^2}{4a^2-x^2}$. **28.** $(a+3)\,x^2-(3a+1)\,x+a=0$.

29. $3x^2-5ax-2a^2=0$. **30.** $\dfrac{x+2}{a+1}=\dfrac{2x-a-1}{x-2}$.

31. $\dfrac{x}{x-a}-\dfrac{2a}{x+a}=\dfrac{8a^2}{x^2-a^2}$.

32. A motor-boat on a river took five hours to travel 42 km with the current and 20 km against the current. Find the speed of the motor-boat if it is known that the flow rate of the river was 2 km/h everywhere along the route.

33. Having increased the speed of the train by 10 km/h, they succeeded in reducing the time necessary for the train to cover the path 720 km long by an hour. Find the original speed of the train.

34. When driven along a road, a vehicle covered 80 m. During this time, its front wheel revolved eight times more than the rear wheel. Find the circumference of each wheel if the circumference of the front wheel is 0.5 m less than that of the rear wheel.

35. One tractor team had to plough 240 hectares of soil, the other by 35% more than the first one. Ploughing daily by 3 hectares less than the second team, the first team, nevertheless, completed the job two days earlier than the second team. How many hectares did each team plough daily?

36. Two typists were given a manuscript to retype. After two hours working together one was given another task, while the second typist completed the job alone in 80 minutes. How much time would it take each typist to retype the entire manuscript if the first typist could have completed the job 70 minutes earlier than the second could have done?

37. A rectangular lawn is located within a second fenced rectangular plot 150 m long by 110 m wide, the lawn's edges being equidistant from the fence. Find the length and width of the lawn if its area is 4/11 of the area of the plot.

38. A box (without a cover) has the shape of a rectangular parallelepiped with a square base. The area of its bottom exceeds the area of the surface of

its lateral walls by 176 cm². The height of the box is 10 cm. Find the area of its bottom.

39. A 12 cm × 18 cm photograph is put into a frame of constant width. Find the width of the frame if its area is equal to the area of the photograph.

40. By what number should 73 be divided so that the quotient is by 3 greater than the divisor, the remainder being less than the quotient by 7.

41. The price of a camera was twice reduced by the same percentage, the overall reduction being from 300 to 192 roubles. By how many per cent was the price of the camera cut each time?

42. The distance between stations A and B is 240 km. A train left B for A and half an hour later another train left A for B with a speed exceeding that of the first train by 12 km/h. Find the speed of each train if it is known that they met at the midpoint between A and B.

43. A motor-boat with tourists started from a landing upstream the river planning to return in six hours after the departure. The speed of the boat in stagnant water is 12 km/h. What is the rate of flow of the river if the boat covered a distance of 22.5 km, made a two hours' stop, and returned to the same landing by the assigned time?

44. A tank is filled up with water through two pipes during 2 hours and 55 minutes. The first pipe alone can fill up the tank two hours faster than the second. How much time would it take each pipe to fill up the tank?

§ 4.2. Equations of Higher Degrees

In this section, we shall consider in short the solution of equations of the form $P(x) = 0$, where $P(x)$ is a polynomial of a degree higher than two and also the solution of equations reducible to the indicated form. We shall consider two principal methods for solving equations of the form $P(x) = 0$: (1) factoring the left-hand side of an equation and (2) introducing new variables. Note that there are common methods for solving equations of the third and fourth degree (the formulas of roots), but we shall not consider them here.

1. Factoring. It consists in the following. Suppose we have to solve an equation $P(x) = 0$, where $P(x)$ is a polynomial of degree $n > 2$. Assume that we have succeeded in factoring this polynomial: $P(x) = P_1(x) \cdot P_2(x) \ldots P_k(x)$, where $P_1(x)$, $P_2(x)$, . . ., $P_k(x)$ are polynomials of power lower than n. Then the equation $P(x) = 0$ will take the form

$$P_1(x) \cdot P_2(x) \ldots P_k(x) = 0$$

and will be equivalent to the collection of equations

$$P_1(x) = 0, \quad P_2(x) = 0, \quad \ldots,, \quad P_k(x) = 0.$$

To factorize a polynomial $P(x)$, we shall use the technique mentioned above (see § 2.1), it will be the method of selection of integral roots.

Example 1. Solve the equation $x^3 + 2x^2 + 3x + 6 = 0$.

○ We factor the left-hand side of the equation:

$$x^2(x + 2) + 3(x + 2) = 0 \quad \text{or} \quad (x + 2)(x^2 + 3) = 0.$$

The last equation is equivalent to the collection of equations: $x + 2 = 0$; $x^2 + 3 = 0$. From the first equation we find $x = -2$, while the second equation has no real roots. Thus, we get the answer: $x = -2$. ●

Example 2. Solve the equation $x^3 + 4x^2 - 24 = 0$.

○ Let us try to find an integral root of the given equation. It was established above (see § 2.1, Item 3) that only a divisor of a constant term can be the integral root of a polynomial with integral coefficients. Let us write the divisors of the constant term: ± 1, ± 2, ± 3, ± 4, ± 6, ± 12, ± 24. Substituting the value $x = 1$ into the given equation, we get $1^3 + 4 \cdot 1^2 - 24 \neq 0$; thus, $x = 1$ is not a root of the equation. Further, for $x = -1$ and $x = 2$ we have: $(-1)^3 + 4(-1)^2 - 24 \neq 0$ and $2^3 + 4 \cdot 2^2 - 24 = 0$, respectively. Thus, $x_1 = 2$ is the integral root of the equation.

We know (see § 2.1, Item 4) that if a is a root of the polynomial $P(x)$, then $P(x)$ is divisible by $x - a$. With this fact in view, we divide the polynomial $x^3 + 4x^2 - 24$ by $x - 2$:

$$
\begin{array}{r|l}
x-2 & x^2+6x+12 \\
\cline{2-2}
 & x^3+4x^2-24 \\
 & \underline{x^3-2x^2} \\
 & \quad 6x^2 \\
 & \quad \underline{6x^2-12x} \\
 & \qquad 12x-24 \\
 & \qquad \underline{12x-24} \\
 & \qquad\quad 0
\end{array}
$$

Hence, $x^3 + 4x^2 - 24 = (x - 2)(x^2 + 6x + 12)$, and, consequently, the original equation takes the form

$$(x - 2)(x^2 + 6x + 12) = 0.$$

This equation is equivalent to the collection of equations (one of which has been already solved): $x - 2 = 0$, $x^2 + 6x + 12 = 0$. The second equation has no real roots. Thus, we get the answer: $x = 2$. ●

2. Introducing a New Variable. In order to explain this method, let us consider the following example. Suppose we have to solve the equation $x^4 - 10x^2 + 9 = 0$. We set $x^2 = y$. Then the equation takes the form $y^2 - 10y + 9 = 0$, whence we find $y = 1$ and $y = 9$. Since $y = x^2$, we get the collection of equations: $x^2 = 1$, $x^2 = 9$. From the first equation we find $x = 1$ and $x = -1$, from the second: $x = 3$ and $x = -3$.

Remark. An equation of the form $ax^4 + bx^2 + c = 0$, where $a \neq 0$, is referred to as *biquadratic*. In the above example, we solved a biquadratic equation by introducing a new variable ($x^2 = y$). As it will be shown below, this method can be successfully used for solving many equations.

Example 1. Solve the equation $(x^2 - 3x)^2 + 3(x^2 - 3x) - 28 = 0$.

○Setting $x^2 - 3x = y$, we get $y^2 + 3y - 28 = 0$, whence $y = -7$ and $y = 4$. Now, the problem is reduced to solving the collection of equations: $x^2 - 3x = -7$, $x^2 - 3x = 4$. The first equation of the collection has no real roots; from the second equation we find $x_1 = 4$ and $x_2 = -1$.●

Example 2. Solve the equation $(x - 1)(x - 2)(x - 3)(x - 4) = 3$.

○We have

$$(x - 1)(x - 4) = x^2 - 5x + 4, \quad (x - 2)(x - 3) = x^2 - 5x + 6.$$

Thus, the given equation can be rewritten as follows:

$$(x^2 - 5x + 4)(x^2 - 5x + 6) = 3.$$

Let us introduce a new variable by setting $x^2 - 5x + 4 = y$. Note that $x^2 - 5x + 6 = (x^2 - 5x + 4) + 2 = y + 2$. We then get

$$y(y + 2) = 3 \text{ or } y^2 + 2y - 3 = 0,$$

whence $y = -3$ and $y = 1$. Now, it remains to solve the collection of equations

$$x^2 - 5x + 4 = -3, \quad x^2 - 5x + 4 = 1.$$

The first equation has no solution, from the second equation we find $x = (5 + \sqrt{13})/2$, $x = (5 - \sqrt{13})/2$.●

Example 3. Solve the equation $\frac{24}{x^2+2x-8} - \frac{15}{x^2+2x-3} = 2$.

○Let us set $x^2 + 2x - 3 = y$, then $x^2 + 2x - 8 = (x^2 + 2x - 3) - 5 = y - 5$, and the equation takes the form

$$\frac{24}{y-5} - \frac{15}{y} = 2,$$

whence we find: $y_1 = 12.5$ and $y_2 = -3$. Since $y = x^2 + 2x - 3$, we have to solve the collection of equations

$$x^2 + 2x - 3 = 12.5, \quad x^2 + 2x - 3 = -3.$$

From the first equation we find: $x_{1,2} = (-2 \pm \sqrt{66})/2$, from the second equation, $x_3 = 0$ and $x_4 = -2$.●

3. Finding Fractional Roots of Equations with Integral Coefficients. Up to this point we have used the selection method to find only integral roots of equations. In this item, we are going to consider some methods for finding fractional roots. These methods are based on the following theorem.

Theorem. *If a reduced equation with integral coefficients has a rational root, then this root is necessarily an integer.*

□For the sake of simplicity, let us consider an equation of the third degree

$$x^3 + bx^2 + cx + d = 0,$$

where b, c, and d are whole numbers. Suppose that it has a rational root $r = p/q$, where G.C.D. $(p, q) = 1$. Substituting this value into the equation, we get

$$r^3 + br^2 + cr + d = 0, \text{ that is, } \frac{p^3}{q^3} + \frac{bp^2}{q^2} + \frac{cp}{q} + d = 0.$$

Multiplying both sides of the last equality by q^3, we have

$$p^3 + bp^2q + cpq^2 + dq^3 = 0 \text{ or } p^2 = q(-bp^2 - cpq - dq^2).$$

Consequently, p^3 is divisible by q.

By hypothesis, p and q are relatively prime numbers, hence, p^3 is divisible by q if and only if $q = 1$. Thus, $r = p$, that is, the rational root of the equation is an integer.

The proof for an equation of an arbitrary degree is carried out in a similar way. ■

In practice, this theorem is used as follows: if a non-reduced equation with integral coefficients is given, then, using a suitable substitution, the equation is transformed into a reduced one. Then we find the integral root of the reduced equation and, with its help, the rest of the roots. Finally, returning to the original variable, we find the roots of the given equation.

Example 1. Solve the equation

$$21x^3 + x^2 - 5x - 1 = 0. \tag{1}$$

○Equations whose left-hand side is a polynomial with integral coefficients and a constant term equal to 1 or -1 are readily transformed into reduced equations by dividing them termwise by x in higher power and by making the substitution $1/x = y$. In our example we get

$$21 + \frac{1}{x} - \frac{5}{x^2} - \frac{1}{x^3} = 0.$$

Setting $1/x = y$, we arrive at the equation

$$21 + y - 5y^2 - y^3 = 0 \text{ or } y^3 + 5y^2 - y - 21 = 0.$$

Having found, as in the previous examples, the integral root of the equation $y_1 = -3$ and dividing the polynomial $y^3 + 5y^2 - y - 21$ by $y + 3$, we get the quadratic trinomial $y^2 + 2y - 7$ whose roots are $-1 + 2\sqrt{2}$ and $-1 - 2\sqrt{2}$. Since $x = 1/y$, we find $x_1 = -1/3$, $x_2 = 1/(-1 + 2\sqrt{2}) = (1 + 2\sqrt{2})/7$, $x_3 = 1/(-1 - 2\sqrt{2}) = (1 - 2\sqrt{2})/7$. ●

Note that in the process of solution we actually replaced the given equation by

$$\frac{21x^3+x^2-5x-1}{x^3}=0, \tag{2}$$

that is, by the equation with a variable in the denominator. According to the aforesaid (see § 4.1, Item 5), it is equivalent to the following mixed system:

$$\begin{cases} 21x^3+x^2-5x-1=0, \\ x^3\neq 0. \end{cases} \tag{3}$$

This system is satisfied by nonzero roots of equation (1). But since 0 cannot be a root of the equation whose constant term is nonzero, system (3) and, hence, also equation (2) are satisfied by all roots of equation (1).

Thus, equations (1) and (2) are equivalent. This note proves the validity of the transformation used in this example.

Example 2. Solve the equation $4x^3 - 10x^2 + 14x - 5 = 0$.

○Here we shall apply another method of transforming a nonreduced equation to a reduced one. We multiply both sides of the given equation by a number such that the coefficient of x^3 becomes the cube of a certain integer. In this case, the number 2 can serve as such a multiplier. Multiplying both sides of the equation by 2, we get

$$8x^3 - 20x^2 + 28x - 10 = 0.$$

Now, we set $2x = y$. Then the equation takes the form

$$y^3 - 5y^2 + 14y - 10 = 0. \tag{4}$$

The same as in the previous examples, we find the integral root of the reduced equation given by (4): $y = 1$. Dividing $y^3 - 5y^2 + 14y - 10$ by $y - 1$, we get the quadratic trinomial $y^2 - 4y + 10$ which has no real roots. Thus, equation (4) has a unique root 1. Since $x = y/2$, the original equation has only one root 1/2. ●

Exercises

In Exercises 1 to 5, solve the given equations by factoring the left-hand side:

1. $x^3 + x^2 - 4x - 4 = 0$. 2. $3x^3 - 7x^2 - 7x + 3 = 0$.
3. $(2z + 3)^3 + (z - 1)^3 - 27z^3 - 8 = 0$.
4. $3x^4 + x^3 - 12x^2 - 4x = 0$. 5. $x^3 - 4x^2 + x + 6 = 0$.

In Exercises 6 to 17, solve the given equations by introducing a new variable:

6. $x^4-5x^2+6=0$. 7. $(x^2-9)^2-8(x^2-9)+7=0$.
8. $(x^2-3x+1)(x^2-3x+3)=3$. 9. $\frac{x^2+x-5}{x}+\frac{3x}{x^2+x-5}+4=0$.
10. $16x(x+1)(x+2)(x+3)=9$. 11. $x^2+\frac{1}{x^2}+x+\frac{1}{x}=4$.

12. $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$.
13. $x^4 + x^3 - 4x^2 + x + 1 = 0$.
14. $8x^3 + 16x - 9 = 0$. 15. $4x^3 - 16x^2 + 17x - 3 = 0$.
16. $10x^3 - 3x^2 - 2x + 1 = 0$. 17. $2(x^2 + x + 1)^2 - 7(x - 1)^2 = 13(x^3 - 1)$.

§ 4.3. Irrational Equations

1. Additional Data on Equivalence of Equations. In § 1.5, Item 2, we introduced the concept of equivalence of equations and formulated two theorems on equivalent equations. We introduce some more data on equivalent equations.

Let there be given two equations

$$f_1(x) = g_1(x), \tag{1}$$

$$f_2(x) = g_2(x). \tag{2}$$

If each root of equation (1) is at the same time a root of equation (2) then equation (2) is said to be the *redundant equation* (1). Note that equivalence of equations means that each of the equations is the consequence of another.

In the process of solving an equation, we often have to use such transformations that lead to an equation which is a consequence of the original equation. A redundant equation is satisfied by all the roots of the original equation, but in addition to them, a redundant equation can also have such solutions which are not the roots of the original equation; these are the so-called *extraneous roots.* In order to reveal and reject extraneous solutions, we usually check all the found roots of a redundant equation by substituting them in turn into the original equation.

If, when solving an equation we replaced it by a redundant equation, then the above check is an integral part of the solution. Therefore, it is important to know what transformations make the given equation a redundant equation. Below, we shall consider some of such transformations.

Let there be given an equation

$$f(x) = g(x). \tag{3}$$

Multiplying both sides by the same expression $h(x)$, we get the equation

$$f(x)\,h(x) = g(x)\,h(x), \tag{4}$$

whose roots are both the roots of equation (3) and the roots of the equation $h(x) = 0$. Hence, equation (4) is a redundant equation (3). It is clear that equations (3) and (4) are equivalent if the "extraneous" equation $h(x) = 0$ has no root.

Thus, the following theorem is valid.

Theorem 4.3.1. *If both sides of an equation are multiplied by $h(x)$, then an equation is obtained which is redundant. If the equation*

If the equation $h(x) = 0$ has no root, then the obtained equation is equivalent to the original one.

Note that the inverse transformation, that is, changing from equation (4) to equation (3) by dividing both sides of equation (4) by $h(x)$, is, as a rule, inadmissible since this can lead to a loss of roots; in this case the roots of the equation $h(x) = 0$ can be "lost". For instance, the equation $(x - 2)(x - 3) = 2(x - 3)$ has too roots: 3 and 4. The division of both sides of the equation by $x - 3$ yields the equation $x - 2 = 2$ with only one root equal to 4, that is, the other root has been lost.

Let us return to equation (3) and square both of its sides. We get the equation

$$(f(x))^2 = (g(x))^2, \tag{5}$$

which has both the roots of equation (3) and the roots of the "extraneous" equation $f(x) = -g(x)$. It is clear that equations (3) and (5) are equivalent provided that the "extraneous" equation has no root. This occurs, in particular, when $f(x)$ and $g(x)$ take on only nonnegative values.

Theorem 4.3.2. *When squaring both sides of an equation (and, in general, when raising both sides of an equation to any even power), the equation obtained is a consequence of the original equation.*

Hence, this transformation may result in the appearance of extraneous roots.

Note that raising both sides of an equation to one and the same odd power leads to an equation equivalent to the given one.

2. Methods of Solving Irrational Equations. *Irrational* are the equations containing the variable under the radical signs or with fractional exponents. When solving irrational equations, we use two principal methods: (1) raising both sides of an equation to the same power, (2) introducing new (auxiliary) variables. However, we sometimes have to use artificial technique as well. When raising both sides of an equation to the same power, one should bear in mind that if n is an odd number, then the equations $f(x) = g(x)$ and $(f(x))^n = (g(x))^n$ are equivalent, and if n is an even number, then the equation $(f(x))^n = (g(x))^n$ is a consequence of the equation $f(x) = g(x)$, that is, the changing from the equation $f(x) = g(x)$ to the equation $(f(x))^n = (g(x))^n$ may lead to extraneous roots.

When solving irrational equations, we often use the formula $(\sqrt[n]{f(x)})^n = f(x)$; if n is even its application may lead to an extension of the domain of definition of an equation (with an even n the restriction $f(x) \geqslant 0$ is natural for $(\sqrt[n]{f(x)})^n$ whereas when $(\sqrt[n]{f(x)})^n$ is replaced by $f(x)$ this restriction is removed).

For these and other reasons, in most cases of solving irrational equations a check of found solutions is necessary. The method of such

a check depends on the kind of the found solutions (simple or complicated) and also on the method of solution.

Example 1. Solve the equation $\sqrt[5]{x-3}=2$.

○Raising both sides of the equation to the fifth power, we get $x-3=32$, whence $x=35$. In this case, to verify the found value by substitution into the original equation is unnecessary since, as it was noted above, raising both sides of an equation to the same odd power is an equivalent transformation of the equation. Thus, we get the answer: $x=35$. ●

Example 2. Solve the equation $\sqrt{x-1}+\sqrt{2x+6}=6$.

○Squaring both sides of the equation, we get the redundant equation

$$x-1+2\sqrt{(x-1)(2x+6)}+2x+6=36,$$

that is,

$$2\sqrt{2x^2+4x-6}=-3x+31.$$

Squaring both sides of the obtained equation, we get

$$8x^2+16x-24=9x^2-186x+961 \text{ or } x^2-202x+985=0,$$

whence $x_1=5$, $x_2=197$.

Here, the found solutions must be necessarily verified. This is easily carried out by substituting the found solutions into the original equation.

For $x=5$ we have $\sqrt{5-1}+\sqrt{2\cdot 5+6}=6$. Thus, $x_1=5$ is a root of the given equation. For $x=197$ we have $\sqrt{197-1}+\sqrt{2\cdot 197+6}\neq 6$. Therefore $x_2=197$ is an extraneous root. Thus, we get the answer: $x=5$. ●

Example 3. Solve the equation $\sqrt{x^2+x-5}+\sqrt{x^2+8x-4}=5$.

○Squaring both sides of the given equation yields rather a complicated equation, therefore we transform the equation to the form

$$\sqrt{x^2+x-5}=5-\sqrt{x^2+8x-4}$$

and square both sides. We have

$$(\sqrt{x^2+x-5})^2=(5-\sqrt{x^2+8x-4})^2,$$

$$x^2+x-5=25-10\sqrt{x^2+8x-4}+x^2+8x-4,$$

$$10\sqrt{x^2+8x-4}=7x+26. \tag{6}$$

Squaring both sides of the equation once again we get

$$(10\sqrt{x^2+8x-4})^2=(7x+26)^2$$

$$100x^2+800x-400=49x^2+364x+676$$

$$51x^2+436x-1076=0, \quad x_1=2, \quad x_2=-538/51.$$

The verification shows that the value $x = 2$ satisfies the original equation. As to the second value ($x = -538/51$), its substitution into the original equation leads to very complicated computations. However, we can avoid such a substitution if we notice that for this value the right-hand side of equation (6) attains a negative value: $7 \cdot (-538/51) + 26 = -2440/51$, whereas the left-hand side of the equation cannot be negative. Thus, $-538/51$ is not a root even of equation (6) which is a consequence of the given equation. The more so, this value cannot be a root of the given equation. Thus, we get the answer: $x = 2$. ●

Example 4. Solve the equation $x^2 + 3 - \sqrt{2x^2 - 3x + 2} = 1.5(x + 4)$.

○ Isolating the radical and squaring both sides of the equation thus obtained would lead to an awkward equation. At the same time, a little attention shows us that the given equation is readily reduced to a quadratic one. Indeed, multiplying both sides of the given equation by 2, we get

$$2x^2 + 6 - 2\sqrt{2x^2 - 3x + 2} = 3x + 12$$

and, further,

$$2x^2 - 3x + 2 - 2\sqrt{2x^2 - 3x + 2} - 8 = 0.$$

Setting $\sqrt{2x^2 - 3x + 2} = y$, we get $y^2 - 2y - 8 = 0$, whence $y_1 = 4$, $y_2 = -2$. Hence, the given equation is equivalent to the following collection of equations:

$$\sqrt{2x^2 - 3x + 2} = 4, \quad \sqrt{2x^2 - 3x + 2} = -2.$$

From the first equation we find: $x_1 = 7/2$, $x_2 = -2$, while the second equation has no solution.

Since the original equation is equivalent to the equation $\sqrt{2x^2 - 3x + 2} = 4$ (since the second equation of the collection had no solution), the found values can be verified by substituting them into the equation $\sqrt{2x^2 - 3x + 2} = 4$. This substitution shows that both values are the roots of the indicated, and, hence, the original equation. ●

Example 5. Solve the equation

$$\sqrt{2x^2 + 3x + 5} + \sqrt{2x^2 - 3x + 5} = 3x. \tag{7}$$

○ We multiply both sides of the given equation by the expression $\sqrt{2x^2 + 3x + 5} - \sqrt{2x^2 - 3x + 5}$, which is conjugate to the expression $\sqrt{2x^2 + 3x + 5} + \sqrt{2x^2 - 3x + 5}$.

Since

$$(\sqrt{2x^2 + 3x + 5} + \sqrt{2x^2 - 3x + 5}) \cdot (\sqrt{2x^2 + 3x + 5} - \sqrt{2x^2 - 3x + 5}) = (2x^2 + 3x + 5) - (2x^2 - 3x + 5) = 6x,$$

equation (7) will take the from

$$6x = 3x\left(\sqrt{2x^2+3x+5} - \sqrt{2x^2-3x+5}\right)$$

or

$$x\left(\sqrt{2x^2+3x+5} - \sqrt{2x^2-3x+5} - 2\right) = 0. \tag{8}$$

As is easily seen, $x_1 = 0$ is one of the roots of equation (8). It remains to solve the equation

$$\sqrt{2x^2+3x+5} - \sqrt{2x^2-3x+5} = 2. \tag{9}$$

Adding together equations (7) and (9), we get the redundant equation

$$2\sqrt{2x^2+3x+5} = 3x+2. \tag{10}$$

Squaring both sides of equation (10), we get

$$8x^2 + 12x + 20 = 9x^2 + 12x + 4,$$

and, further, $x^2 = 16$, whence $x_2 = 4$, $x_3 = -4$.

Substituting, in turn, the found values $x_1 = 0$, $x_2 = 4$, and $x_3 = -4$ into equation (7), we make sure that the latter is satisfied only by the value $x_2 = 4$. Thus, $x = 4$ is the only root of equation (7). ●

Exercises

1. Prove that the following irrational equations have no solution: (a) $\sqrt{2x+3}+\sqrt{x-3}=0$, (b) $\sqrt{x+2}+\sqrt{x-1}=-2$, (c) $\sqrt{4-x}-\sqrt{x-6}=2$, (d) $\sqrt{x}+\sqrt{x+9}=2$, (e) $\sqrt{x^2-4}+3\sqrt{4-x^2}+\sqrt{x+4}=2.5$, (f) $\sqrt{x}+\sqrt{x-2}=1-x$.

In Exercises 2 to 7, solve the given equations by raising both sides to an appropriate power:

2 $\sqrt{2-x}+\sqrt{x+3}=3$. **3.** $\sqrt{3x+4}+\sqrt{x-4}=2\sqrt{x}$.

4. $\sqrt{15-x}+\sqrt{3-x}=6$. **5.** $\dfrac{x-2}{\sqrt{2x-5}}-\sqrt{x-3}=\sqrt{2x-5}$.

6. $\sqrt[3]{x+34}-\sqrt[3]{x-3}=1$. **7.** $\sqrt[3]{2x-1}+\sqrt[3]{x-1}=1$.

In Exercises 8 to 12, solve the given equations by introducing a new variable:

8. $x^2+2x+\sqrt{x^2+2x+8}-12=0$. **9.** $\sqrt[7]{\dfrac{5-x}{x+3}}+\sqrt[7]{\dfrac{x+3}{5-x}}=2$.

10. $\sqrt{x^3+8}+\sqrt[4]{x^3+8}=6$.

11. $\sqrt{x-2+\sqrt{2x-5}}+\sqrt{x+2+3\sqrt{2x-5}}=7\sqrt{2}$.

12. $\sqrt{x^3+x^2-1}+\sqrt{x^3+x^2+2}=3$.

13. Solve the equation $(\sqrt{x+5}+\sqrt{x})(\sqrt{x}-1)=5$ by multiplying both sides by the same expression.

§ 4.4. Systems of Algebraic Equations

1. Basic Concepts. In § 1.6, we gave the definition for a system of two equations in two variables or simultaneous equations, introduced the notion of equivalence of systems, and formulated two theorems on equivalence of systems. Let us consider simultaneous equations in more detail.

Let there be given two simultaneous equations

$$\begin{cases} f_1(x, y) = g_1(x, y), \\ f_2(x, y) = g_2(x, y) \end{cases} \tag{1}$$

and

$$\begin{cases} \varphi_1(x, y) = \psi_1(x, y), \\ \varphi_2(x, y) = \psi_2(x, y). \end{cases} \tag{2}$$

If each solution of system (1) is at the same time a solution of system (2), then (2) is called a *redundant system* (1) *of equations.* A redundant system of equations may sometimes consist of only one equation. For instance, the equation $3x - 2y = 3$ is a consequence of the system

$$\begin{cases} 2x + y = 5, \\ x - 3y = -2 \end{cases}$$

which is the sum of the equations entering this system. In general, a redundant system of equations may contain either less or more equations than the original system. Thus, the system

$$\begin{cases} 2x + y = 5 \\ x - 3y = -2 \\ 3x - 2y = 3 \end{cases}$$

is derived from the system

$$\begin{cases} 2x + y = 5 \\ x - 3y = -2 \end{cases}$$

which, in turn, is derived from the system

$$\begin{cases} 2x + y = 5 \\ x - 3y = -2 \\ 3y - 2y = 3. \end{cases}$$

It is clear that two *systems of equations are equivalent if and only if the second system is a consequence of the first, and the first is a consequence of the second.*

In § 1.6 we noted that, replacing each equation of a system by an equivalent equation, we get a system equivalent to the given one. The following theorem generalizes this result.

Theorem 4.4.1. *If the equation* $f(x, y) = g(x, y)$ *is a consequence of equations* $f_1(x, y) = g_1(x, y)$ *and* $f_2(x, y) = g_2(x, y)$ *or of one of these equations, then the system*

$$\begin{cases} f_1(x, y) = g_1(x, y), \\ f(x, y) = g(x, y) \end{cases}$$

is a redundunt system of equations

$$\begin{cases} f_1(x, y) = g_1(x, y), \\ f_2(x, y) = g_2(x, y). \end{cases} \tag{3}$$

In particular, the following systems are redundant systems of equations (3):

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ a_1 f_1(x, y) + a_2 f_2(x, y) = a_1 g_1(x, y) + a_2 g_2(x, y), \end{cases} \tag{4}$$

$$\begin{cases} f_1(x, y) = g_1(x, y), \\ f_1(x, y) \cdot f_2(x, y) = g_1(x, y) \cdot g_2(x, y), \end{cases} \tag{5}$$

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ (f_2(x, y))^2 = (g_2(x, y))^2. \end{cases} \tag{6}$$

Note that for $a_2 \neq 0$ system (4) is equivalent to system (3).

If, solving a system, we transformed it into a consequence of the original system, then the found solutions of the new system must be undoubtedly verified (say, by substituting them into the original system). If the obtained system is equivalent to the original one, then such a verification is not needed.

Let us note one more result which is useful for solving systems of equations.

Theorem 4.4.2. *If the collection of equations*

$$f_{21}(x, y) = g_{21}(x, y), \quad f_{22}(x, y) = g_{22}(x, y), \ \ldots, \quad f_{2k}(x, y) = g_{2k}(x, y)$$

is equivalent to the equation $f_2(x, y) = g_2(x, y)$ *(or is its consequence), then the collection of systems*

$$\begin{cases} f_1(x, y) = g_1(x, y), \\ f_{21}(x, y) = g_{21}(x, y), \end{cases} \quad \begin{cases} f_1(x, y) = g_1(x, y), \\ f_{22}(x, y) = g_{22}(x, y), \end{cases} \ \ldots, \quad \begin{cases} f_1(x, y) = g_1(x, y), \\ f_{2k}(x, y) = g_{2k}(x, y) \end{cases}$$

is equivalent to system (3) *(or is its consequence).*

In particular, the following collection of systems

$$\begin{cases} f_1(x, y) = g_1(x, y), \\ f_{21}(x, y) = 0, \end{cases} \quad \begin{cases} f_1(x, y) = g_1(x, y), \\ f_{22}(x, y) = 0, \ \ldots, \end{cases} \quad \begin{cases} f_1(x, y) = g_1(x, y), \\ f_{2k}(x, y) = 0 \end{cases}$$

is a consequence of the system

$$\begin{cases} f_1(x, y) = g_1(x, y), \\ f_{21}(x, y) \cdot f_{22}(x, y) \ldots f_{2k}(x, y) = 0. \end{cases}$$

2. Methods of Solving Systems of Equations. Substitution, addition, and introduction of new variables are the principal methods of solving systems of algebraic equations. The first two methods have been already considered in application to systems of linear equations. The examples given below will show how these methods are applied to systems of arbitrary algebraic equations. In a number of cases, the multiplication method proves to be useful (see Example 6).

Example 1. Solve the two simultaneous equations

$$\begin{cases} x - 3y = 10, \\ x^2 - 2y = 100. \end{cases}$$

○ Here it is convenient to use the **substitution method.** From the first equation we find $x = 3y + 10$. Substituting the expression $3y + 10$ for x into the second equation of the system, we get $(3y + 10)^2 - 24y = 100$, whence $y_1 = 0$ and $y_2 = -4$. The corresponding values of x are found from the equation $x = 3y + 10$: if $y = 0$, then $x = 10$; if $y = -4$, then $x = -2$. Thus, the system has two solutions $(-2, -4)$ and $(10, 0)$. ●

Example 2. Solve the system of equations

$$\begin{cases} x^2 + y^2 - 2x + 3y - 9 = 0, \\ 2x^2 + 2y^2 - 3x + 5y - 17 = 0. \end{cases}$$

○ Let us use the **addition method.** If both sides of the first equation of the system are multiplied by 2, and the obtained equation is subtracted from the second equation, then the terms containing the variables in the second power are mutually annihilated:

$$(2x^2 + 2y^2 - 3x + 5y - 17) - (2x^2 + 2y^2 - 4x + 6y - 18) = 0,$$

$$x - y + 1 = 0.$$

As a result, we get a simpler system:

$$\begin{cases} x^2 + y^2 - 2x + 3y - 9 = 0, \\ x - y + 1 = 0, \end{cases}$$

which is readily solved by the substitution method:

$$\begin{cases} y = x + 1, \\ x^2 + (x+1)^2 - 2x + 3(x+1) - 9 = 0. \end{cases}$$

From the second equation of the last system we find: $x_1 = 1$, $x_2 = -2.5$. Since $y = x + 1$, we have: $y_1 = 2$, $y_2 = -1.5$.

Thus, the given system has two solutions: (1, 2) and (—2,5, —1,5). ●

Example 3. Solve the system of equations

$$\begin{cases} 3x^2 - 2xy = 160, \\ x^2 - 3xy - 2y^2 = 8. \end{cases}$$

○We multiply both sides of the second equation by 20 and subtract the obtained result from the first equation:

$$\begin{cases} 3x^2 - 2xy = 160, \\ (3x^2 - 2xy) - 20(x^2 - 3xy - 2y^2) = 160 - 8 \cdot 20. \end{cases} \tag{7}$$

Consider the second equation of system (7). It is transformed to the form

$$17x^2 - 58xy - 40y^2 = 0. \tag{8}$$

The obtained equation may be considered quadratic with respect to x. Using the formula for the roots of a quadratic equation, we get

$$x = \frac{29y \pm \sqrt{841y^2 + 680y^2}}{17} = \frac{29y \pm 39y}{17}, \quad x = 4y, \quad x = -\frac{10}{17}y.$$

Thus, equation (8) disintegrates into two equations: $x = 4y$ and $x = -10y/17$, and, in accordance with this, system (7) reduces to two systems (that is, a collection of two systems is obtained):

$$\begin{cases} 3x^2 - 2xy = 160, \\ x = 4y; \end{cases} \qquad \begin{cases} 3x^2 - 2xy = 160, \\ x = -\frac{10}{17}y. \end{cases}$$

Each of these systems is easily solved by the substitution method. Thus, we get the answer: (8, 2), (—8, —2); (—5, 8.5), (5, —8.5). ●

Example 4. Solve the system

$$\begin{cases} \frac{1}{x+y} + \frac{1}{x-y} = 2, \\ \frac{3}{x+y} + \frac{4}{x-y} = 7. \end{cases}$$

○Here, we use the **method of introducing new variables.** We set $1/(x + y) = u$ and $1/(x - y) = v$. Then the system takes the

form

$$\begin{cases} u+v=2, \\ 3u+4v=7, \end{cases}$$

whence we find $\begin{cases} u=1 \\ v=1. \end{cases}$ Returning to the variables x and y, we get

$$\begin{cases} 1/(x+y)=1, \\ 1/(x-y)=1, \end{cases} \text{ whence } \begin{cases} x+y=1, \\ x-y=1. \end{cases}$$

The last system and, hence, the given system have only one solution (1, 0). ●

Example 5. Solve the system

$$\begin{cases} y^2+\sqrt{3y^2-2x+3}=\frac{2}{3}x+5, \\ 3x-2y=5. \end{cases}$$

○ We transform the first equation to the form

$$(3y^2-2x+3)+3\sqrt{3y^2-2x+3}-18=0 \qquad (9)$$

and set $\sqrt{3y^2-2x+3}=z$. Then equation (9) takes the form: $z^2+3z-18=0$. We find the roots of this quadratic equation: -6 and 3. Now, equation (9) disintegrates into two equations: $\sqrt{3y^2-2x+3}=-6$; $\sqrt{3y^2-2x+3}=3$. The first equation of this collection has no solution. Squaring both sides of the second equation, we get $3y^2-2x+3=9$, that is, $3y^2-2x=6$. Thus, the equation $3y^2-2x=6$ is a consequence of the first equation of the given system. Hence, the system

$$\begin{cases} 3y^2-2x=6, \\ 3x-2y=5 \end{cases}$$

is also a consequence of the given system.

Solving the last system (for instance, by the substitution method), we get (3, 2) and (17/27, —14/9). A verification shows that both pairs found satisfy the given system. ●

Note that the verification of the found pairs with respect to the second equation of the given system involves no difficulties. It is more difficult to verify the found pairs with respect to the first equation; however, we can make the verification easier by noting that the first equation of the original system is equivalent to the equation $\sqrt{3y^2-2x+3}=3$. Hence, the verification may be carried out with the help of this last equation.

Example 6. Solve the system

$$\begin{cases} xy - 6 = \dfrac{y^3}{x}, \\ xy + 24 = \dfrac{x^3}{y}. \end{cases}$$

○Let us use the **multiplication method.** Multiplying the equations of the given system by each other, we get the system

$$\begin{cases} xy - 6 = \dfrac{y^3}{x}, \\ (xy + 24)(xy - 6) = \dfrac{x^3}{y} \cdot \dfrac{y^3}{x}, \end{cases} \tag{10}$$

which is a consequence of the original system. With the aid of rather simple transformations, the second equation of this system is reduced to the equation $xy = 8$ which is a consequence of the second equation of system (10). Therefore the system

$$\begin{cases} xy - 6 = \dfrac{y^3}{x}, \\ xy = 8 \end{cases} \tag{11}$$

is a consequence of system (10).

To solve system (11), let us apply the substitution method. From the second equation we find $y = 8/x$; then the first equation takes the form $8-6 = 8^3/x^4$, whence $x^4 = 2^8$, that is, $x = \pm 4$. Accordingly, from the equation $y = 8/x$ we find $y = \pm 2$. Thus, we have found two solutions of system (11): (4, 2) and (— 4, — 2).

Since, in the long run, system (11) is a consequence of the given system, the found solutions of system (11) must be verified by substituting these solutions into the given system. The verification shows that both of the found solutions of system (11) are also solutions of the given system. ●

3. Symmetric Systems. A polynomial $P(x, y)$ is said to be *symmetric* if interchanging the variables x and y leaves the expression $P(x, y)$ unchanged. Given below is a couple of symmetric polynomials:

$$P_1(x, y) = x^3 + 3xy + y^3, \quad P_2(x, y) = x^2 + y^2 + 3x + 3y.$$

The expression $F(x, y) = \dfrac{1}{x} + \dfrac{1}{y}$ is also symmetric.

We regard $x + y$ and xy as basic symmetric polynomials in two variables. All other symmetric polynomials in two variables can be expressed in terms of the basic ones. Setting, for brevity, $x + y = u$ and $xy = v$, we get, for instance,

$$x^2 + y^2 = (x + y)^2 - 2xy = u^2 - 2v,$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) = u(u^2 - 3v),$$

$$x^4 + y^4 = (x^2 + y^2)^2 - 2x^2y^2 = (u^2 - 2v)^2 - 2v^2 = u^4 - 4u^2v + 2v^2$$

and so forth.

A system both equations of which are symmetric is called *symmetric*. A symmetric system can be solved by changing the variables taking the basic symmetric polynomials as new variables.

Example. Solve the system of equations

$$\begin{cases} x^3+x^3y^3+y^3=17, \\ x+xy+y=5. \end{cases}$$

● We set $x+y=u$, $xy=v$. Since $x^3+y^3=u^3-3uv$, the given system is reduced to the following:

$$\begin{cases} u^3-3uv+v^3=17, \\ u+v=5. \end{cases}$$

From this system, which can be easily solved by the substitution method we find $u_1=3$ and $v_1=2$, $u_2=2$ and $v_3=3$. Now, it remains to solve the following collection of systems:

$$\begin{cases} x+y=3, \\ xy=2; \end{cases} \qquad \begin{cases} x+y=2, \\ xy=3. \end{cases}$$

The solutions of the first system of this collection are: (1, 2) and (2, 1). The second system has no solution. Thus, we get the answer: (1, 2), (2, 1).●

4. Systems of n Equations in n Variables, Where $n>2$. Up to this point we have considered systems of two equations in two variables. Let us now consider a system of three equations in three variables.

$$\begin{cases} f(x, y, z)=0, \\ g(x, y, z)=0, \\ h(x, y, z)=0. \end{cases}$$

Any triple of numbers (a, b, c) satisfying each equation of the system is a solution of such a system.

The technique of solution similar to that considered for systems of two equations in two variables is also valid for systems of three equations in three variables.

Example 1. Solve the system of equations

$$\begin{cases} x+y+z=2, \\ 2x+3y+z=1, \\ x^2+(y+2)^2+(z-1)^2=9. \end{cases}$$

○Let us apply the substitution method. Expressing (from the first equation) x in terms of y and z and substituting the result into the second and third equations of the system, we have

$$\begin{cases} x=2-y-z, \\ 2(2-y-z)+3y+z=1, \\ (2-y-z)^2+(y+2)^2+(z-1)^2=9 \end{cases}$$

and, further,

$$\begin{cases} x=2-y-z, \\ y-z=-3, \\ y^2+z^2+yz-3z=0. \end{cases}$$

The last two equations of the obtained system, in turn, form a system of two equations in two variables. Solving this system by the substitution method, we have

$$\begin{cases} y=z-3, \\ (z-3)^2+z^2+(z-3)z-3z=0, \end{cases} \text{ that is, } \begin{cases} y=z-3, \\ z^2-4z+3=0. \end{cases}$$

From the equation $z^2-4z+3=0$ we find $z_1=1$, $z_2=3$. From the equation $y=z-3$ we get $y_1=-2$, $y_2=0$, and from the equation $x=2-y-z$ we find $x_1=3$, $x_2=-1$. Thus, we have obtained the following solutions: $(3, -2, 1)$ and $(-1, 0, 3)$. ●

Example 2. Solve the system of equations

$$\begin{cases} 5x-6y+3z-u=4, \\ 3x+3y-2z+u=5, \\ -x+y+2z-u=6, \\ x+y+z+u=7. \end{cases}$$

○ We replace the second equation of the system by the sum of the first and second equations, the third equation by the difference of the first and third equations, the fourth equation by the sum of the first and fourth equations. Then we get a system (equivalent to the given) in which the variable u will be eliminated from the second, third, and fourth equations:

$$\begin{cases} 5x-6y+3z-u=4, \\ 8x-3y+z=9, \\ 6x-7y+z=-2, \\ 6x-5y+4z=11. \end{cases}$$

The second, third, and fourth equations of the obtained system contain only the variables x, y, and z. Let us replace the third equation by the difference between the second and third equations and then multiply both sides of the second equation by -4 adding the result to the fourth equation of the system. This enables us to exclude the variable z from the last two equations. Finally, we get

$$\begin{cases} 5x-6y+3z-u=4, \\ 8x-3y+z=9, \\ 2x+4y=11, \\ -26x+7y=-25. \end{cases}$$

The third and fourth equations of the obtained system contain only the variables x and y. We then multiply both sides of the third equation by -7, both sides of the fourth equation by 4 and add together the results. This enables us to exclude the variable y from the fourth equation:

$$\begin{cases} 5x-6y+3z-u=4, \\ 8x-3y+z=9, \\ 2x+4y=11, \\ -118x=-177. \end{cases}$$

We then transform this system to a form such that the coefficients of the variables which lie along the diagonal are equal to unity:

$$\begin{cases} -5x+6y-3z+u=-4, \\ 8x-3y+z=9, \\ 0.5x+y=2.75, \\ x=1.5. \end{cases} \tag{12}$$

This system is readily solved. We have

$$\begin{cases} x=1.5, \\ y=2.75-0.5x=2, \\ z=9-8x+3y=3, \\ u=-4+5x-6y+3z=0.5. \end{cases}$$

Thus, we get the answer: (1.5, 2, 3, 0.5). ●

A system of the form (12) is termed *triangular*. Solving a system of linear equations by reducing it to a triangular form is called the *method of successive elimination of variables* (or **Gauss' method**).

5. Problems on Setting Up Systems of Equations. The solution of many practical problems is reduced to the solution of systems of n algebraic equations in n variables.

Example 1. At the end of a year a savings-bank calculated the interest due to a depositor as 6 roubles. Adding another 44 roubles, the depositor left his money for another year. At the end of the second year the due interest was calculated again, and the deposit together with all the due interest amounted to 257 roubles and 50 kopecks. How much money was originally deposited?

○ Let x roubles be the original deposit, and $y\%$ the annual interest paid by the savings-bank. Then $\frac{y}{100} \cdot x$ roubles will be added to the original deposit at the end of the first year. Thus, we get the equation $\frac{yx}{100} = 6$.

At the end of the first year, the depositor added another 44 roubles. Thus, at the beginning of the second year the deposit amounted to

$x + 6 + 44 = x + 50$ roubles. At the end of the second year, the deposit together with all the interest amounted to $\left(x + 50 + \frac{y}{100}(x + 50)\right)$ roubles, and, the problem states that this equals 257.5 roubles. This enables us to write one more equation:

$$x + 50 + \frac{y}{100}(x + 50) = 257.5.$$

Thus, we have arrived at the following system of two equations in two variables:

$$\begin{cases} \frac{yx}{100} = 6, \\ x + 50 + \frac{y}{100}(x + 50) = 257.5. \end{cases}$$

Transforming both equations, we get

$$\begin{cases} xy = 600, \\ 100x + 50y + xy = 20\,750. \end{cases}$$

Replacing the second equation of the system by the difference between the second and the first equation, we get

$$\begin{cases} xy = 600, \\ 100x + 50y = 20\,150. \end{cases}$$

This system can be solved by the substitution method. We find $y = 403 - 2x$ from the second equation. Substituting the expression $403-2x$ for y into the first equation, we get $x(403-2x) = 600$, whence we find two roots: 200 and 1.5. It is clear that only the first value satisfies the sense of the problem.

Substituting the found value 200 instead of x into the equation $y = 403 - 2x$, we get $y = 3$. Thus, the original deposit amounted to 200 roubles, the annual interest paid by the savings-bank being equal to 3%. ●

Example 2. A pool has three drain pipes. Through the first and second pipes together, with the third pipe being closed, the filled pool is emptied in 15 minutes, through the first and third pipes, with the second pipe closed, in 12 minutes, and through the second and third pipes, with the first pipe closed, in 10 minutes. How much time would it take to empty the filled pool through each pipe separately?

○ Let x (min) be the time required to empty the filled pool through the first pipe, y (min) through the second pipe, and z (min) through the third pipe. We assume the capacity of the pool to be equal to unity.

By hypothesis, the first and second pipes, operating together, empty the pool in 15 minutes. During this time the first pipe empties

a part of the pool, expressed by the fraction $15/x$, the second pipe by the fraction $15/y$. Consequently $\frac{15}{x} + \frac{15}{y} = 1$. Reasoning in a similar way, we get two more equations $\frac{12}{x} + \frac{12}{z} = 1$ and $\frac{10}{y} + \frac{10}{z} = 1$. Thus, we have obtained the system of three equations in three variables

$$\begin{cases} \frac{15}{x} + \frac{15}{y} = 1, \\ \frac{12}{x} + \frac{12}{z} = 1, \\ \frac{10}{y} + \frac{10}{z} = 1. \end{cases}$$

Let us use the method of introducing new variables. Setting $3/x = u$, $5/y = v$, and $2/z = w$, we get the system

$$\begin{cases} 5u + 3v = 1, \\ 4u + 6w = 1, \\ 2v + 5w = 1. \end{cases}$$

From the last equation we express $w = \frac{1}{5} - \frac{2}{5}v$. Substituting the result into the second equation of the system, we get

$$4u + 6\left(\frac{1}{5} - \frac{2}{5}v\right) = 1 \quad \text{or} \quad 12v - 20u = 1.$$

Solving the system of two linear equations in two variables

$$\begin{cases} 5u + 3v = 1, \\ -20u + 12v = 1, \end{cases}$$

we get $u = 3/40$ and $v = 5/24$. From the equation $w = \frac{1}{5} - \frac{2}{5}v$ we find $w = \frac{7}{60}$.

Now, the problem is reduced to the following system of equations:

$$\begin{cases} 3/x = 3/40, \\ 5/y = 5/24, \\ 2/z = 7/60. \end{cases}$$

This system has only one solution: $\left(40,\ 24,\ 17\ \frac{1}{7}\right)$. Thus, through the first pipe alone the pool is emptied in 40 minutes, through the second pipe alone in 24 minutes, and through the third pipe in $17\ \frac{1}{7}$ minutes. ●

Exercises

In Exercises 1 to 5, solve the given systems of equations using the methods of substitution and addition:

1. $\begin{cases} x+y=-2, \\ x^2+y^2=100. \end{cases}$ 2. $\begin{cases} 2x^2-3y=23, \\ 3y^2-8x=59. \end{cases}$ 3. $\begin{cases} x^2+y^2+x+y=2, \\ 2x^2-y^2+2x-y=4. \end{cases}$

4. $\begin{cases} x^2+xy=36, \\ xy+y^2=45. \end{cases}$ 5. $\begin{cases} 2x^2-3xy+y^2=3, \\ x^2+2xy-2y^2=6. \end{cases}$

In Exercises 6 to 11, solve the given systems of equations by introducing new variables:

6. $\begin{cases} \sqrt{\frac{x}{y}}-\sqrt{\frac{y}{x}}=\frac{5}{6}, \\ x-y=5. \end{cases}$ 7. $\begin{cases} x+y+\frac{x}{y}=15, \\ \frac{(x+y)\,x}{y}=56. \end{cases}$

8. $\begin{cases} \frac{1}{x}-\frac{1}{y}=2, \\ \frac{1}{x^2}-\frac{1}{y^2}=16. \end{cases}$ 9. $\begin{cases} (x+2y)^2-(y-2x)^2=168, \\ (x+2y)+(y-2x)=12. \end{cases}$

10. $\begin{cases} \frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{5}{2}, \\ x^2+y^2=20. \end{cases}$ 11. $\begin{cases} x^2+2y+\sqrt{x^2+2y+1}=1, \\ xy=-4. \end{cases}$

In Exercises 12 to 14, solve the indicated symmetric systems:

12. $\begin{cases} x+y=5, \\ x^3+y^3=35. \end{cases}$ 13. $\begin{cases} x^3+y^3=8, \\ x^2+y^2=4. \end{cases}$ 14. $\begin{cases} x^4+x^2y^2+y^4=13, \\ x^2-xy+y^2=7. \end{cases}$

In Exercises 15 to 18, solve the given systems of equations:

15. $\begin{cases} x+y+z=-2, \\ x-y+2z=-7, \\ 2x+3y-z=1. \end{cases}$ 16. $\begin{cases} x-2y+3z=-1, \\ 2x+y-5z=9, \\ 4x-3y+z=7. \end{cases}$

17. $\begin{cases} x+2y-z=0, \\ 3x+5y+z=-10, \\ x+y+3z=-10. \end{cases}$ 18. $\begin{cases} x-y-z=5, \\ 2x+y+3y=3, \\ x-4y-6z=7. \end{cases}$

19. Solve the system of equations:

$$\begin{cases} x+y+z+t=2, \\ 2x+y-z+2t=5, \\ 3x+2y-5z+t=3, \\ x-2y+3z+t=5. \end{cases}$$

20. The hypotenuse of a right triangle is equal to 82 cm, and its area to 720 cm^2. Find its legs.

21. The distance between two towns A and B is 270 km. Two trains start simultaneously one from each town and meet in three hours. It takes one of the trains 1.35 hours more than the other to cover the entire distance. Find the speed of each train.

22. A collective farm gathered in 12 000 centners of winter wheat. If the cropping capacity were 25% higher, then the same harvest could be gathered

in from a field by 80 hectares less than from the actually sown area. How many hectares were sown, and what harvest was obtained from one hectare?

23. A body moves with uniform acceleration (without initial velocity). Having covered a path of 270 m, it acquires a speed of 9 m/s. With what acceleration and how much time was the body in motion?

24. Moving with the current, a boat covers a distance of 16 km in two hours, and against the current in eight hours. Determine the speed of the boat in stagnant water and the flow rate of the river.

25. A swimming pool is filled with water through three pipes. The first and second pipes together can fill this pool in 54/7 hours, the second and third pipes in 54/5 hours. How much time would it take each pipe to fill the pool if it is known that the first pipe fills it by 13.5 hours faster than the third pipe?

Chapter 5

INEQUALITIES

§ 5.1. Proving Inequalities

In the present section, we shall consider inequalities whose validity must be proved on a given set of values of variables. If such a set is not given, then it is implied that the variables may take on any real values.

There is no unique technique for proving inequalities, therefore we shall consider certain of the widespread methods illustrating them by examples.

1. Estimating the Sign of a Difference. The method consists in the following: in order to establish the truth of an inequality $f(x, y, \ldots, z) > g(x, y, \ldots, z)$ $(f < g,\ f \geqslant g,\ f \leqslant g)$ on a given set of values of variables $x, y, \ldots, z$, we write the difference $f(x, y, \ldots, z) - g(x, y, \ldots, z)$ and prove that it is positive (negative, nonnegative, nonpositive, respectively) for all corresponding collections $(x, y, \ldots, z)$.

Example 1. Prove that if $x \geqslant 0$, $y \geqslant 0$, then $\frac{x+y}{2} \geqslant \sqrt{xy}$ (that is, the *arithmetic mean of two nonnegative numbers is not less than their geometric mean*; this inequality is called **Cauchy's inequality**).

○ From the difference $\frac{x+y}{2} - \sqrt{xy}$, we have

$$\frac{x+y}{2} - \sqrt{xy} = \frac{x+y-2\sqrt{xy}}{2} = \frac{(\sqrt{x}-\sqrt{y})^2}{2}.$$

But the inequality $(\sqrt{x}-\sqrt{y})^2 \geqslant 0$ is true for any nonnegative values of x and y. Hence $\frac{x+y}{2} \geqslant \sqrt{xy}$, the equality being satisfied only when $x = y$. ●

Example 2. Prove that if $x > 0$, then $x + \frac{1}{x} \geqslant 2$ (that is, the sum of a positive number and its inverse is not less than 2).

○ Set up the difference $x + \frac{1}{x} - 2$. We have

$$x + \frac{1}{x} - 2 = \frac{x^2-2x+1}{x} = \frac{(x-1)^2}{x}.$$

Since $\frac{(x-1)^2}{x} \geqslant 0$ for all positive values of x, hence follows the truth of the inequality being proved. The equality holds when $x = 1$. ●

Example 3. Prove that $x^2 + 4y^2 + 3z^2 + 14 > 2x + 12y + 6z$.

○We form the difference between the left-hand and right-hand sides of the given inequality and carry out all necessary transformations

$$\begin{aligned}(x^2+4y^2+3z^2+4)-(2x+12y+6z) &= (x^2-2x+1)\\ &+(4y^2-12y+9)+(3z^2-6z+3)+1\\ &=(x-1)^2+(2y-3)^2+3(z-1)^2+1.\end{aligned}$$

The obtained sum is positive for all real values of x, y, z, whence follows the validity of the inequality being proved. ●

2. Deductive Method of Proving Inequalities. The principle of this method consists in the following: the required inequality is derived from some familiar (reference) inequalities with the aid of a number of transformations. For example, Cauchy's inequality $\frac{x+y}{2} \geqslant \sqrt{xy}$ $(x \geqslant 0,\ y \geqslant 0)$, the inequality $x + \frac{1}{x} \geqslant 2$ $(x > 0)$, the inequalities $-1 \leqslant \sin\alpha \leqslant 1$, $-1 \leqslant \cos\alpha \leqslant 1$ are frequently used as such reference inequalities.

Example 1. Prove that $\frac{a+b+c+d}{4} > \sqrt[4]{abcd}$, where a, b, c, d are nonnegative numbers.

○We shall use here as a reference inequality Cauchy's inequality set up for nonnegative numbers $x = \frac{a+b}{2}$, $y = \frac{c+d}{2}$. We have

$$\frac{a+b+c+d}{4} = \frac{1}{2}\left(\frac{a+b}{2}+\frac{c+d}{2}\right) \geqslant \sqrt{\frac{a+b}{2}\cdot\frac{c+d}{2}}.$$

Applying now Cauchy's inequality to the numbers a, b and c, d, we get

$$\sqrt{\frac{a+b}{2}\cdot\frac{c+d}{2}} \geqslant \sqrt{\sqrt{ab}\cdot\sqrt{cd}}.$$

But $\sqrt{\sqrt{ab}\cdot\sqrt{cd}} = \sqrt[4]{abcd}$. Thus, $\frac{a+b+c+d}{4} \geqslant \sqrt[4]{abcd}$. The equality holds when $a = b = c = d$. ●

Example 2. Prove that if $a > 0$, $b > 0$, $c > 0$, then

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geqslant 9.$$

○We write three reference inequalities:

$$\frac{a}{b}+\frac{b}{a} \geqslant 2, \quad \frac{a}{c}+\frac{c}{a} \geqslant 2, \quad \frac{b}{c}+\frac{c}{b} \geqslant 2.$$

Adding them together, we get

$$\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b} \geqslant 6,$$

or

$$\frac{a+c}{b}+\frac{b+c}{a}+\frac{a+b}{c}\geqslant 6,$$

$$\left(1+\frac{a+c}{b}\right)+\left(1+\frac{b+c}{a}\right)+\left(1+\frac{a+b}{c}\right)\geqslant 9.$$

Further, we have

$$\frac{a+b+c}{b}+\frac{a+b+c}{a}+\frac{a+b+c}{c}\geqslant 9,$$

that is,

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geqslant 9,$$

which was required to be proved (the equality sign occurs only when $a=b=c$). ●

3. Proof by Contradiction. The method consists in the following. Let there be required to prove the truth of the inequality

$$f(x, y, \ldots, z) > g(x, y, \ldots, z). \tag{1}$$

We suppose the contrary, that is, that the inequality

$$f(x, y, \ldots, z) \leqslant g(x, y, \ldots, z) \tag{2}$$

is true. Using the properties of numerical inequalities, we transform inequality (2). If these transformations yield a false inequality, then this means that our supposition is false, and therefore inequality (1) is true.

Example. Prove that if $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$, $d\geqslant 0$, then

$$\sqrt{(a+c)(b+d)}\geqslant \sqrt{ab}+\sqrt{cd}.$$

○Suppose the contrary, that is, that the inequality

$$\sqrt{(a+c)(b+d)}<\sqrt{ab}+\sqrt{cd}$$

is true. Squaring both of its sides, we get

$$ab+bc+ad+cd<ab+2\sqrt{abcd}+bd,$$

whence

$$bc+ad<2\sqrt{abcd}, \text{ that is, } \frac{bc+ad}{2}<\sqrt{bc\cdot ad}.$$

But this contradicts Cauchy's inequality set up for nonnegative numbers bc and ad. Hence, our supposition is false, that is, the inequality $\sqrt{(a+c)(b+d)}\geqslant \sqrt{ab}+\sqrt{cd}$ is true. ●

4. Comparing the Values of Numerical Expressions. In most cases we can immediately say which of two real numbers is greater, for example, $8>3$, $\sqrt{6}>\sqrt{5}$. It is not difficult to find out that $\sqrt[5]{5}<\sqrt[7]{1000}$. Indeed, $\sqrt[5]{5}<2$, and $\sqrt[7]{1000}>2$, hence, $\sqrt[5]{5}<\sqrt[7]{1000}$.

Let now $a = \sqrt[3]{3}$, $b = \sqrt{2}$. Both numbers belong to the open interval (1, 2), but which of them is greater is not clear yet. To determine the inequality sign between these numbers, let us reason as follows. Suppose that $a > b$, that is, $\sqrt[3]{3} > \sqrt{2}$. Raising both members of the last inequality to the sixth power, we get $(\sqrt[3]{3})^6 > (\sqrt{2})^6$, that is, $9 > 8$.

Thus*, $a > b \Leftrightarrow (\sqrt[3]{3})^6 > (\sqrt{2})^6 \Leftrightarrow 9 > 8$.

Since $9 > 8$ is a true inequality, the equivalent inequality $a > b$ is also true.

If we supposed that $a < b$, then we would obtain: $a < b \Leftrightarrow (\sqrt[3]{3})^6 < (\sqrt{2})^6 \Leftrightarrow 9 < 8$. Since $9 < 8$ is an untrue inequality, $a < b$ is also untrue, and since $a \neq b$, only one possibility remains: $a > b$.

Example. Compare the numbers a and b if: (a) $a = \sqrt{26} + \sqrt{6}$, $b = \sqrt{13} + \sqrt{17}$, (2) $a = \sqrt{5} + \sqrt{30} + \sqrt{50}$, $b = \sqrt{10} + \sqrt{20} + \sqrt{60}$.

○(1) Let us assume that $a > b$. Then, using the properties of numerical inequalities, we successively get:

$$(\sqrt{26} + \sqrt{6})^2 > (\sqrt{13} + \sqrt{17})^2, \quad 33 + 2\sqrt{156} > 30 + 2\sqrt{221},$$

$$1 + \sqrt{156} > \sqrt{221}, \quad (1 + \sqrt{156})^2 > 221, \quad \sqrt{156} > 32.$$

Thus, $a > b \Leftrightarrow \sqrt{156} > 32$. But $\sqrt{156} < 32$; hence, the original numbers are related by the same inequality sign, that is, $a < b$.

(2) Let us first estimate each of the terms of the sum a to within 1. We have $2 < \sqrt{5} < 3$, $5 < \sqrt{30} < 6$, $7 < \sqrt{50} < 8$, that is, $14 < \sqrt{5} + \sqrt{30} + \sqrt{50} < 17$. Thus, $14 < a < 17$.

Now we estimate each of the terms of the sum b also to within 1. We have $3 < \sqrt{10} < 4$, $4 < \sqrt{20} < 5$, $7 < \sqrt{60} < 8$, that is, $14 < \sqrt{10} + \sqrt{20} + \sqrt{60} < 17$. Thus, $14 < b < 17$.

We have obtained the same estimates for the numbers a and b, which keeps us from comparing the numbers a and b.

Let us increase the accuracy of estimation of each term of the sums a and b. Taking the accuracy 0.1, we have:

$$\begin{array}{rr} 2.2 < \sqrt{5} < 2.3 & 3.1 < \sqrt{10} < 3.2 \\ +\,5.4 < \sqrt{30} < 5.5 & +\,4.4 < \sqrt{20} < 4.5 \\ 7 < \sqrt{50} < 7.1 & 7.7 < \sqrt{60} < 7.8 \\ \hline 14.6 < a < 14.9, & 15.2 < b < 15.5 \end{array}.$$

Thus, $a \in (14.6,\ 14.9)$, $b \in (15.2,\ 15.5)$. Hence $a < b$. ●

* The symbol $\Leftrightarrow$ indicates that two sentences, statements are equivalent (equivalence sign).

Exercises

1. Prove the following inequalities:
(a) $a^4 + b^4 \geqslant a^3b + ab^3$, (b) $a^2 + b^2 + c^2 + 3 \geqslant 2(a + b + c)$,
(c) $(a^2+1)(b^2+1)(c^2+1) \geqslant 8abc$, (d) $\dfrac{a^2+a+2}{\sqrt{a^2+a+1}} \geqslant 2$.

2. Prove that for all positive numbers a and b the inequality $\sqrt{a}+\sqrt{b} > \sqrt{a+b}$ is satisfied.

3. Prove that if a, b, and c are the legs and the hypotenuse of a right triangle, then $a + b \leqslant c\sqrt{2}$.

4. Prove that if the same positive number is added to the numerator and denominator of the same positive number, then a positive proper fraction increases, while an improper fraction decreases.

5. Prove that if m, n and, p denote the lengths of the sides of a triangle, then $m^2 + n^2 + p^2 < 2(mn + mp + np)$.

6. Find the least value of the function $y = \dfrac{8}{x^2} + \dfrac{x^2}{2}$.

7. Which of the numbers a and b is greater: (a) $= \sqrt{12} - \sqrt{5}$, $b = \sqrt{10} - \sqrt{6}$, (b) $a = \sqrt{29}$, $b = \sqrt{10} + \sqrt{19}$, (c) $a = \sqrt{27}$, $b = \sqrt{20} + \sqrt{7}$?

8. Prove that, in a triangle, the sum of the length of a side and the altitude drawn to this side is greater than half of its perimeter.

9. A motor ship covered a path with the stream and the same path against the stream. Prove that the average speed of the motor ship was less than its proper speed (both the proper speed of the motor ship and the rate of flow of the river are assumed to be constant).

§ 5.2. Solving Rational Inequalities

The present section will be dedicated to solving inequalities of the form $P(x)/Q(x) > 0$ (instead of the sign $>$ any other inequality sign can be encountered), where $P(x)$ and $Q(x)$ are polynomials, as well as to solving inequalities reduced to this form.

Inequalities of the form $P(x)/Q(x) > 0$ can be solved as follows. Consider the function

$$h(x) = \frac{(x-a)(x-b)}{(x-c)(x-d)},$$

where $a < b < c < d$. If $x > d$, then each of the factors $x - a$, $x - b$, $x - c$, and $x - d$, is positive, and, consequently, $h(x) > 0$. If $c < x < d$, then $x - d < 0$, the other factors being positive also. Hence on the open interval (c, d) we have $h(x) < 0$. Similarly, on the interval (b, c) we have $h(x) > 0$, etc. (Fig. 141).

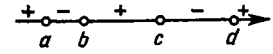


Fig. 141

The sign alternation of the function $h(x)$ can be illustrated with a curve which runs from top right to left, as shown in Fig. 142. This figure should be understood as follows: in the intervals where the curve is above the number line the inequality

$h(x) > 0$ is satisfied; and on the intervals where the curve lies below the number line the inequality $h(x) < 0$ holds. This wavy curve will be called the *sign curve*.

For the above reasoning, the number of linear factors in the numerator and denominator and mutual position of the roots ($x = a$, $x = b$) and poles* ($x = c$, $x = d$) of the function $h(x)$ on the number line were unessential. Therefore it is applicable for a function of the following form:

$$f(x) = \frac{(x-a_1)(x-a_2)\ldots(x-a_n)}{(x-b_1)(x-b_2)\ldots(x-b_k)}, \qquad (1)$$

where the numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_k$ are **pairwise different**. We can also illustrate the sign alternation of the func-

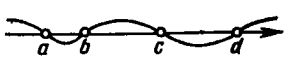


Fig. 142

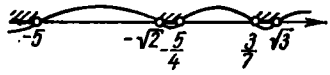


Fig. 143

tion $f(x)$ with a sign curve running from top right to left as shown in Fig. 143. It passes through all the roots and poles of the function $f(x)$ marked on the number line.

Example 1. Solve the inequality $\frac{(x+5)(x-\sqrt{3})(x+\sqrt{2})}{(2x-3)(4x+5)} < 0$.

○Transforming the left-hand side of the inequality, we get

$$\frac{(x+5)(x-\sqrt{3})(x+\sqrt{2})}{2(x-3/2)\cdot 4(x+5/4)} < 0, \quad \frac{(x+5)(x-\sqrt{3})(x+\sqrt{2})}{(x-3/2)(x+5/4)} < 0.$$

The alternation of signs of the function $f(x) = \frac{(x+5)(x-\sqrt{3})(x+\sqrt{2})}{(x-3/2)(x+5/4)}$ is illustrated by the sine curve in Fig. 143. The values of x for which $f(x) < 0$ (the corresponding intervals are hatched) satisfy the following collection of simple inequalities: $x < -5$, $-\sqrt{2} < x < -5/4$, $3/2 < x < \sqrt{3}$. This is the solution of the given inequality.●

Example 2. Solve the inequality $\frac{x^2-3x-18}{13x-x^2-42} \geqslant 0$.

○By multiplying both sides of the inequality by -1, we get

$$\frac{x^2-3x-18}{x^2-13x+42} \leqslant 0, \text{ that is } \frac{(x-6)(x+3)}{(x-6)(x-7)} \leqslant 0.$$

The method used in the preceding example cannot be applied to this inequality since both the numerator and the denominator contain the **same** factor $x - 6$. It is clear that the fraction on the left-hand side of the inequality can be reduced by $x - 6$ on condition

* *Poles* of a rational function are defined as roots of its denominator.

that $x \neq 6$. As a result of this transformation, we get $\frac{x+3}{x-7} \leqslant 0$. Using the sign curve (Fig. 144), we find $-3 \leqslant x < 7$. Excluding the value $x = 6$ from the found set, we get $-3 \leqslant x < 6$; $6 < x < 7$. ●

Example 3. Solve the inequality $\frac{(x-3)(x+2)}{x^2-1} < 1$.

○ We have

$$\frac{x^2-x-6}{x^2-1} - 1 < 0, \text{ that is, } \frac{x+5}{(x-1)(x+1)} > 0.$$

We graw the curve of signs for the function $f(x) = \frac{x+5}{(x-1)(x+1)}$ (Fig. 145). By this curve, we find the solutions for the inequality: $-5 < x < -1$, $x > 1$. ●

Now we pass to the case when, on reducing a rational inequality to the form

$$\frac{(x-a_1)(x-a_2)\ldots(x-a_n)}{(x-b_1)(x-b_2)\ldots(x-b_k)} > 0, \tag{2}$$

there are equal factors either in the numerator or the denominator.

Consider the expression $(x - a)^3$. It is clear that for any values of x this expression has the same sign as the expression $x - a$. A si-

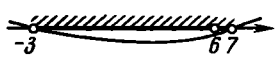


Fig. 144

Fig. 145

milar conclusion can also be drawn for any expression of the form $(x - a)^n$, where n is an odd number. Therefore if the numerator or the denominator of the left-hand side of inequality (2) contains a factor of the form $(x - a)^n$, where n is an odd number, then, replacing it by the factor $x - a$, the sign of the expression on the left-hand side of the inequality will not change. This means that such a replacement will lead to a simpler inequality equivalent to the original one.

Consider the expression $(x - a)^2$. It takes on positive values if $x \neq a$ and vanishes if $x = a$. If the numerator or denominator of the left-hand side of inequality (2) contains the factor $(x - a)^2$, then on condition $x \neq a$, both sides of inequality (2) can be multiplied or divided by $(x - a)^2$ with the sign of the inequality retained. A similar conclusion may be drawn as far as expressions of the form $(x - a)^n$ are concerned, where n is any even number.

Example 4. Solve the inequality $\frac{x^2(3x-4)^3(x-2)^4}{(x-5)^5(2x-7)^6} \leqslant 0$.

○ Replacing the factors with odd exponents by the corresponding factors of the first power on the left-hand side of the given inequality, we get an equivalent inequality

$$\frac{3x-4}{x-5}\cdot\frac{x^2(x-2)^4}{(2x-7)^6}\leqslant 0. \tag{3}$$

The left-hand member of the obtained inequality contains three factors with even exponents: x^2, $(x-2)^4$, and $(2x-7)^6$. These factors vanish for the values $x=0$, $x=2$, and $x=3.5$, respec-

Fig. 146

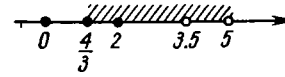


Fig. 147

tively. Note that the values 0 and 2 do satisfy the given nonstrict inequality, while the value 3.5 does not. Thus, $x=0$ and $x=2$ are solutions of the given inequality.

Let us now consider the case when x takes on values other than 0, 2, and 3.5. Then the expression $\frac{x^2(x-2)^4}{(2x-7)^6}$ is positive dividing both sides of inequality (3) by this expression, we get an equivalent inequality

$$\frac{3x-4}{x-5}\leqslant 0, \text{ that is, } \frac{3(x-4/3)}{x-5}\leqslant 0 \text{ or } \frac{x-4/3}{x-5}\leqslant 0.$$

Drawing the sign curve for the last inequality (Fig. 146), we get $4/3\leqslant x<5$.

Now, we should account for the fact that $x=0$ and $x=2$ are solutions of the inequality, while $x=3.5$ is not. Since the value $x=0$ is a solution of the inequality and does not belong to the interval [4/3, 5), it should be additionally included into the answer. Further, the value $x=2$ is a solution of the inequality and belongs to the interval [4/3, 5); therefore this value should not be singled out in the answer. Finally, the value $x=3.5$ is not a solution of the inequality, but belongs to the interval [4/3, 5); hence, this value should be excluded.

Using a geometrical illustration (Fig. 147), we write the solutions of the given inequality: $x=0$, $4/3\leqslant x<3.5$, $3.5<x<5$. ●

Note that in all the examples considered the inequalities reduce to a canonical inequality $f(x)>0$, where $f(x)$ is a function of form (1).

However, such a reduction is not always possible; for example, the inequality $x^2+2x+2<0$ cannot be reduced to canonical form, since the quadratic trinomial x^2+2x+2 has no real roots and therefore cannot be factored into linear factors.

In such cases, the following result is used: if the discriminant $D = b^2 - 4ac$ of the quadratic trinomial $ax^2 + bx + c$ is negative, and the leading coefficient a is positive, then the inequality $ax^2 + bx + c > 0$ is satisfied for all values of x (see § 3.9, Item 2).

Example 5. Solve the inequality $\frac{x^2-3x+5}{3+4x} < 0$.

○ The discriminant of the quadratic trinomial in the numerator $D = 9 - 20 < 0$, and the leading coefficient $a = 1 > 0$. Hence, the numerator of the left-hand member of the given inequality is positive for any values of x. Therefore, dividing both sides of the inequality by $x^2 - 3x + 5$ and keeping the sign of the inequality unchanged, we get the equivalent inequality $\frac{1}{3x+4} < 0$, whence $x < -4/3$. Thus, we get the answer: $(-\infty, -4/3)$. ●

The technique used in the above examples is called the **method of intervals.**

Exercises

In Exercises 1 to 14, solve the given inequalities:

1. $x^2 - 25 > 0$. 2. $x^2 + 6x \geqslant 0$. 3. $9x^2 + 6x + 1 > 0$.
4. $4x^2 - 12x + 9 \leqslant 0$. 5. $-x^2 + 4x - 5 < 0$.
6. $\frac{(2-x)x}{x} > x + \frac{1-3x}{4}$.
7. $x^2 - 4x - 5 \geqslant 0$. 8. $\frac{x(x-1)(x+2)}{(x-3)(x+4)} > 0$. 9. $\frac{0.2}{x - x^2 - 10} > 0$.
10. $\frac{x^2-7x+10}{12-x^2+x} \leqslant 0$. 11. $\frac{3-2x}{4+x^2} > 1$. 12. $x - 2 < \frac{3}{x}$.
13. $\frac{x^2-3x+1}{x^2-1} > 1$. 14. $\frac{x}{x^2+3x+2} > \frac{x}{x^2+7x+12}$.
15. Find the domain of definition of each of the given functions:

(a) $y = \sqrt{81 - x^2}$ (b) $y = \sqrt{\frac{1}{x} - 3}$.

16. The flow rate of a river is 3 km/h, and the distance between points A and B is 28 km. What must the speed of a motor boat be so that it took the boat not more than seven hours to cover the distance from A to B and back?

17. A man going to catch a train covered a one-third of the distance to the railway station during the first hour. If he walked at the same speed, he would reach the station exactly when the train was due to depart. With what speed must the man move to cover the remaining 7 km to the station at least 15 minutes before the train's departure?

18. For what values of the parameter a is the inequality $(a + 1)x^2 + (a + 1)x + 1 > 0$ valid for any real value of x?

19. For what values of the parameter a does the equation $\frac{5}{a-3x} = \frac{2}{ax-2}$ have a negative root?

§ 5.3. Systems and Collections of Rational Inequalities

1. Solving Systems and Collections of Rational Inequalities. In § 1.7, we introduced the notion of a system of inequalities in one variable and a collection of inequalities in one variable and considered several examples illustrating the solution of systems and collections of linear inequalities. In the present section, we shall deal with solving systems and collections of rational inequalities and inequalities with moduli and irrational inequalities as well which, as we shall see later, are reduced in the process of solution to systems or collections of systems of rational inequalities.

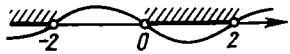


Fig. 148

Fig. 149

There is nothing essentially new in solving systems and collections of rational inequalities as compared with solving systems and collections of linear inequalities. Therefore we immediately turn to considering examples.

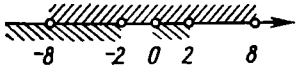


Fig. 150

Fig. 151

Example 1. Solve the system of inequalities $\begin{cases} \dfrac{x^2+x-4}{x}<1, \\ x^2<64. \end{cases}$

○Beginning with the first inequality, we have

$$\frac{x^2+x-4}{x}-1<0, \quad \frac{(x-2)(x+2)}{x}<0.$$

Using the curve of signs (Fig. 148), we find the solutions of this inequality: $x<-2$, $0<x<2$.

Now we solve the second inequality of the given system. We have $x^2-64<0$, or $(x-8)(x+8)<0$. With the aid of the curve of signs (Fig. 149), we find the solutions of the second inequality: $-8<x<8$.

Marking the found solutions to both inequalities on a common number line (Fig. 150), we find the intervals where these solutions intersect: $-8<x<-2$, $0<x<2$. This is the solution of the given system. ●

Example 2. Solve the collection of inequalities

$$x^5 \geqslant 100x^3, \quad \frac{(x+9)(5x-x^2-18)}{x^2-18x+45} \geqslant 0.$$

○ Let us transform the first inequality:

$$x^3(x-10)(x+10) \geqslant 0 \text{ or } x(x-10)(x+10) \geqslant 0$$

(it was noted in the preceding section that factors in odd powers may be replaced by appropriate factors in the first power), we illustrate the solutions of the last inequality geometrically: $-10 \leqslant x \leqslant 0$, $x \geqslant 10$ (Fig. 151).

Consider the second inequality of the collection; we have

$$\frac{(x+9)(x^2-5x+18)}{(x-3)(x-15)} \leqslant 0.$$

Since the discriminant of the quadratic trinomial $x^2 - 5x + 18$ is negative, and the leading coefficient is positive, $x^2 - 5x + 18 > 0$

Fig. 152

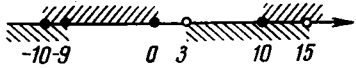


Fig. 153

for all values of x, and, consequently, we may divide both sides of the inequality by $x^2 - 5x + 18$, preserving the sign of the inequality:

$$\frac{x+9}{(x-3)(x-15)} \leqslant 0.$$

We find (Fig. 152) $x \leqslant -9$; $3 < x < 15$.

Combining the solutions found we get (Fig. 153) $x \leqslant 0$, $x > 3$. ●

When solving systems and collections of inequalities, we use the notion of consequence of inequality which is similar to the notion of redundant equation (see § 4.3). We shall say that the inequality $f_1(x) > g_1(x)$ is a *redundant inequality* $f(x) > g(x)$ if each solution of the inequality $f(x) > g(x)$ is, at the same time, a solution of the inequality $f_1(x) > g_1(x)$.

Let the inequality $f_1(x) > g_1(x)$ be a redundant inequality $f(x) > g(x)$. Then the system of inequalities

$$\begin{cases} f(x) > g(x), \\ f_1(x) > g_1(x) \end{cases}$$

is equivalent to the inequality $f(x) > g(x)$, and the collection of inequalities

$$\left[\begin{array}{l} f(x) > g(x), \\ f_1(x) > g_1(x) \end{array}\right.$$

is equivalent to the redundant inequality $f_1(x) > g_1(x)$.

This result implies that *if in a system of inequalities one inequality is a consequence of another (or other inequalities), then the redundant inequality may be omitted. If in a collection of ineqalities one is a consequence of the other, then the second inequality may be omitted (the redundant inequality alone being left).*

In what follows, we shall make a repeated use of this remark.

2. Inequalities with Moduli. When solving inequalities containing a variable under the sign of modulus and when solving irrational inequalities, the following equivalence theorem may be of use (we give it without proof).

Theorem. *Let there be given an inequality $f(x) > g(x)$, and the expressions $f(x)$ and $g(x)$ take on only nonnegative values for any $x \in M$. If both sides of the inequality are squared and the sign of the inequality is left unchanged, then the inequality $(f(x))^2 > (g(x))^2$ results which is equivalent to the one given on the set M.*

Remark. The theorem also holds when both sides of an inequality are raised to any **even** power.

Let us show how this theorem is used for solving inequalities with moduli. Let there be required to solve the inequality

$$|f(x)| > |g(x)|. \tag{1}$$

Since for any x from the domains of definition of the expressions $f(x)$ and $g(x)$ the relationships $|f(x)| \geqslant 0$, $|g(x)| \geqslant 0$, $(|f(x)|)^2 = (f(x))^2$ and $(|g(x)|)^2 = (g(x))^2$ are valid, making use of the theorem, we conclude that inequality (1) is equivalent in its domain of definition to the inequality $(f(x))^2 > (g(x))^2$. Such method is called the **method of squaring**.

Example 1. Solve the inequality $|x - 1| < 3$.

○Squaring both sides of the given inequality, we get an equivalent inequality: $(x-1)^2 < 9$. Solving the last inequality, we find $x^2 - 2x - 8 < 0$, whence $-2 < x < 4$. Thus, we get $(-2, 4)$. ●

Example 2. Solve the inequality $|2x - 1| \leqslant |3x + 1|$.

○Squaring both sides of the inequality, we get the inequality $(2x-1)^2 \leqslant (3x+1)^2$ equivalent to the given one. Transforming the last inequality, we have $5x^2 + 10x \geqslant 0$, whence $x \leqslant -2$, $x \geqslant 0$. ●

The method of squaring is applicable only to solve relatively simple inequalities with moduli. A more general method of solving inequalities with moduli is based on the **removal of the modulus sign by definition**. This method consists in the following.

Let there be required to solve the inequality

$$|f(x)| > g(x). \tag{2}$$

Since

$$|f(x)| = \begin{cases} f(x) & \text{for } f(x) \geqslant 0, \\ -f(x) & \text{for } f(x) < 0, \end{cases}$$

inequality (2) is equivalent to the following collection of systems of inequalities:

$$\begin{cases} f(x) \geqslant 0, \\ f(x) > g(x), \end{cases} \quad \begin{cases} f(x) \geqslant 0, \\ -f(x) > g(x) \end{cases}$$

(see also § 1.7, Item 4).

Example 3. Solve the inequality $\left|\frac{2x+3}{3x-2}\right| > 1$.

○The given inequality is equivalent to the following collection:

$$\begin{cases} \frac{2x+3}{3x-2} \geqslant 0, \\ \frac{2x+3}{3x-2} > 1, \end{cases} \quad \begin{cases} \frac{2x+3}{3x-2} < 0, \\ -\frac{2x+3}{3x-2} > 1. \end{cases}$$

The first inequality of each of the systems is a consequence of the second, and therefore both first inequalities can be omitted. As a result, we get the collection

$$\frac{2x+3}{3x-2} > 1, \quad \frac{2x+3}{3x-2} < -1.$$

Solving the inequalities of this collection, we find $2/3 < x < 5$; $-1/5 < x < 2/3$. ●

Example 4. Solve the inequality $\frac{x^2+|x|-12}{x-3} \geqslant 2x$.

○Here, we have to consider two cases: $x \geqslant 0$ and $x < 0$. In the first case $|x| = x$, in the second $|x| = -x$. Hence, the given inequality is equivalent to the collection of two systems:

$$\begin{cases} x \geqslant 0, \\ \frac{x^2+x-12}{x-3} \geqslant 2x, \end{cases} \quad \begin{cases} x < 0, \\ \frac{x^2-x-12}{x-3} \geqslant 2x. \end{cases}$$

Let us solve the first system. Since $x^2 + x - 12 = (x + 4)(x - 3)$, the left-hand side of the second inequality of the system can be reduced by $x - 3$ provided that $x \neq 3$. Then we get the system

$$\begin{cases} x \geqslant 0, \\ x + 4 \geqslant 2x, \\ x \neq 3, \end{cases}$$

which has the following solutions: $0 \leqslant x < 3$; $3 < x \leqslant 4$.

Let us solve the second system of the collection. Since we consider the case $x < 0$, the more so $x - 3 < 0$, and therefore we may

multiply both sides of the second inequality by $x - 3$ by changing the sign of the inequality. Then we get

$$\begin{cases} x < 0, \\ x^2 - x - 12 \leqslant 2x(x-2), \end{cases} \text{ that is } \begin{cases} x < 0, \\ x^2 - 5x + 12 \geqslant 0. \end{cases}$$

The discriminant of the quadratic trinomial $x^2 - 5x + 12$ is negative and the leading coefficient is positive. Hence, the inequality $x^2 - 5x + 12 \geqslant 0$ is satisfied for all x, and the set of solutions of the system coincides with the set of solutions of the first inequality, that is, $x < 0$.

Combining the found sets of solutions of the first and second systems of the collection, we get $(-\infty, 3) \cup (3, 4)$. ●

3. Irrational Inequalities. Consider an inequality of the form

$$\sqrt{f(x)} < g(x). \tag{3}$$

It is clear that every solution of this inequality is, at the same time, a solution of the inequality $f(x) \geqslant 0$ and a solution of the inequality $g(x) > 0$ (it follows from inequality (3) that $g(x) > \sqrt{f(x)} \geqslant 0$). Then, according to the remark made at the end of Item I, we conclude that inequality (3) is equivalent to the system of inequalities

$$\begin{cases} f(x) \geqslant 0, \\ g(x) > 0, \\ \sqrt{f(x)} < g(x). \end{cases}$$

Since on the set, described by the first two inequalities of the system, both sides of the third inequality are defined and take on only nonnegative values, their squaring on the indicated set is an equivalent transformation of the inequality. As a result, we get that the *inequality* $\sqrt{f(x)} < g(x)$ *is equivalent to the system of inequalities*

$$\begin{cases} f(x) \geqslant 0, \\ g(x) > 0, \\ f(x) < (g(x))^2. \end{cases}$$

In much the same way, the *inequality* $\sqrt{f(x)} \leqslant g(x)$ *is equivalent to the system of inequalities*

$$\begin{cases} f(x) \geqslant 0, \\ g(x) \geqslant 0, \\ f(x) \leqslant (g(x))^2. \end{cases}$$

Consider now an inequality of the form

$$\sqrt{f(x)} > g(x). \tag{4}$$

As before, we conclude that this inequality is equivalent to the system of inequalities

$$\begin{cases} \sqrt{f(x)} > g(x), \\ f(x) \geqslant 0, \end{cases}$$

but, as distinct from the preceding case, here $g(x)$ can attain both positive and negative values. Therefore we shall consider the system in question in each of the cases $g(x) < 0$ and $g(x) \geqslant 0$ separately. Then, we shall get the collection of systems

$$\begin{cases} g(x) < 0, \\ f(x) \geqslant 0, \\ \sqrt{f(x)} > g(x), \end{cases} \qquad \begin{cases} g(x) \geqslant 0, \\ f(x) \geqslant 0, \\ \sqrt{f(x)} > g(x). \end{cases}$$

In the first of these systems, we may omit the last inequality, since it is a consequence of the first two, in the second system, both sides of the last inequality may be squared. Then in the second system we get $f(x) > (g(x))^2$, the inequality $f(x) \geqslant 0$ being its consequence, that is, in the second system the inequality $f(x) \geqslant 0$ may be omitted.

Finally, we get the following result: the *inequality* $\sqrt{f(x)} > g(x)$ *is equivalent to the collection of two systems of inequalities*:

$$\begin{cases} g(x) < 0, \\ f(x) \geqslant 0, \end{cases} \qquad \begin{cases} g(x) \geqslant 0, \\ f(x) > (g(x))^2. \end{cases}$$

Example 1. Solve the inequality $\sqrt{x^2 - x - 12} < x$.

○ This inequality is equivalent to the following system:

$$\begin{cases} x^2 - x - 12 \geqslant 0, \\ x > 0, \\ x^2 - x - 12 < x^2. \end{cases}$$

Solving this system, we get $x \geqslant 4$. ●

Example 2. Solve the inequality $\sqrt{x^2 - 3x + 2} > x + 3$.

○ This inequality is equivalent to the collection of two systems:

$$\begin{cases} x + 3 < 0, \\ x^2 - 3x + 2 \geqslant 0, \end{cases} \qquad \begin{cases} x + 3 \geqslant 0, \\ x^2 - 3x + 2 > (x + 3)^2. \end{cases}$$

Solving the first system, we get $x < -3$, from the second system we find $-3 \leqslant x < -7/9$. Combining the solutions of the systems, we egt $(-\infty, -7/9)$. ●

Exercises

1. Solve the system of inequalities

$$\begin{cases} 2-\dfrac{5+x}{7}<1-\dfrac{9-x}{14}, \\ 12-\dfrac{1}{3}\left(47-\dfrac{60}{x}\right)<3. \end{cases}$$

2. Find the integral solutions of the system

$$\begin{cases} \dfrac{2x-11}{4}+\dfrac{19-2x}{2}<2x, \\ \dfrac{2x+15}{9}>\dfrac{1}{5}(x-1)+\dfrac{x}{3}. \end{cases}$$

3. Find the integral values of x which satisfy the system of inequalities

$$\begin{cases} \dfrac{x^2-7x+6}{3x^2-x+1}<0, \\ x^2<36. \end{cases}$$

4. For what values of the parameter a is the inequality $-3<\dfrac{x^2+ax-2}{x^2-x+1}<2$ satisfied by any real numbers?

5. Solve the following collection of inequalities:

$$3x-1<\frac{x}{2}+2,\ 2x-3<\frac{x}{4}+1,\ 3-x>2+4x.$$

In Exercises 6 to 14, solve the given inequalities:

6. $\left|\dfrac{x}{x-2}\right|>1$. 7. $|x-3|+\dfrac{2}{3}x>2.5$.

8. $|2x+3|<|3x-1|$. 9. $|x+2|+|5x-2|>|3x+4|$.

10. $\dfrac{x^2-2x+1}{x^2-4x+4}+\left|\dfrac{1-x}{x-2}\right|-12<0$. 11. $|x^2-3x-15|<2x^2-x$.

12. $|2x^2+x+11|>x^2-5x+6$. 13. $x^2+2|x|\leqslant 63$. 14. $-2<\left|\dfrac{x-2}{x+1}\right|<1$.

15. Find the domain of definition for each of the given functions:

(a) $y=\sqrt{|x-5|-7}$, (b) $y=\sqrt{-|2x-3|+2}$, (c) $y=\sqrt{x^2-3}+\sqrt[4]{-x^2+4}$.

(d) $y=\sqrt{\dfrac{3x-4}{x+2}}+\sqrt[6]{(x^4-5x^3+6x^2)(1-x^2)}$.

16. For what integral values of a does the system $\begin{cases} 6x-3y=a+3, \\ 5x+2y=2a \end{cases}$ have the solution $\begin{cases} x<2, \\ y>0 \end{cases}$?

In Exercises 17 to 21, solve the given inequalities;

17. $\sqrt{x^2+3x}>2$. 18. $\sqrt{x^2-7x}>-3$. 19. $\sqrt{3x-1}<2$.

20. $\sqrt{3x-1}<\sqrt{x}$. 21. $\sqrt{x^2-7x}>-3$.

§ 5.4. Inequalities and Systems of Inequalities in Two Variables

In the present section, we shall consider inequalities of the form

$$f(x, y) > g(x, y).$$

The solution of such an inequality is a pair of real numbers (x_0, y_0) which satisfies the inequality, that is, a pair (x_0, y_0) such that $f(x_0, y_0) > g(x_0, y_0)$ is a true numerical inequality. As is known, a pair of real numbers (x_0, y_0) uniquely defines a point in the coordinate plane. This makes it possible to represent the set of solutions of an inequality or a system of inequalities in two variables geometrically, as some set of points in the coordinate plane.

The set of solutions of an inequality in two variables, and systems of such inequalities as well can be written with the aid of systems of the form

$$\begin{cases} a < x < b, \\ f(x) < y < g(x) \end{cases} \quad \text{or} \quad \begin{cases} c < y < d, \\ \varphi(y) < x < \psi(y) \end{cases}$$

(inequality signs in the systems may be nonstrict). These systems will be called *systems of the first and second forms*, respectively.

Example 1. Give a geometrical interpretation of the solution of the inequality $x + y - 1 > 0$ and indicate both forms of notation of the solutions.

○ We transform the given inequality to the form $y > -x + 1$. Then we construct the straight line $y = -x + 1$ in the rectangular coordinate system (Fig. 154). Since the ordinate of any point lying above the straight line $y = -x + 1$ is greater than the ordinate of a point on the straight line with the same abscissa, the set of points in the plane above this straight line serves as a geometrical interpretation of the solution of the given inequality (we may use the term "a geometrical solution of an inequality").

The geometrical interpretation enables us to write the solutions in the form

$$\begin{cases} -\infty < x < +\infty \\ 1 - x < y < +\infty \end{cases} \quad \text{or} \quad \begin{cases} -\infty < y < +\infty, \\ 1 - y < x < +\infty \end{cases}$$

(for the second notation, we have to transform the equation $y = -x + 1$ to the form which is solved with respect to x). ●

Example 2. Find the geometrical solution of the inequality $x(x - 2) \leqslant y - 3$ and write both forms of the notation of the solutions.

○ We transform the given inequality to the form which is solved with respect to y, we have $y \geqslant x^2 - 2x + 3$. Then, we construct in the coordinate plane a parabola which represents the graph of the

function $y = x^2 - 2x + 3$ (Fig. 155). Since the ordinate of any point lying above the parabola $y = x^2 - 2x + 3$ is greater than that of the point lying on the parabola with the same abscissa, the geometrical solution of the given inequality is the set of points in the plane lying on the parabola $x^2 - 2x + 3$ and above it. The analytical notation of the solutions is as follows

$$\begin{cases} -\infty < x < +\infty, \\ x^2 - 2x + 3 \leqslant y < +\infty. \end{cases}$$

Now we form the second version of the notation of solutions. We have $2 \leqslant y < +\infty$ (Fig. 155). We fix y (that is, we assume that y is a constant number from the interval $[2, +\infty)$) and see how x

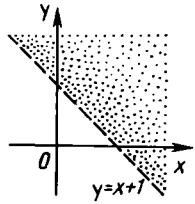


Fig. 154

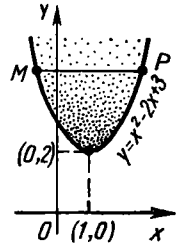


Fig. 155

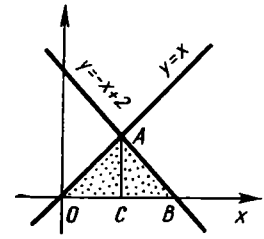


Fig. 156

varies. The least value of x (for the chosen value of y) corresponds to the point M lying on the left-hand branch of the parabola, while the greatest value to the point P on its right-hand branch. From the equation $y = x^2 - 2x + 3$ we get $x^2 - 2x + (3 - y) = 0$. This equation, quadratic with respect to x, has two roots $1 - \sqrt{y-2}$ and $1 + \sqrt{y-2}$. Consequently, the point M has the abscissa $x = 1 - \sqrt{y-2}$, and the point P the abscissa $x = 1 + \sqrt{y-2}$. Hence, with a fixed y from the interval $[2, +\infty)$, the abscissa x runs over the closed interval $[1 - \sqrt{y-2}, 1 + \sqrt{y-2}]$. This enables us to write the solutions in the form

$$\begin{cases} 2 \leqslant y < +\infty \\ 1 - \sqrt{y-2} \leqslant x \leqslant 1 + \sqrt{y-2}. \end{cases} \bullet$$

Example 3. Specify the domain which is the union of the domains

$$\begin{cases} 0 \leqslant x \leqslant 1, \\ 0 \leqslant y \leqslant x \end{cases} \quad \text{and} \quad \begin{cases} 1 \leqslant x \leqslant 2, \\ 0 \leqslant y \leqslant -x+2, \end{cases}$$

with the aid of a system of the second form.

○ Construct the given domains in the coordinate plane (Fig. 156). The system $\begin{cases} 0 \leqslant x \leqslant 1 \\ 0 \leqslant y \leqslant x \end{cases}$ corresponds to the intersection of four half-planes: $x \geqslant 0$ (the straight line $x = 0$ and points lying to the right of it), $x \leqslant 1$ (the straight line $x = 1$ and points lying to the left of it), $y \leqslant x$ (the straight line $y = x$ and points situated below it), and $y \geqslant 0$ (the straight line $y = 0$ and points lying above it). Hence, the geometrical interpretation of the system being considered is the set of points lying inside the triangle OAC (an interior domain) and on its boundary.

In much the same way the system $\begin{cases} 1 \leqslant x \leqslant 2 \\ 0 \leqslant y \leqslant 2 - x \end{cases}$ corresponds to the intersection of the half-planes $x \geqslant 1$, $x \leqslant 2$, $y \geqslant 0$, $y \leqslant 2 - x$, that is, the interior of the triangle BAC and its boundary. The union of the found sets is the set of points lying inside the triangle OAB bounded by the straight lines $y = x$, $y = 2 - x$, $y = 0$ and on its boundary.

Thus, the ordinates of all the points of the obtained domain belong to the closed interval $[0, y_A]$, where y_A is the ordinate of the point A. In order to find y_A, we solve the system of equations $\begin{cases} y = 2 - x, \\ y = x, \end{cases}$ whence we find $y_A = 1$; hence, the ordinate varies within the limits $0 \leqslant y \leqslant 1$. The figure shows that the abscissas of the points of the given domain lie between the abscissas of the points of the straight lines $x = y$ and $x = 2 - y$ ($y = -x + 2$). Thus, the domain can be defined by the following system of inequalities: $\begin{cases} 0 \leqslant y \leqslant 1, \\ y \leqslant x \leqslant 2 - y. \end{cases}$ ●

Example 4. Solve the system of inequalities

$$\begin{cases} x > 0, \\ y > 0, \\ xy > 4, \\ x + y < 5. \end{cases}$$

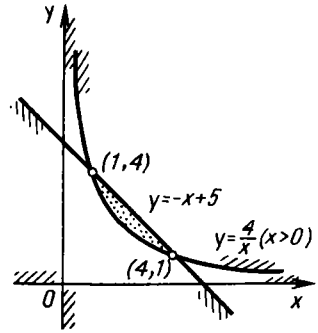


Fig. 157

○ Having found the intersection of the domains $x > 0$, $y > 0$, $y > 4/x$, $y < -x + 5$ in the coordinate plane, we get the geometrical solution of the given system of inequalities (Fig. 157). In order to write the solutions, we find the coordinates of the points of intersection of the straight lines $y = 4/x$ and $y = -x + 5$.

Solving the system of equations

$$\begin{cases} y = 4/x, \\ y = -x + 5, \end{cases}$$

we find the coordinates of the desired points: (1, 4) and (4, 1). Thus, we arrive at the system $\begin{cases} 1 < x < 4, \\ 4/x < y < 5 - x. \end{cases}$ ●

Exercises

1. Determine whether the following pairs of values of variables x and y are the solutions to the inequality $x + 3y + 1 < 0$: (a) $x = 0$, $y = 0$, (b) $x = 0$, $y = -3$, (c) $x = -3$, $y = 0$, (d) $x = 1$, $y = 1$, (e) $x = 1$, $y = -1$, (f) $x = 1$, $y = -2/3$.

2. Find geometrical solutions to the following inequalities and, at least, one of the forms of notation of solutions: (a) $y \leqslant 2x - 8$, (b) $y \geqslant -x - 6$, (c) $x + y - 2 \geqslant 0$, (d) $5x + 2y - 4 < 0$, (e) $y \geqslant 0.5x^2$, (f) $y \leqslant -1.5x^2$, (g) $y \geqslant 6/x$, (h) $y < 8/x$, (i) $x^2 + y^2 \leqslant 100$, (j) $|x| + |y| \leqslant 1$.

In Exercises 3 to 7, indicate in the coordinate plane the set of points (x, y) for which:

3. (a) $x \geqslant 4$, (b) $1 < x < 5$, (c) $x < -1$, (d) $y \leqslant 3.5$, (e) $y > -1$, (f) $-2 < y < 6$.

4. (a) $|x| \leqslant 2$, (b) $|y| \geqslant 3$.

5. (a) $\begin{cases} x \geqslant 0 \\ y \leqslant 0 \end{cases}$ (b) $\begin{cases} x > 2, \\ y < -1, \end{cases}$ (c) $\begin{cases} 2 \leqslant x \leqslant 4.5 \\ 1 \leqslant y \leqslant 5, \end{cases}$ (d) $\begin{cases} |x| \leqslant 2, \\ |y| \geqslant 5. \end{cases}$

6. (a) $x^2 + 2x + 4 - y \leqslant 0$, (b) $y + (x - 3)(x + 5) \leqslant 0$.

7. (a) $\begin{cases} y \geqslant 0, \\ x - y = 0, \end{cases}$ (b) $\begin{cases} 3x - 6y + 6 \leqslant 0, \\ x + y = 0. \end{cases}$

8. Solve the given systems of inequalities:

(a) $\begin{cases} y \geqslant x \\ y \leqslant -x, \end{cases}$, (b) $\begin{cases} y \geqslant x + 1, \\ 0 \leqslant x \leqslant 2, \\ y \leqslant 2, \end{cases}$ (c) $\begin{cases} y > x^2 - 4, \\ y < 5. \end{cases}$

9. Solve the given collections of systems of inequalities:

(a) $\begin{cases} -4 \leqslant x \leqslant -2, \\ 1 \leqslant y \leqslant 2, \end{cases}$ $\begin{cases} -2 \leqslant x \leqslant 0, \\ 1 \leqslant y \leqslant 5, \end{cases}$

(b) $\begin{cases} y \leqslant x, \\ y \geqslant 0, \\ 0 \leqslant x \leqslant 2, \end{cases}$ $\begin{cases} y \leqslant 4 - x, \\ y \geqslant 0, \\ 2 \leqslant x \leqslant 4. \end{cases}$

Chapter 6

BASIC FORMULAS OF TRIGONOMETRY AND THEIR APPLICATIONS

§ 6.1. Addition Theorem and Its Corollaries

1. Addition Theorem. *For any real numbers* α *and* β *the following formulas are valid*:

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta, \tag{1}$$

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta. \tag{2}$$

This theorem is given without proof.

Example. Compute $\sin 75°$.

○ We have $\sin 75° = \sin(30° + 45°)$. Making use of formula (2) for $\alpha = 30°$, $\beta = 45°$, we get

$$\sin(30° + 45°) = \sin 30° \cos 45° + \cos 30° \sin 45°.$$

As is known, $\sin 30° = 1/2$, $\sin 45° = \cos 45° = \sqrt{2}/2$, $\cos 30° = \sqrt{3}/2$. Hence

$$\sin 75° = \sin(30° + 45°) = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + \sqrt{6}}{4}. \quad \bullet$$

2. Addition Formulas. Double-angle Functions. Replacing β by $-\beta$ in formulas (1) and (2) and taking into consideration that $\sin x$ is an odd function and $\cos x$ an even one, we get:

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta, \tag{3}$$

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta. \tag{4}$$

Setting $\alpha = \beta$ in formulas (1) and (2), we get

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha, \tag{5}$$

$$\sin 2\alpha = 2 \sin\alpha \cos\alpha. \tag{6}$$

Let us prove now that

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}, \tag{7}$$

where*

$$\alpha \neq \pi/2 + \pi k,\ \beta \neq \pi/2 + \pi k,\ \alpha + \beta \neq \pi/2 + \pi k,\ k \in \mathbf{Z}$$

* Henceforward, when operating with trigonometric expressions containing variables in denominators, we shall assume that in the transformations to be carried out all the denominators of the fractions are other than zero and shall not stipulate this each time (for brevity).

Ineed,

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\sin\beta\cos\alpha}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}.$$

Dividing both the numerator and denominator by $\cos\alpha\cos\beta$ termwise, we get formula (7).

The formula

$$\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}. \tag{8}$$

is derived similarly.

Replacing β by $-\beta$ in formulas (7) and (8) and bearing in mind that $\tan x$ and $\cot x$ are odd functions, we have

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}, \tag{9}$$

$$\cot(\alpha-\beta)=\frac{1+\cot\alpha\cot\beta}{\cot\beta-\cot\alpha}. \tag{10}$$

Finally, setting in formulas (7) and (8) $\alpha=\beta$, we get

$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}, \tag{11}$$

$$\cot 2\alpha=\frac{\cot^2\alpha-1}{2\cot\alpha}. \tag{12}$$

Example 1. Compute $\cos 15°$.

○ Since $15°=45°-30°$, using formula (3), we get

$$\cos 15°=\cos(45°-30°)=\cos 45°\cos 30°+\sin 45°\sin 30°$$

$$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}. \quad ●$$

Example 2. Compute $\tan 2\alpha$ if $\tan\alpha=3/4$.

○ Applying formula (11) for the tangent of a double angle, we get

$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}=\frac{2\cdot 3/4}{1-(3/4)^2}=\frac{24}{7}. \quad ●$$

3. Reduction Formulas. By reduction formulas we usually understand formulas expressing, in terms of $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, and $\cot\alpha$, the values of trigonometric functions for the values of an argument obtained by adding the quantity $n\pi/2$, $n\in\mathbf{Z}$ to α or $-\alpha$. Since 2π is the period of each of the functions $\sin x$, $\cos x$, $\tan x$, and $\cot x$, it suffices to confine ourselves to the values of $n=1$, 2, 3, and 4. Let us, for instance, compute $\sin\left(\frac{\pi}{2}+\alpha\right)$. We have

$$\sin\left(\frac{\pi}{2}+\alpha\right)=\sin\frac{\pi}{2}\cos\alpha+\sin\alpha\cos\frac{\pi}{2}=\cos\alpha.$$

Analogously,

$$\sin(\pi-\alpha)=\sin\pi\cos\alpha-\sin\alpha\cos\pi=\sin\alpha,$$

$$\cos\left(\frac{\pi}{2}+\alpha\right)=\cos\frac{\pi}{2}\cos\alpha-\sin\frac{\pi}{2}\sin\alpha=-\sin\alpha,$$

$$\tan\left(\frac{\pi}{2}+\alpha\right)\frac{\sin\left(\frac{\pi}{2}+\alpha\right)}{\cos\left(\frac{\pi}{2}+\alpha\right)}=\frac{\cos\alpha}{-\sin\alpha}=-\cot\alpha.$$

The other reduction formulas are derived in a similar way. They are tabulated below:

Function \ Argument	$\frac{\pi}{2}-\alpha$	$\frac{\pi}{2}+\alpha$	$\pi-\alpha$	$\pi+\alpha$	$\frac{3\pi}{2}-\alpha$	$\frac{3\pi}{2}+\alpha$	$2\pi-\alpha$
sin	$\cos\alpha$	$\cos\alpha$	$\sin\alpha$	$-\sin\alpha$	$-\cos\alpha$	$-\cos\alpha$	$-\sin\alpha$
cos	$\sin\alpha$	$-\sin\alpha$	$-\cos\alpha$	$-\cos\alpha$	$-\sin\alpha$	$\sin\alpha$	$\cos\alpha$
tan	$\cot\alpha$	$-\cot\alpha$	$-\tan\alpha$	$\tan\alpha$	$\cot\alpha$	$-\cot\alpha$	$-\tan\alpha$
cot	$\tan\alpha$	$-\tan\alpha$	$-\cot\alpha$	$\cot\alpha$	$\tan\alpha$	$-\tan\alpha$	$-\cot\alpha$

To ease the task of memorizing the above reduction formulas, we may recommend the following technique:

(1) if the arc α is measured from the horizontal diameter ($\pi\pm\alpha$, $2\pi-\alpha$), then the name of function is retained; and if the arc α is measured from the vertical diameter ($\pi/2\pm\alpha$, $3\pi/2\pm\alpha$), then the names sine, cosine, tangent, and cotangent are changed for cosine, sine, cotangent, and tangent, respectively;

(2) regarding α as an arc in the first quadrant ($0<\alpha<\pi/2$) find in which quadrant the end point of the arc $\pi n/2\pm\alpha$ is located; then determine the sign of the given expression in this quadrant and put it before the result obtained.

Example 1. Simplify $\cos(3\pi/2-\alpha)$.

○ (1) Since the arc α is measured from the vertical diameter, we change cosine for sine.

(2) Regarding α as an arc of the first quadrant, we find that $3\pi/2-\alpha$ is a point belonging to the third quadrant; the given expression is negative in the third quadrant, therefore we put the minus sign before the result. Thus, $\cos(3\pi/2-\alpha)=-\sin\alpha$. ●

Example 2. Reduce $\tan 302^\circ$ to the function of an acute angle.

○ We have $\tan 302^\circ=\tan(360^\circ-58^\circ)$. Since the angle of 58° is laid off from the horizontal diameter, the name of the function is retained. The angle of 302° is found in the fourth quadrant, and

the tangent takes on only negative values in this quadrant. Therefore $\tan 302^\circ = -\tan 58^\circ$. ●

4. Relationships Between Trigonometric Functions of the Same Argument. Setting $\beta = \alpha$ in formula (3), we get

$$\cos(\alpha - \alpha) = \cos\alpha \cdot \cos\alpha + \sin\alpha \cdot \sin\alpha.$$

But $\cos(\alpha - \alpha) = \cos 0 = 1$, $\cos\alpha \cdot \cos\alpha = \cos^2\alpha$, and $\sin\alpha \times \sin\alpha = \sin^2\alpha$. Thus, we get

$$\cos^2\alpha + \sin^2\alpha = 1. \tag{13}$$

Let us now consider the expression $1 + \tan^2\alpha$. We have

$$1 + \tan^2\alpha = 1 + \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{\cos^2\alpha + \sin^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha}.$$

Thus,

$$1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}. \tag{14}$$

Similarly, we derive the formula

$$1 + \cot^2\alpha = \frac{1}{\sin^2\alpha}. \tag{15}$$

Example. It is known that $\sin x = a$, where $|a| < 1$. Find the values of other trigonometric functions.

○ From formula (13) we get $\cos^2 x = 1 - a^2$, whence $\cos x_1 = \sqrt{1 - a^2}$ or $\cos x_2 = -\sqrt{1 - a^2}$. We see that for a given value of sine the function $\cos x$ can take on two values because there are two points on a circle with the given ordinate a (Fig. 158). The abscissas of these points are equal in absolute value but have opposite signs. Therefore if $\sin x = a$, then for $\cos x$ we get two values equal in absolute value (modulus) but having opposite signs.

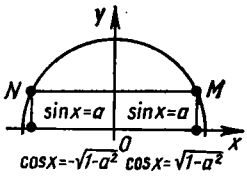


Fig. 158

In order to get a uniquely determined solution, the quadrant to which the point $M(x)$ belongs should be indicated. For instance, if the point $M(x)$ lies in the first or fourth quadrant, then $\cos x = \sqrt{1 - a^2}$. If the point $M(x)$ belongs to the second or third quadrant, then $\cos x = -\sqrt{1 - a^2}$.

Knowing $\sin x$ and $\cos x$, we can find $\tan x$ and $\cot x$:

$$\tan x_1 = \frac{a}{\sqrt{1-a^2}} \text{ or } \tan x_2 = -\frac{a}{\sqrt{1-a^2}},$$

$$\cot x_1 = \frac{\sqrt{1-a^2}}{a} \text{ or } \cot x_2 = -\frac{\sqrt{1-a^2}}{a}.$$

The two values of cosine obtained above determine two sets of answers:

$$(1)\ \begin{cases} \cos x_1 = \sqrt{1-a^2}. \\ \tan x_1 = \dfrac{a}{\sqrt{1-a^2}}, \\ \cot x_1 = \dfrac{\sqrt{1-a^2}}{a}, \end{cases} \quad (2)\ \begin{cases} \cos x_2 = -\sqrt{1-a^2}, \\ \tan x_2 = -\dfrac{a}{\sqrt{1-a^2}}, \\ \cot x_2 = -\dfrac{\sqrt{1-a^2}}{a}. \end{cases} \bullet$$

5. Half-angle Functions. Let us derive the formulas relating the functions of the arguments α and $\alpha/2$.

Let us prove that

$$1+\cos\alpha = 2\cos^2\frac{\alpha}{2}. \tag{16}$$

Indeed, using formulas (13) and (5) for the argument $\alpha/2$, we get

$$1+\cos\alpha = \left(\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}\right)+\left(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\right) = 2\cos^2\frac{\alpha}{2}.$$

Similarly, we prove that

$$1-\cos\alpha = 2\sin^2\frac{\alpha}{2}. \tag{17}$$

In practice, formulas (16) and (17) are used "from right to left", that is, in the form

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}, \quad \sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}.$$

In such a case, they are called the **power reduction formulas.**

Example. Compute $\cos\alpha$, $\sin\alpha$, $\tan\alpha$, and $\cot\alpha$ if $\alpha = 112°30'$.

○ Since $112°30' = 225° \div 2$, using formula (16) for $\alpha = 225°$, we get $1 + \cos 225° = 2\cos^2 112°30'$. Hence,

$$\cos^2 112°30' = \frac{1+\cos 225°}{2} = \frac{1-\sqrt{2}/2}{2} = \frac{2-\sqrt{2}}{4}.$$

Then

$$|\cos 112°30'| = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

But $\cos 112°30' < 0$, hence $\cos 112°30' = -\dfrac{\sqrt{2-\sqrt{2}}}{2}$.

Similarly, using formula (17) and bearing in mind that the angle $112°30'$ belongs to the second quadrant, we get

$$\sin 112°30' = \sqrt{\frac{1-\cos 225°}{2}} = \sqrt{\frac{1+\sqrt{2}/2}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

Further, we have

$$\tan 112°30' = \frac{\sin 112°30'}{\cos 112°30'} = -\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$$

$$= -\frac{\sqrt{2+\sqrt{2}}\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}\sqrt{2+\sqrt{2}}}$$

$$= -\frac{2+\sqrt{2}}{\sqrt{2}} = -(\sqrt{2}+1),$$

$$\cot 112°30' = \frac{1}{\tan 112°30'} = -\frac{1}{\sqrt{2}+1} = -(\sqrt{2}-1). \bullet$$

6. Transforming a Product of Trigonometric Functions into a Sum. Let us derive formulas enabling us to transform the products $\sin\alpha \times \cos\beta$, $\sin\alpha \sin\beta$, and $\cos\alpha \cos\beta$ into sums of trigonometric functions.

Consider the expression $\frac{\sin(\alpha-\beta)+\sin(\alpha+\beta)}{2}$. Applying the addition formulas to $\sin(\alpha-\beta)$ and $\sin(\alpha+\beta)$, we get

$$\frac{\sin(\alpha-\beta)+\sin(\alpha+\beta)}{2} = \frac{\sin\alpha\cos\beta-\sin\beta\cos\alpha+\sin\alpha\cos\beta+\sin\beta\cos\alpha}{2}$$

$$= \frac{2\sin\alpha\cos\beta}{2} = \sin\alpha\cos\beta.$$

Consequently,

$$\sin\alpha\cos\beta = \frac{\sin(\alpha-\beta)+\sin(\alpha+\beta)}{2}. \tag{18}$$

Similarly, we derive the following formulas:

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}, \tag{19}$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}. \tag{20}$$

Example 1. Transform the product $\sin 43° \cos 19°$ into a sum.

○ Using formula (18) for $\alpha = 43°$, $\beta = 19°$, we get

$$\sin 43° \cos 19° = \frac{\sin(43°-19°)+\sin(43°+19°)}{2} = \frac{1}{2}(\sin 24° + \sin 62°). \bullet$$

Example 2. Compute $16 \sin\frac{\alpha}{2}\sin\frac{3\alpha}{2}$ if $\cos\alpha = \frac{3}{4}$.

○ By formula (19), we have

$$16\sin\frac{\alpha}{2}\sin\frac{3\alpha}{2} = 16\cdot\frac{\cos\left(\frac{\alpha}{2}-\frac{3\alpha}{2}\right)-\cos\left(\frac{\alpha}{2}+\frac{3\alpha}{2}\right)}{2}$$

$$= 8(\cos(-\alpha)-\cos 2\alpha).$$

Since $\cos(-\alpha) = \cos\alpha$, and $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \cos^2\alpha - (1 - \cos^2\alpha) = 2\cos^2\alpha - 1$, the given expression equals $8(\cos\alpha - 2\cos^2\alpha + 1)$. But $\cos\alpha = 3/4$, therefore

$$16\sin\frac{\alpha}{2}\sin\frac{3\alpha}{2} = 8\left(\cos\alpha - 2\cos^2\alpha + 1\right) = 8\left(\frac{3}{4} - 2\cdot\frac{9}{16} + 1\right) = 5. \bullet$$

7. Transforming a Sum of Trigonometric Functions into a Product. Let us derive formulas enabling us to transform the sums $\sin\alpha \pm \sin\beta$, $\cos\alpha \pm \cos\beta$, $\tan\alpha \pm \tan\beta$, $\cot\alpha \pm \cot\beta$ into appropriate products of trigonometric functions.

Consider the sum $\sin\alpha + \sin\beta$. Setting $\alpha = x + y$, $\beta = x - y$, we get

$$\begin{aligned}\sin\alpha + \sin\beta &= \sin(x+y) + \sin(x-y)\\ &= \sin x\cos y + \sin y\cos x + \sin x\cos y\\ &\quad - \sin y\cos x = 2\sin x\cos y.\end{aligned}$$

Solving the system $\begin{cases} x+y=\alpha, \\ x-y=\beta \end{cases}$ with respect to x and y, we find $x = (\alpha+\beta)/2$; $y = (\alpha-\beta)/2$. Consequently,

$$2\sin x\cos y = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2},$$

that is,

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}. \tag{21}$$

Similarly, we derive the following formulas:

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}, \tag{22}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}, \tag{23}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}. \tag{24}$$

Let us now consider the sum of tangents $\tan\alpha + \tan\beta$. We have

$$\tan\alpha + \tan\beta = \frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta} = \frac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\cos\alpha\cos\beta} = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}.$$

Thus,

$$\tan\alpha + \tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}. \tag{25}$$

Similarly, we derive the following formulas:

$$\tan\alpha - \tan\beta = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}, \tag{26}$$

$$\cot\alpha + \cot\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}, \tag{27}$$

$$\cot\alpha - \cot\beta = \frac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta}. \tag{28}$$

Example 1. Transform $\sin 42^\circ - \cos 62^\circ$ into a product.

○ We have $\cos 62^\circ = \cos(90^\circ - 28^\circ) = \sin 28^\circ$. Hence, $\sin 42^\circ - \cos 62^\circ = \sin 42^\circ - \sin 28^\circ$. Applying formula (22) to the difference of sines, we get

$$\sin 42^\circ - \sin 28^\circ = 2\sin\frac{42^\circ-28^\circ}{2}\cos\frac{42^\circ+28^\circ}{2} = 2\sin 7^\circ\cos 35^\circ. \quad \bullet$$

Example 2. Transform $\sin x + \cos 2x - \sin 3x$ into a product.

○ We have

$$\sin x + \cos 2x - \sin 3x = \cos 2x - (\sin 3x - \sin x)$$

$$= \cos 2x - 2\sin\frac{3x-x}{2}\cos\frac{3x+x}{2}$$

$$= \cos 2x - 2\sin x\cos 2x = 2\cos 2x\left(\frac{1}{2} - \sin x\right)$$

$$= 2\cos 2x\cdot\left(\sin\frac{\pi}{6} - \sin x\right) = 2\cos 2x\cdot 2\sin\frac{\pi/6-x}{2}\cos\frac{\pi/6+x}{2}$$

$$= 4\cos 2x\sin\left(\frac{\pi}{12} - \frac{x}{2}\right)\cos\left(\frac{\pi}{12} + \frac{x}{2}\right). \quad \bullet$$

8. Transforming the Expression $a\cos\omega t + b\sin\omega t$ to the form $A\sin(\omega t + \alpha)$. As is known, the law of motion of a harmonically oscillating point has the form

$$s = A\sin(\omega t + \alpha),$$

where A is amplitude, ω frequency of oscillations, and α the initial phase (see § 3.7, Item 8). Applying the formula for the sine of the sum of two angles, we get

$$s = A\sin\omega t\cos\alpha + A\sin\alpha\cos\omega t.$$

Denoting $A\cos\alpha$ by b and $A\sin\alpha$ by a, we get

$$s = a\cos\omega t + b\sin\omega t.$$

Let us show that the converse is also true, that is, that any expression of the form $a\cos\omega t + b\sin\omega t$ can be represented as $A\sin(\omega t + \alpha)$. For this purpose, we factor out the expression

$\sqrt{a^2+b^2}$:

$$s = a\cos\omega t + b\sin\omega t = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos\omega t + \frac{b}{\sqrt{a^2+b^2}}\sin\omega t\right).$$

But $\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$. This means that the point with coordinates $\frac{a}{\sqrt{a^2+b^2}}$ and $\frac{b}{\sqrt{a^2+b^2}}$ lies on the number circle. Therefore, there is an α such that

$$\frac{a}{\sqrt{a^2+b^2}} = \sin\alpha, \quad \frac{b}{\sqrt{a^2+b^2}} = \cos\alpha.$$

Denoting $\sqrt{a^2+b^2}$ by A, we get

$$a\cos\omega t + b\sin\omega t = A(\sin\alpha\cos\omega t + \sin\omega t\cos\alpha).$$

Applying formula (2) to the expression in the parentheses, we get

$$a\cos\omega t + b\sin\omega t = A\sin(\omega t + \alpha).$$

The numbers a, b, A, α are related as follows:

$$a = A\sin\alpha, \quad b = A\cos\alpha, \quad A = \sqrt{a^2+b^2},$$

$$\sin\alpha = \frac{a}{\sqrt{a^2+b^2}}, \quad \cos\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

For example, $3\cos 2t + 4\sin 2t = 5\sin(2t+\alpha)$, where $\sin\alpha = 3/5$, $\cos\alpha = 4/5$.

9. Transforming Expressions Containing Inverse Trigonometric Functions.

Example 1. Simplify the expression $\cos(\arcsin x)$, where $-1 \leqslant x \leqslant 1$.

○ We set $\arcsin x = y$, then $\sin y = x$, $-\pi/2 \leqslant y \leqslant \pi/2$. We have to find $\cos y$.

It is known that $\cos^2 y = 1 - \sin^2 y$. Hence, $\cos^2 y = 1 - x^2$. But $-\pi/2 \leqslant y \leqslant \pi/2$, and on the closed interval $[-\pi/2, \pi/2]$ cosine takes on nonnegative values. Therefore $\cos y = \sqrt{1-x^2}$, that is, $\cos(\arcsin x) = \sqrt{1-x^2}$, where $-1 \leqslant x \leqslant 1$. ●

Example 2. Simplify the expression $\sin(\arctan x)$.

○ We set $\arctan x = y$; then $\tan y = x$, $-\pi/2 < y < \pi/2$. We have to find $\sin y$.

It is known that $1 + \tan^2 y = 1/\cos^2 y$, whence $\cos^2 y = 1/(1+\tan^2 y)$. But $-\pi/2 < y < \pi/2$, and on the open interval $(-\pi/2, \pi/2)$ cosine takes on only positive values. Therefore $\cos y = 1/\sqrt{1+\tan^2 y}$, that is, $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$. Since $\sin y = \tan y \cos y$, we have $\sin(\arctan x) = \frac{1}{\sqrt{1+x^2}}$. ●

Example 3. Prove that for any $x \in [-1, 1]$ the following identity is valid:

$$\arcsin x = \pi/2 - \arccos x. \tag{29}$$

○ We compute the value of sine of both sides of the equality in question:

$$\sin(\arcsin x) = x, \quad \sin(\pi/2 - \arccos x) = \cos(\arccos x) = x.$$

As is seen, the sines of both sides are equal, therefore, to make sure that equality (29) is valid, it remains to show that $\arcsin x$ and $\pi/2 - \arccos x$ belong to the same interval of monotonicity of the function $y = \sin x$ (without checking this condition, we can get an incorrect result since, as a matter of fact, trigonometric functions can also take on equal values even for different values of the argument; for instance, $\pi/6 \neq 5\pi/6$, but, nevertheless, $\sin(\pi/6) = \sin(5\pi/6) = 1/2$).

We have $-\pi/2 \leqslant \arcsin x \leqslant \pi/2$; besides, $0 \leqslant \arccos x \leqslant \pi$, and therefore $-\pi/2 \leqslant \pi/2 - \arccos x \leqslant \pi/2$. Thus, $\arcsin x$ and $\pi/2 - \arccos x$ belong to the same interval of monotonicity $[-\pi/2, \pi/2]$ of the function $y = \sin x$. Now, we may state that identity (29) has been proved. ●

Similarly, we can prove that $\arctan x = \pi/2 - \operatorname{arccot} x$.

10. Proving Trigonometric Inequalities.

Example 1. Prove that $\cos\gamma < \cos\alpha + \cos\beta$ if $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\alpha + \beta + \gamma = \pi/2$.

○ We take $\sin\alpha < 1$ and $\sin\beta < 1$ as two reference inequalities and multiply both sides of the first of them by $\cos\beta$ and both sides of the second by $\cos\alpha$. Since, by hypothesis, $\cos\alpha > 0$ and $\cos\beta > 0$, we have

$$\sin\alpha\cos\beta < \cos\beta, \quad \sin\beta\cos\alpha < \cos\alpha.$$

Adding together the last two inequalities, we get

$$\sin\alpha\cos\beta + \sin\beta\cos\alpha < \cos\alpha + \cos\beta,$$

that is,

$$\sin(\alpha + \beta) < \cos\alpha + \cos\beta.$$

The hypothesis implies that $\alpha + \beta = \pi/2 - \gamma$. Therefore $\sin(\alpha + \beta) = \sin(\pi/2 - \gamma) = \cos\gamma$. Hence, $\cos\gamma < \cos\alpha + \cos\beta$. ●

Example 2. Prove that $\sin x < x < \tan x$ if $0 < x < \pi/2$.

○ Let M be a point of the number circle corresponding to the number x. We erect a perpendicular at the point A to intersect the ray OM at the point K and compare the areas of the three figures obtained: the triangle OAM, sector OAM, and triangle OAK (Fig. 159). We have

$$S_{\Delta OAM} < S_{\text{sector } OAM} < S_{\Delta OAK}$$

(reference inequality). Further, we find

$$S_{\Delta OAM} = \frac{1}{2} OA \cdot MH = \frac{1}{2} \cdot 1 \cdot \sin x = \frac{1}{2} \sin x,$$

$$S_{\text{sector } OAM} = \frac{1}{2} OA^2 \cdot x = \frac{1}{2} x, \quad S_{\Delta OAK} = \frac{1}{2} OA \cdot AK = \frac{1}{2} AK.$$

From similarity of the triangles OHM and OAK we get

$$\frac{MH}{OH} = \frac{AK}{OA}, \text{ that is, } \frac{\sin x}{\cos x} = \frac{AK}{1}.$$

Hence, $AK = \tan x$ and therefore $S_{\Delta OAK} = \frac{1}{2} AK = \frac{1}{2} \tan x$. Using the obtained results, we rewrite the reference inequality in the form $\frac{1}{2} \sin x < \frac{1}{2} x < \frac{1}{2} \tan x$, whence $\sin x < x < \tan x$. ●

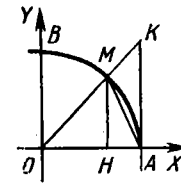


Fig. 159

Example 3. Prove that $|\sin x| \leqslant |x|$ for any x.

○ If $0 < x < \pi/2$, then, by virtue of the aforeproved (see the preceding example), $\sin x < x$. In the interval $(0, \pi/2)$ we have $|\sin x| = \sin x$, and $|x| = x$. Consequently, $|\sin x| < |x|$.

If $-\pi/2 < x < 0$, then $0 < -x < \pi/2$, and therefore $\sin(-x) < -x$. Bearing in mind that the function $y = \sin x$ is odd, we get $-\sin x < -x$. In the interval $(-\pi/2, 0)$ we have $\sin x < 0$, $x < 0$, and therefore $|\sin x| = -\sin x$, $|x| = -x$. Hence, $|\sin x| < |x|$.

If $x = 0$, then $|\sin x| = |x| = 0$.

Let, finally, $|x| \geqslant \pi/2$. Then $|x| > 1$, and $|\sin x| \leqslant 1$. Consequently, $|\sin x| < |x|$.

Generalizing the aforesaid, we get the inequality $|\sin x| \leqslant |x|$ is true for any values of the variable x. ●

Example 4. Prove that $\alpha - \sin \alpha < \beta - \sin \beta$ if $0 < \alpha < \beta < \pi/2$.

○ Here, as a reference inequality, we use the inequality proved in the preceding example. We have

$$\sin \frac{\beta - \alpha}{2} < \frac{\beta - \alpha}{2}, \text{ that is, } 2 \sin \frac{\beta - \alpha}{2} < \beta - \alpha.$$

Multiplying both sides of the last inequality by the positive number $\cos \frac{\beta + \alpha}{2}$, we get

$$2 \sin \frac{\beta - \alpha}{2} \cos \frac{\beta + \alpha}{2} < (\beta - \alpha) \cdot \cos \frac{\beta + \alpha}{2}.$$

Since

$$2 \sin \frac{\beta - \alpha}{2} \cos \frac{\beta + \alpha}{2} = \sin \beta - \sin \alpha,$$

and

$$(\beta-\alpha)\cos\frac{\beta+\alpha}{2}<(\beta-\alpha)\cdot 1=\beta-\alpha,$$

we get $\sin\beta-\sin\alpha<\beta-\alpha$, whence $\alpha-\sin\alpha<\beta-\sin\beta$. ●

Example 5. Prove the inequality $\cos(\alpha+\beta)\cos(\alpha-\beta)\leqslant\cos^2\alpha$.

○ Suppose the contrary, that is, that the inequality $\cos(\alpha+\beta)\cos(\alpha-\beta)>\cos^2\alpha$ is fulfilled. Using the formulas

$$\cos(\alpha+\beta)\cos(\alpha-\beta)=\frac{\cos 2\alpha+\cos 2\beta}{2},\quad \cos^2\alpha=\frac{1+\cos 2\alpha}{2},$$

we get

$$\frac{\cos 2\alpha+\cos 2\beta}{2}>\frac{1+\cos 2\alpha}{2},$$

that is, $\cos 2\beta>1$. Since $\cos 2\beta\leqslant 1$ for any values of β, we have arrived at a contradiction. Hence, our supposition is false, and therefore the inequality $\cos(\alpha+\beta)\cos(\alpha-\beta)\leqslant\cos^2\alpha$ is true. ●

Exercises

1. Simplify the given expressions:

(a) $\dfrac{2\sin^2\alpha-1}{\sin\alpha+\cos\alpha}$, (b) $\sin^4\alpha+\sin^2\alpha\cos^2\alpha-\sin^2\alpha+1$,

(c) $\tan\alpha+\dfrac{\cos^3\alpha-\sin^3\alpha}{(1+\cos\alpha\sin\alpha)\cos\alpha}$, (d) $\cos^6\alpha+\sin^6\alpha-1+3\sin^2\alpha\cos^2\alpha$.

2. Prove the validity of the given equalities:

(a) $1+\dfrac{\sin^4\alpha+\sin^2\alpha\cos^2\alpha}{\cos^2\alpha}=\dfrac{1}{\cos^2\alpha}$, (b) $\dfrac{\cos\alpha+\cot\alpha}{\cot\alpha}-\sin^2\alpha-\cos^2\alpha=\sin\alpha$.

3. Given the value of one of the trigonometric functions $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, and $\cot\alpha$ and the interval in which α is located, find the values of the three other functions: (a) $\sin\alpha=5/13$, $\pi/2<\alpha<\pi$, (b) $\cos\alpha=3/5$, $3\pi/2<\alpha<2\pi$, (c) $\tan\alpha=9/40$, $\pi<\alpha<3\pi/2$, (d) $\cot\alpha=-8/15$, $\pi/2<\alpha<\pi$.

4. Prove the given identities:

(a) $\dfrac{\cos\alpha\sin(\alpha-3)-\sin\alpha\cos(3-\alpha)}{\cos\left(3-\dfrac{\pi}{6}\right)-0.5\sin 3}=-\dfrac{2\sqrt{3}}{3}\tan 3$,

(b) $\dfrac{\sin^2(\alpha+\beta)-\sin^2(\alpha-\beta)}{4\sin^2\alpha\cos^2\beta}=\cot\alpha\tan\beta$.

5. Compute:

(a) $\dfrac{\sin 10^\circ\cos 20^\circ+\cos 10^\circ\sin 20^\circ}{\cos 19^\circ\cos 11^\circ-\sin 19^\circ\sin 11^\circ}$,

(b) $\dfrac{\sin(x+5^\circ)\cos(25^\circ-x)+\sin(25^\circ-x)\cos(5^\circ+x)}{\cos(50^\circ+y)\cos(10^\circ-y)-\sin(50^\circ+y)\sin(10^\circ-y)}$.

6. Prove the identity $\sin\left(\alpha+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha)$.

7. Compute:

(a) $\cos^2\frac{\pi}{8}-\sin^2\frac{\pi}{8}$, (b) $2\sin 15^\circ\sin 75^\circ$,

(c) $\frac{2\tan 75^\circ}{1-\tan^2 75^\circ}$, (d) $\frac{1-2\sin^2 22^\circ 30'}{2\cos^2 15^\circ-1}$, (e) $\sin 285^\circ$.

8. Prove the given identities:

(a) $\sin 3\alpha=3\sin\alpha\cos^2\alpha-\sin^3\alpha$,

(b) $\frac{1}{1+\tan\alpha\tan 2\alpha}=\cos 2\alpha$, (c) $\sin(45^\circ+\alpha)=\cos(45^\circ-\alpha)$.

9. Simplify the given expressions:

(a) $\cos\left(\frac{\pi}{2}+\alpha\right)\sin(\pi+\alpha)+\cos(\pi-\alpha)\sin(-\alpha)$

$+\cos(\pi+\alpha)\cos\left(\alpha-\frac{\pi}{2}\right)-\tan\left(\frac{\pi}{2}+\alpha\right)\cot\left(\frac{3\pi}{2}+\alpha\right)$,

(b) $2\sin 40^\circ+2\cos 130^\circ-3\sin 160^\circ-3\cos(-110^\circ)$.

10. Prove the equality $\tan 55^\circ-\tan 35^\circ=2\tan 20^\circ$.

11. Prove the given identities:

(a) $1+\sin 6\alpha=2\cos^2\left(\frac{\pi}{4}-3\alpha\right)$, (b) $\tan^2\frac{\alpha}{2}=\frac{2\sin\alpha-\sin 2\alpha}{2\sin\alpha+\sin 2\alpha}$.

12. Find $\sin\alpha$, $\cos\alpha$, and $\cot\alpha$ if $\cot(\alpha/2)=2$.

13. Graph the function $y=\frac{2\tan(x/2)}{1+\tan^2(x/2)}$.

14. Solve the inequality $\sin 7x\cos 6x<\cos 7x\sin 6x$.

15. Transform the following into a product: (a) $\sin 9^\circ-\sin 7^\circ$, (b) $\cos(\pi/8)-\sin(\pi/8)$, (c) $1+\cos\alpha$, (d) $1-2\cos\alpha$, (e) $\sin\alpha+\cos 2\alpha+\sin 3\alpha+\cos 4\alpha$.

16. Reduce the given expressions to the form $A\sin(\omega t+\alpha)$:

(a) $5\sin 3t-12\cos 3t$, (b) $7\cos 5t+24\sin 5t$.

17. Find the set of values of the function $y=3\cos(x/2)+4\sin(x/2)$.

In Exercises 18 to 22, prove the given identities:

18. $\frac{\sin\alpha+\sin 3\alpha}{\cos\alpha+\cos 3\alpha}=\tan 2\alpha$.

19. $\cos 4\alpha+4\cos 2\alpha+3=8\cos^4\alpha$.

20. $\frac{2\sin\alpha-\sin 3\alpha+\sin 5\alpha}{\cos\alpha-2\cos 2\alpha+\cos 3\alpha}=-\frac{2\cos 2\alpha}{\tan(\alpha/2)}$.

21. $\frac{\sin^2 2\alpha-4\sin^2\alpha}{\sin^2 2\alpha+4\sin^2\alpha-4}=\tan^4\alpha$.

22. $4\sin\left(\alpha+\frac{\pi}{3}\right)\sin\left(\alpha-\frac{\pi}{3}\right)=4\sin^2\alpha-3$.

23. Transform into a product:

(a) $\sin\alpha+\sin 3\alpha-\sin 5\alpha+\sin 9\alpha$,

(b) $3-4\cos 2\alpha+\cos 4\alpha$.

24. Compute without using tables: $\sin 10^\circ\sin 50^\circ\sin 70^\circ$.

25. Compute: (a) $\sin(\arctan 2)$, (b) $\cot(\arccos(-7/25))$.

26. Check the validity of the following equalities:

(a) $\arctan(1/2)=\arcsin(\sqrt{5}/5)$,

(b) $-\arctan(7/24)=\arcsin(-7/25)$.

27. Simplify the given expressions: (a) $\sin(2 \arcsin x)$, (b) $\tan(\arctan x - \arctan y)$.

28. Prove the identity $\arctan x = \arcsin(x/\sqrt{1+x^2})$.

29. Find the greatest and least values of the given functions:

(a) $y = \dfrac{3}{2\cos^2 3x + 5}$, (b) $y = \dfrac{1}{2\cos x + 5}$.

In Exercises 30 to 37, prove the given inequalities:

30. $\sqrt{\cos\varphi} < \sqrt{2}\cos(\varphi/2)$ if $-\pi/2 < \varphi < \pi/2$.

31. $\cot(\alpha/2) \geqslant 1 + \cot\alpha$ if $0 < \alpha < \pi$.

32. $\tan\alpha \tan\beta < 1$ if α and β are acute angles of an obtuse triangle.

33. $\tan\alpha \tan\beta > 1$ if α and β are angles of an acute triangle.

34. $\sin^4\alpha + \cos^4\alpha \geqslant 1/2$.

35. $\sin^4 x - 6\sin^2 x + 5 > 0$.

36. $\tan\alpha + \cot\alpha \geqslant 2$ if $0 < \alpha < \pi/2$.

37. $-\sqrt{2} \leqslant \sin\alpha + \cos\alpha \leqslant \sqrt{2}$.

§ 6.2. Trigonometric Equations

1. Solving Simple Trigonometric Equations. Let us derive formulas for solving simplest trigonometric equations, that is, equations of the form $T(x) = a$, where $T(x)$ is a trigonometric function.

First consider the equation $\sin x = a$, where $-1 \leqslant a \leqslant 1$. By definition, $\sin x$ is the ordinate of such a point on the number circle which corresponds to the number x. There are two points on the circle M and N symmetric to each other with respect to the OY-axis whose ordinates are equal to a (Fig. 160). One of these points lies on the semicircle DAB, the other on the semicircle BCD. If $a = 1$ or $a = -1$, then these points coincide (that is, merge into one point). One of the numbers to which the point M corresponds on the number circle is $\arcsin a$; therefore all the numbers to which the point M corresponds have the form*

$$\arcsin a = 2k\pi.$$

The point N also corresponds to an infinite set of numbers. From the equality $\sin(\pi - \alpha) = \sin\alpha$ it is seen that one of the numbers of this set is equal to $\pi - \arcsin\alpha$, and therefore all the numbers of the set have the form

$$\pi - \arcsin a + 2k\pi = -\arcsin a + (2k+1)\pi.$$

Thus, the roots of the equation $\sin x = a$ are written in the following form:

$$x = \arcsin a + \pi \cdot 2k, \tag{1}$$

$$x = -\arcsin a + \pi(2k+1). \tag{2}$$

* Here and elsewhere, when writing the answer to a trigonometric equation, we shall assume that the parameter (k, n, or m) takes on any integral values without mentioning this each time (for brevity).

Note that these formulas can be united into one formula:

$$x = (-1)^n \arcsin a + \pi n. \qquad (3)$$

Indeed, for an even n ($n = 2k$) formula (3) yields formula (1) and for an odd n ($n = 2k + 1$) formula (2).

Consider, further, the equation $\cos x = a$, $-1 \leqslant a \leqslant 1$. By definition $\cos x$ is the abscissa of such a point of the number circle which corresponds to the number x. There are two points M and N on the

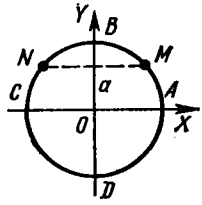


Fig. 160

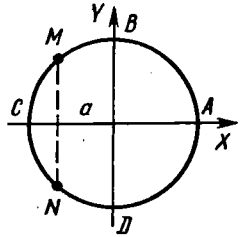


Fig. 161

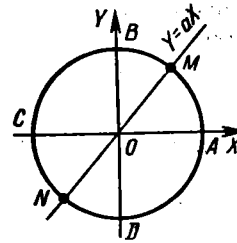


Fig. 162

number circle whose abscissas are equal to a. These points are symmetric to each other with respect to the axis of abscissas, and therefore the arc lengths of AM and AN are equal in absolute value but have opposite signs (Fig. 161). Hence it follows that the point M corresponds to numbers of the form $\arccos a + 2k\pi$, and the point N to numbers of the form $-\arccos a + 2k\pi$. Therefore the solution of the equation $\cos x = a$ has the form

$$x = \pm \arccos a + 2\pi n. \qquad (4)$$

Consider now the equation $\tan x = a$. We know that $\tan x = \sin x/\cos x$ that is, $\tan x$ is equal to the ratio of the ordinate of the point M (x) of the number circle to its abscissa. Therefore, we first find the points on the circle for which $Y/X = a$. These points lie at the intersection of the circle and the straight line $Y = aX$. We get two diametrically opposite points M and N one of which (M) lies on the semicircle DAB, and the other (N) on the semicircle BCD (Fig. 162). The point M corresponds to numbers of the form $\arctan a + 2k\pi$. Since the arc length of MCN is equal to π, the point N corresponds to numbers of the form $\arctan a + \pi + 2k\pi$. These formulas may be combined into one: $\arctan a + \pi n$.

Thus, the solution of the equation $\tan x = a$ has the form

$$x = \arctan a + \pi n. \qquad (5)$$

In exactly the same manner, for the equation $\cot x = a$ we get

$$x = \operatorname{arccot} a + \pi n. \qquad (6)$$

Example 1. Solve the equation $\sin x = 1/2$.

○ By formula (3), we get $x = (-1)^n \arcsin (1/2) + \pi n$. Since $\arcsin (1/2) = \pi/6$, we finally have

$$x = (-1)^n \frac{\pi}{6} + \pi n. \quad \bullet$$

Example 2. Solve the equation $\cos 3x = -\sqrt{2}/2$.

○ Using formula (4), we get $3x = \pm \arccos (-\sqrt{2}/2) + 2\pi n$. Since $\arccos (-\sqrt{2}/2) = \pi - \arccos (\sqrt{2}/2) = \pi - \pi/4 = 3\pi/4$, we have

$$3x = \pm \frac{3\pi}{4} + 2\pi n, \text{ that is, } x = \pm \frac{\pi}{4} + \frac{2\pi n}{3}. \quad \bullet$$

Example 3. Solve the equation $\tan \left(x - \frac{\pi}{6}\right) = -\sqrt{3}$.

○ Using formula (5), we get $x - \frac{\pi}{6} = \arctan (-\sqrt{3}) + \pi n$. Since $\arctan (-\sqrt{3}) = -\arctan \sqrt{3} = -\pi/3$, we have

$$x - \frac{\pi}{6} = -\frac{\pi}{3} + \pi n, \text{ that is, } x = -\frac{\pi}{6} + \pi n. \quad \bullet$$

Note that in some cases it is more convenient to use particular formulas which are readily derived with the help of the number circle:

(1) $\sin x = 0$, $x = \pi n$.
(2) $\sin x = 1$, $x = \pi/2 + 2\pi n$.
(3) $\sin x = -1$, $x = -\pi/2 + 2\pi n$.
(4) $\cos x = 0$, $x = \pi/2 + \pi n$.
(5) $\cos x = 1$, $x = 2\pi n$.
(6) $\cos x = -1$, $x = \pi + 2\pi n$.

2. Solving Trigonometric Equations by Introducing a New Variable. There are two principal methods for simplifying trigonometric equations: (1) introduction of a new variable and (2) factorization. Both methods are used to reduce a given trigonometric equation to a simpler one or to a collection of simpler equations. Consider the **method of introducing a new variable.**

Let a trigonometric equation have the form $f(\sin x) = 0$, where f is an algebraic function. Then, by making substitution $\sin x = z$, it is reduced to the algebraic equation $f(z) = 0$. Let $z_1, z_2, \ldots, z_n$ be such roots of this equation for which $|z_k| \leqslant 1$. Then the equation $f(\sin x) = 0$ is equivalent to the collection of equations:

$$\sin x = z_1, \ \sin x = z_2, \ \ldots, \ \sin x = z_n.$$

Equations of the form $f(\cos x) = 0$, $f(\tan x) = 0$, and $f(\cot x) = 0$ are solved in a similar way.

Prior to making a substitution, we often have to carry out certain identity transformations. If a given equation contains trigonometric

functions of the same argument, we have to express all these functions in terms of one of them, say, in terms of $\sin x$, and then, by making use of the substitution $\sin x = z$, to reduce this equation to an algebraic one.

Example 1. Solve the equation $2\cos^2 x + 4\cos x = 3\sin^2 x$.

○ Since $\sin^2 x = 1 - \cos^2 x$, the given equation can be rewritten as follows:

$$2\cos^2 x + 4\cos x = 3(1 - \cos^2 x), \quad \text{that is,}$$
$$5\cos^2 x + 4\cos x - 3 = 0.$$

Let us set $\cos x = z$, then we get the quadratic equation $5z^2 + 4z - 3 = 0$. Solving this equation, we find $z_1 = (-2 + \sqrt{19})/5$, $z_2 = -(2 + \sqrt{19})/5$. The second root does not suit since $|(2 + \sqrt{19})/5| > 1$. Hence, it remains only to solve the equation $\cos x = (-2 + \sqrt{19})/5$. Solving this equation, we get

$$x = \pm \arccos \frac{-2+\sqrt{19}}{5} + 2\pi n. \quad \bullet$$

Let us give some rules facilitating the choice of a new variable for trigonometric equations. $R(\cos x, \sin x)$ will denote a rational function of $\cos x$ and $\sin x$, that is, a function obtained from $\cos x$, $\sin x$, and constants by adding, multiplying, and dividing. For example,

$$\frac{1+\cos^2 x+\sin x}{\cos^2 x-\sin^2 x}, \qquad \frac{(\cos x+\sin x)(\cos^3 x+\sin^3 x-1)}{2\cos x \sin x}.$$

Consider an equation of the form $R(\cos x, \sin x) = 0$. In some cases we succeed in reducing such an equation to a rational equation with respect to $\cos x$ or $\sin x$. Namely, *if* $\cos x$ *enters an equation only in even powers, then, by replacing everywhere* $\cos^2 x$ *by* $1 - \sin^2 x$, *we get an algebraic equation with respect to* $\sin x$. In similar fashion, *if* $\sin x$ *enters an equation only in even powers, then, by replacing* $\sin^2 x$ by $1 - \cos^2 x$, *the given equation is reduced to an algebraic equation with respect to* $\cos x$.

Example 2. Solve the equation $\cos^4 x + 3\sin x - \sin^4 x - 2 = 0$.

○ This equation contains $\cos x$ only in the fourth power. According to the mentioned rule, it is advisable to replace $\cos^2 x$ by $1 - \sin^2 x$, that is, to replace $\cos^4 x$ by $(1 - \sin^2 x)^2$. Then the given equation takes the form

$$(1 - \sin^2 x)^2 + 3\sin x - \sin^4 x - 2 = 0.$$

The substitution $\sin x = z$ reduces the last equation to the algebraic equation $(1 - z^2)^2 + 3z - z^4 - 2 = 0$ and, further, to $2z^2 - 3z + 1 = 0$, whence $z_1 = 1$, $z_2 = 1/2$. Now, we have to solve the collection

of equations $\sin x = 1$, $\sin x = 1/2$, whence we get

$$x = \frac{\pi}{2} + 2\pi n, \quad x = (-1)^k \frac{\pi}{6} + \pi k. \bullet$$

3. Homogeneous Equations. In some cases, a trigonometric equation of the form $R(\cos x, \sin x) = 0$ can be reduced to an algebraic equation with respect to $\tan x$. Such equations are characterized by the following property: they remain unchanged upon simultaneous replacement of $\cos x$ by $-\cos x$ and $\sin x$ by $-\sin x$. The following homogeneous equations may serve as examples of such equations:

$$a \sin x + b \cos x = 0,$$

$$a \sin^2 x + b \sin x \cos x + c \cos^2 x = 0,$$

$$a \sin^3 x + b \sin^2 x \cos x + c \sin x \cos^2 x + d \cos^3 x = 0.$$

Consider the case when $a \neq 0$. Dividing both sides of the first equation by $\cos x$, both sides of the second equation by $\cos^2 x$, and both sides of the third by $\cos^3 x$, we get in succession the following algebraic equations with respect to $\tan x$, which are therefore solvable by making the substitution $\tan x = z$:

$$a \tan x + b = 0,$$

$$a \tan^2 x + b \tan x + c = 0,$$

$$a \tan^3 x + b \tan^2 x + c \tan x + d = 0.$$

It is easily seen that for $a \neq 0$ homogeneous equations are not satisfied by the values of x for which $\cos x = 0$. Therefore the division of both sides of a homogeneous equation by $\cos x$, $\cos^2 x$, and $\cos^3 x$ does not lead to a loss of roots if $a \neq 0$.

Example 1. Solve the equation $8 \sin x - 7 \cos x = 0$.

○ Dividing both sides of the equation termwise by $\cos x$, we get $8 \tan x - 7 = 0$. Further, we have $\tan x = 7/8$, whence $x = \arctan(7/8) + \pi n$. ●

Example 2. Solve the equation $\sin^2 x + 2 \sin x \cos x - 3 \cos^2 x = 0$.

○ Dividing both sides of the homogeneous equation of the second degree by $\cos^2 x$, we get $\tan^2 x + 2 \tan x - 3 = 0$. Further, setting $u = \tan x$, we obtain the quadratic equation $u^2 + 2u - 3 = 0$, whence $u_1 = -3$ and $u_2 = 1$.

Solving the collection of equations $\tan x = -3$, $\tan x = 1$, we get $x = \arctan(-3) + \pi k$, $x = \pi/4 + \pi n$. ●

Example 3. Solve the equation $5 \sin^2 x + \sqrt{3} \sin x \cos x + 6 \cos^2 x = 5$.

○ We have

$$5 \sin^2 x + \sqrt{3} \sin x \cos x + 6 \cos^2 x = 5 (\sin^2 x + \cos^2 x),$$

$$\sqrt{3} \sin x \cos x + \cos^2 x = 0. \qquad (7)$$

The obtained equation has no term of the form $a \sin^2 x$, that is, $a = 0$. Here, it is impossible to divide both sides of the equation by $\cos^2 x$ since the values of x for which $\cos^2 x = 0$ satisfy equation (7), and therefore the division by $\cos^2 x$ will lead to a loss of roots. Let us proceed in a different way: factoring the left-hand side of equation (7), we get $\cos x (\sqrt{3} \sin x + \cos x) = 0$.

Now, the problem is reduced to solving the collection of equations

$$\cos x = 0, \quad \sqrt{3} \sin x + \cos x = 0. \tag{8}$$

From the first equation of collection (8) we find $x = \pi/2 + \pi k$. Dividing both sides of the second equation by $\cos x$, we have $\sqrt{3} \tan x + 1 = 0$, whence $\tan x = -1/\sqrt{3}$, $x = \arctan(-1/\sqrt{3}) + \pi n$, that is, $x = -\pi/6 + \pi n$.

Thus, we get two sets of solutions: $x = \pi/2 + \pi n$, $x = -\pi/6 + \pi n$. ●

4. Universal Substitution. The substitutions we have considered are appropriate only for special cases of equations of the form $R(\cos x, \sin x) = 0$. Let us now consider the substitution which enables us to reduce any equation of the form $R(\cos x, \sin x) = 0$ to a rational algebraic equation. This is the substitution $\tan(x/2) = u$. If $x \neq \pi + 2\pi n$, then the following identities are valid:

$$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}, \quad \sin x = \frac{2 \tan(x/2)}{1 + \tan^2(x/2)}. \tag{9}$$

Indeed, we have

$$\frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} = \frac{1 - \dfrac{\sin^2(x/2)}{\cos^2(x/2)}}{1/\cos^2(x/2)} = \cos^2(x/2) - \sin^2(x/2) = \cos x,$$

$$\frac{2 \tan(x/2)}{1 + \tan^2(x/2)} = \frac{2 \cdot \dfrac{\sin(x/2)}{\cos(x/2)}}{1/\cos^2(x/2)} = 2 \sin(x/2) \cos(x/2) = \sin x.$$

Therefore the substitution $\tan(x/2) = u$ transforms the equation $R(\cos x, \sin x) = 0$ into the equation

$$R\left(\frac{1 - u^2}{1 + u^2}, \frac{2u}{1 + u^2}\right) = 0.$$

The left-hand side of this equation is a rational function of u. Hence the substitution has transformed the equation to rational form. The substitution $\tan(x/2) = u$ is said to be *universal*.

Since the universal substitution may be only used for $x \neq \pi + 2\pi n$, we have to see whether numbers of the form $x = \pi + 2\pi n$ are solutions to the given equation.

Example 1. Solve the equation $3 \sin x + 4 \cos x = 5$.

○ Expressing $\sin x$ and $\cos x$ in terms of $\tan (x/2)$ by formulas (7) and setting $\tan (x/2) = u$, we arrive at the rational equation

$$3 \cdot \frac{2u}{1+u^2} + 4 \cdot \frac{1-u^2}{1+u^2} = 5.$$

Solving this equation, we get $u = 1/3$. From the equation $\tan (x/2) = 1/3$ we find

$$\frac{x}{2} = \arctan \frac{1}{3} + \pi n, \text{ that is, } x = 2 \arctan \frac{1}{3} + 2\pi n.$$

We then check to see that the values $x = \pi + 2\pi n$ do not satisfy the given equation. Thus, we get $x = 2 \arctan (1/3) + 2\pi n$. ●

Example 2. Solve the equation $3 \sin 2x + \cos 2x + 1 = 0$.

○ We make use of the universal substitution. Expressing $\sin 2x$ and $\cos 2x$ in terms of $\tan x$ and setting $\tan x = u$, we get the rational equation

$$\frac{6u}{1+u^2} + \frac{1-u^2}{1+u^2} + 1 = 0,$$

whence $u = -1/3$. From the equation $\tan x = -1/3$ we find $x = \arctan (-1/3) + \pi k$.

However, we have to see whether the values of x for which $2x = \pi + 2\pi n$, that is, the values $x = \pi/2 + \pi n$, satisfy the given equation. We have

$$3 \sin (\pi + 2\pi n) + \cos (\pi + 2\pi n) + 1$$
$$= 3 \sin \pi + \cos \pi + 1 = 3 \cdot 0 - 1 + 1 = 0.$$

The verification shows that the values $\pi/2 + \pi n$ are solutions to the equation. Thus, the given equation has the following solutions: $x = \arctan (-1/3) + \pi n$, $x = \pi/2 + \pi n$. ●

5. Solving Trigonometric Equations by the Factorization Method. The **factorization method** consists in the following. Let there be given a trigonometric equation $f(x) = 0$, where $f(x) = g(x) h(x)$. Then all the solutions of the equation $f(x) = 0$ are also the solutions of the collection of equations $g(x) = 0$, $h(x) = 0$. The converse is false, since among the solutions of one of these equations there may be such that do not belong to the domain of definition of the other equation, and, hence, to the domain of the equation $f(x) = 0$. Thus, having transformed the equation $\cos x (\tan x - 1) = 0$ into the collection of equations $\cos x = 0$, $\tan x = 1$, we get $x = \pi/2 + \pi k$, $x = \pi/4 + \pi k$. But the values $x = \pi/2 + \pi k$ are extraneous roots for the equation $\cos x (\tan x - 1) = 0$ since for the indicated values $\tan x$ is not defined.

The change from the equation $g(x) h(x) = 0$ to the collection of equations $g(x) = 0$, $h(x) = 0$ is an equivalent transformation if the domain of definition for each of the equations of the collection

is the same as that of the original equation. This may occur, for instance, when the left-hand side of the equation $g(x)\,h(x) = 0$ is defined for all x's.

Example. Solve the equation $\sin 5x + \sin x + 2\sin^2 x = 1$.

○ Let us apply the formula for the sum of sines to the first two terms and take advantage of the fact that $2\sin^2 x = 1 - \cos 2x$. Then the equation will take the form

$$2\sin 3x\cos 2x + 1 - \cos 2x = 1 \text{ or } \cos 2x\,(2\sin 3x - 1) = 0.$$

Now, the problem is reduced to solving the collection of equations $\cos 2x = 0$, $2\sin 3x - 1 = 0$ which is equivalent to the given equation. Solving the equations of this collection, we get the following two sets of solutions of the given equation:

$$x = \frac{\pi}{4} + \frac{\pi n}{2}, \quad x = (-1)^n \frac{\pi}{18} + \frac{\pi n}{3}. \bullet$$

6. Method of Introducing an Auxiliary Argument. Sometimes, when solving trigonometric equations it turns out to be useful to replace the expression $a\cos x + b\sin x$ by $A\sin(x + \varphi)$, where $A = \sqrt{a^2 + b^2}$, $\sin\varphi = a/\sqrt{a^2 + b^2}$, $\cos\varphi = b/\sqrt{a^2 + b^2}$ (see § 6.1, Item 8). In this case, φ is called the *auxiliary argument*.

Example 1. Solve the equation $8\cos x + 15\sin x = 17$.

○ Dividing both sides of the equation by $\sqrt{8^2 + 15^2} = 17$, we get

$$\frac{8}{17}\cos x + \frac{15}{17}\sin x = 1.$$

Since $(8/17)^2 + (15/17)^2 = 1$, there is a φ such that $8/17 = \sin\varphi$ and $15/17 = \cos\varphi$. Let us rewrite the last equation as follows:

$$\sin\varphi\cos x + \sin x\cos\varphi = 1.$$

But $\sin\varphi\cos x + \sin x\cos\varphi = \sin(x + \varphi)$. Hence $(x + \varphi) = 1$, whence $x = \pi/2 + 2\pi n - \varphi$. Since $\varphi = \arcsin(8/17)$, we finally get the following solutions of the given equation: $x = \pi/2 - \arcsin(8/17) + 2\pi n$. ●

Example 2. Solve the equation $5\sin x - 12\cos x + 13\sin 3x = 0$.

○ We have

$$\sqrt{5^2 + 12^2}\left(\frac{5}{\sqrt{5^2 + 12^2}}\sin x - \frac{12}{\sqrt{5^2 + 12^2}}\cos x\right) = -13\sin 3x,$$

$$\frac{5}{13}\sin x - \frac{12}{13}\cos x = -\sin 3x.$$

Setting $5/13 = \cos\varphi$, $12/13 = \sin\varphi$, we get

$$\sin x\cos\varphi - \cos x\sin\varphi = -\sin 3x$$

and, further,

$$\sin(x - \varphi) + \sin 3x = 0, \quad 2\sin\left(2x - \frac{\varphi}{2}\right)\cos\left(x + \frac{\varphi}{2}\right) = 0.$$

Solving the collection of equations $\sin\left(2x-\frac{\varphi}{2}\right)=0$, $\cos\left(x+\frac{\varphi}{2}\right)=0$, we get: $x=\frac{\varphi}{4}+\frac{\pi n}{2}$, $x=\frac{\pi}{2}-\frac{\varphi}{2}+\pi n$. Finally, bearing in mind that $\varphi=\arccos(5/13)$, we get the following two sets of solutions for the given equation:

$$x=\frac{1}{4}\arccos\frac{5}{13}+\frac{\pi n}{2},\quad x=\frac{\pi}{2}-\frac{1}{2}\arccos\frac{5}{13}+\pi n. \bullet$$

7. Equations with a Variable in the Denominator. When solving equations of the form $T_1(x)/T_2(x)=0$, where $T_1(x)$ and $T_2(x)$ are trigonometric functions, we set up a mixed system

$$\begin{cases} T_1(x)=0, \\ T_2(x)\neq 0, \end{cases}$$

(proceeding from the condition that the fraction is equal to zero) and then, on solving the equation $T_1(x)=0$, choose those of its solutions for which the sentence $T_2(x)\neq 0$ is true and reject the solutions for which this sentence is false (that is, the sentence $T_2(x)=0$ is true). A couple of examples below shows what methods are used to realize this choice in practice.

Example 1. Solve the equation $\frac{\sin x}{1+\cos x}=0$.

○ The given equation is equivalent to the following mixed system

$$\begin{cases} \sin x=0, \\ 1+\cos x\neq 0. \end{cases}$$

From the equation $\sin x=0$ we find $x=\pi k$. From the condition $1+\cos x\neq 0$ we find $x\neq\pi+2\pi n$. Rejecting from the set of numbers of the form $x=\pi k$ the values entering the set $x=\pi+2\pi n$, we get $x=2\pi m$. These are the solutions of the original equation. Thus, we get $x=2\pi m$. ●

Example 2. Solve the equation $\tan x+\tan 2x=\tan 3x$.

○ Using the formula of the sum of tangents for the expression on the left-hand side, we get

$$\frac{\sin 3x}{\cos x\cos 2x}=\frac{\sin 3x}{\cos 3x},$$

and, further,

$$\frac{\sin 3x(\cos 3x-\cos x\cos 3x)}{\cos x\cos 2x\cos 3x}=0.$$

Transforming the product $\cos x \cos 2x$ into a sum, we have in the numerator:

$$\sin 3x\left(\cos 3x-\frac{\cos 3x+\cos x}{2}\right)=\frac{1}{2}\sin 3x\,(\cos 3x-\cos x)$$
$$=-\sin 3x\sin 2x\sin x,$$

and the equation takes the form

$$\frac{\sin 3x\sin 2x\sin x}{\cos x\cos 2x\cos 3x}=0.$$

Now, the problem is reduced to solving the mixed system

$$\begin{cases}\sin 3x\sin 2x\sin x=0,\\ \cos x\cos 2x\cos 3x\neq 0.\end{cases}$$

The equation $\sin 3x \sin 2x \sin x = 0$ is equivalent to the collection of equations $\sin 3x = 0$, $\sin 2x = 0$, $\sin x = 0$. From this collection we get $x = \pi k_1/3$, $x = \pi k_2/2$, $x = \pi k_3$. We mark the solutions found by points on the number circle (closed circles in Fig. 163).

From the conditions $\cos x \cos 2x \cos 3x \neq 0$ we find

$$\begin{cases}\cos x\neq 0,\\ \cos 2x\neq 0,\\ \cos 3x\neq 0,\end{cases}\quad\text{that is,}\quad\begin{cases}x\neq\frac{\pi}{2}+\pi n,\\ x\neq\frac{\pi}{4}+\frac{\pi m}{2},\\ x\neq\frac{\pi}{6}+\frac{\pi l}{3}.\end{cases}$$

On the same number circle, we mark the points corresponding to numbers of the form $\frac{\pi}{2}+\pi n$, $\frac{\pi}{4}+\frac{\pi m}{2}$, $\frac{\pi}{6}+\frac{\pi l}{3}$ (open circles in Fig. 163). From the solutions found above we have to exclude the values to which there correspond the upper and the lower end points of the vertical diameter of the circle. The other closed circles in Fig. 163 correspond to numbers of the form $x = \pi k/3$. These are solutions of the given equation. Thus, we get $x = \pi k/3$. ●

8. Systems of Trigonometric Equations in Two Variables. When solving systems of trigonometric equations, we use the methods considered above (see § 4.4) and formulas of trigonometry. Thus, we have nothing new here. We consider an example of solving a system of trigonometric equations, and thereby show two specific features of such systems.

Example. Solve the system of equations

$$\begin{cases}\sin x\cos y=1/4,\\ \sin y\cos x=3/4.\end{cases}$$

○ Replacing the first equation of the system by the sum of both equations given and the second by their difference, we get a system

which is equivalent to the given one:

$$\begin{cases} \sin x \cos y + \sin y \cos x = 1, \\ \sin x \cos y - \sin y \cos x = -1/2. \end{cases}$$

Since $\sin x \cos y + \sin y \cos x = \sin (x + y)$, and $\sin x \cos y - \sin y \cos x = \sin (x - y)$, we have

$$\begin{cases} \sin (x+y) = 1, \\ \sin (x-y) = -1/2. \end{cases}$$

From the equation $\sin (x + y) = 1$ we find $x + y = \pi/2 + 2\pi k$.

Prior to passing to the equation $\sin (x - y) = -1/2$, we should like to note that the general formulas obtained in subsection 1 for

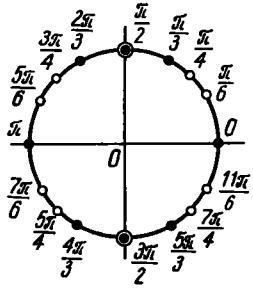


Fig. 163

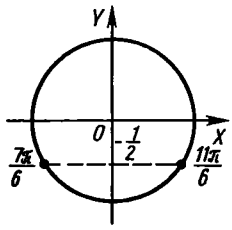


Fig. 164

the equations $\sin x = a$ and $\cos x = a$ are usually inconvenient when solving systems of trigonometric equations. It is more expedious to make use of the unit circle to solve each of such equations (this is the first specific feature).

Thus, consider the equation $\sin (x - y) = -1/2$. Making use of the unit circle (Fig. 164), we find that either $x - y = 7\pi/6 + 2\pi n$ or $x - y = 11\pi/6 + 2\pi n$. Thus, the problem is reduced to solving two linear systems of equations in two variables x and y:

$$\begin{cases} x+y = \pi/2 + 2\pi k, \\ x-y = 7\pi/6 + 2\pi n, \end{cases} \quad \begin{cases} x+y = \pi/2 + 2\pi k, \\ x-y = 11\pi/6 + 2\pi n. \end{cases}$$

Solving the first system, we get

$$x = 5\pi/6 + \pi (k + n), \quad y = -\pi/3 + \pi (k - n),$$

Solving the second system, we get

$$x = 7\pi/6 + \pi (k + n), \quad y = -2\pi/3 + \pi (k - n). \bullet$$

Note that what counts in solving systems of trigonometric equations is the use of different letters to denote the parameter (k, n,

$m, \ldots$) in writing the solutions of the first and second equations of a given system. In other words, if in the first equation of the system the parameter was denoted by the letter k, then in the second equation this letter cannot be used. In our example, we used the letter n for this purpose (this is the second of the two specific features mentioned above).

Exercises

1. Solve the following trigonometric equations:

(a) $2\sin(2x-1)=1$, (b) $3\cos\left(x+\frac{\pi}{4}\right)=1$, (c) $\tan 3x=10^6$,

(d) $\cos x^2=-1$, (e) $\sin(3x-1)=(\sqrt{3}+1)/2$.

In Exercises 2 to 5, solve the given equations by introducing a new variable:

2. $\cos^2 x-3\cos x=4$. 3. $\tan 3x+3\cot 3x=4$.

4. $3\sin^2 3x+7\cos 3x=3$. 5. $\cos 2x-5\sin x-3=0$.

In Exercises 6 to 10, solve the given homogeneous equations:

6. $2\sin 3x+3\cos 3x=0$. 7. $\sin^2 x+3\sin x\cos x-15\cos^2 x=0$.

8. $3\cos^2\frac{x}{2}=4\sin^2\frac{x}{2}$. 9. $3\cos^2 x=4\sin x\cos x$.

10. $\sin^3\frac{x}{2}+3\sin^2\frac{x}{2}\cos\frac{x}{2}-4\cos^3\frac{x}{2}=0$.

In Exercises 11 to 13, solve the given equations using the universal substitution:

11. $2\sin x-3\cos x=3$. 12. $\sin 2x+\tan x=2$. 13. $3\sin 2x+\cos 2x=2$.

In Exercises 14 to 21, solve the given equations:

14. $5\sin x\cot x-\sin x-5\cot x+1=0$. 15. $\sin^6 x+\cos^6 x=7/16$.

16. $2\cos^2 x+\cos 5x=1$. 17. $\sin x-\cos x=1$.

18. $3\sin\frac{x}{2}-4\cos\frac{x}{2}=2$.

19. $\frac{\sin 3x}{\sin x}=0$. 20. $\cos x+\cos 2x=2$. 21. $\sin^{18} x+\cos^{20} x=1$.

In Exercises 22 and 23, solve the given systems of trigonometric equations:

22. $\begin{cases}\tan x+\tan y=1,\\ x+y=\pi/4.\end{cases}$ 23. $\begin{cases}\sin x\sin y=0.75,\\ \tan x\tan y=3.\end{cases}$

24. Solve the given equations with parameter a:

(a) $\sin^6 x+\cos^6 x=a$, (b) $\sin x=a\sin 3x$.

Chapter 7

SEQUENCES

§ 7.1. Method of Mathematical Induction

1. Complete and Incomplete Induction. A deductive and an inductive methods form the basis of any mathematical investigation. The deductive method is a method of reasoning by which concrete applications or consequences are deduced from general principles. In mathematics we apply the following deductive method of reasoning: a given figure is a rectangle, and any rectangle has equal diagonals, consequently, the given rectangle has equal diagonals as well.

By its initial meaning, the word "induction" is applied to reasoning by which general conclusions are deduced from a number of particular assertions. *Complete induction* is the simplest method of such type of reasoning. Consider the following example.

Suppose that we have to ascertain that every even natural number n within the limits $4 \leqslant n \leqslant 20$ is representable in the form of a sum of two prime numbers. To this end, we take all such numbers and write out the corresponding expansions:

$$4 = 2 + 2, \quad 6 = 3 + 3, \quad 8 = 5 + 3, \quad 10 = 7 + 3,$$

$$12 = 7 + 5, \quad 14 = 7 + 7, \quad 16 = 11 + 5, \quad 18 = 11 + 7,$$

$$20 = 13 + 7.$$

These nine equalities show that each of the numbers of interest can actually be represented as a sum of two prime numbers.

Thus, complete induction consists in that a general assertion is proved separately in each of a finite number of cases possible.

Sometimes, we succeed in guessing the general result on having considered not all but a sufficiently large number of particular cases (so-called *incomplete induction*). The result obtained by incomplete induction remains, however, only a hypothesis until it is proved by an exact mathematical reasoning covering **all** particular cases. In other words, incomplete induction in mathematics is not regarded as a legitimate method of a rigorous proof, but is a powerful method of revealing new truths.

Let, for example, the sum of the first n successive odd numbers be found. Consider the following particular cases:

$$1 = 1^2,$$
$$1 + 3 = 4 = 2^2,$$
$$1 + 3 + 5 = 9 = 3^2,$$
$$1 + 3 + 5 + 7 = 16 = 4^2,$$
$$1 + 3 + 5 + 7 + 9 = 25 = 5^2.$$

Having considered these few particular cases, we arrive at the following general conclusion:

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2,$$

that is, the sum of the first n successive odd numbers is equal to n^2.

The results observed may not, of course, serve as a proof of the validity of the considered formula. In what follows, we shall get acquainted with a method by which we can prove that this formula is true.

Complete induction is of limited usefulness in mathematics. Many interesting mathematical statements cover an infinite number of particular cases, however, man is uncapable of verifying an infinite number of cases (any statement pertaining to all natural numbers may serve as an example of such a statement). But incomplete induction often leads to incorrect results.

In many cases, we overcome such difficulties by resorting to a special method of reasoning called the method of mathematical induction.

2. Method of Mathematical Induction. Let the validity of some assertion for any natural number n be proved (for example, that the sum of the first n odd numbers is equal to n^2). A direct check of this assertion for every value of n is impossible, since the set of natural numbers is infinite. In order to prove this assertion, we check its correctness for $n = 1$. Then we prove that for any natural value of k from the validity of the assertion in question for $n = k$ there also follows its validity for $n = k + 1$. Then the assertion is regarded to be proved for all n. Indeed, the assertion is valid for $n = 1$. But then it is also valid for the next number $n = 1 + 1 = 2$. From the validity of the assertion for $n = 2$ there follows its validity for $n = 2 + 1 = 3$. Hence, in turn, there follows the validity of the assertion for $n = 4$, and so on. It is clear that finally, we shall reach any natural number n. Hence, the assertion is valid for any n.

Generalizing the aforesaid, we may formulate the following general principle.

Principle of mathematical induction. *If a proposition $A(n)$ dependent on a natural number n is true for $n = 1$, then from the fact*

that it is true for $n = k$ (where k is any natural number) it follows that it is also true for the next number $n = k + 1$ (in short, this is written as follows*: $A(k) \Rightarrow A(k+1)$), *then the proposition $A(n)$ is true for any natural number n.*

In a number of cases, we have to prove the correctness of a certain assertion not for all natural numbers, but only for $n \geqslant p$, where p is a fixed natural number. In this case, the principle of mathematical induction is formulated as follows.

If a proposition $A(n)$ is true for $n = p$ and if $A(k) \Rightarrow A(k+1)$ for any $k \geqslant p$, then the proposition $A(n)$ is true for any $n \geqslant p$.

This principle underlies the method of proof called the **method of mathematical induction.**

The proof by the method of mathematical induction is carried out in the following way. First the proposition being proved is verified for $n = 1$, that is, the truth of the proposition $A(1)$ is established. This step of the proof is known as the *basis of induction.* The next part of the proof is called the *induction step.* This step proves the validity of the assertion for $n = k + 1$ on the assumption that the assertion is valid for $n = k$ (the supposition of induction), hat is, that $A(k) \Rightarrow A(k+1)$.

Prior to considering examples, let us agree on the following: if we are given n numbers $a_1, a_2, a_3, \ldots, a_n$, then, instead of writing their sum in the form $a_1 + a_2 + a_3 + \ldots + a_n$, we shall sometimes use the notation $\sum_{i=1}^{n} a_n$ ($\sum$ the Greek letter "sigma"). For example, $\sum_{i=1}^{3} a_i$ is another form of notation for the sum $a_1 + a_2 + a_3$, $\sum_{i=1}^{5} i^2$ is another form of notation for the sum $1^2 + 2^2 + 3^2 + 4^2 + 5^2$, while the sum $1 + 3 + 5 + \ldots + (2n - 1)$ of the first successive odd numbers considered in the preceding subsection can be written as follows: $\sum_{i=1}^{n} (2i - 1)$. Note that $\sum_{i=1}^{1} a_i = a_1$.

The following equality holds:

$$\sum_{i=1}^{k+1} a_i = \sum_{i=1}^{k} a_i + a_{k+1}.$$

It is obtained from the fact that $\sum_{i=1}^{k+1} a_i = a_1 + a_2 + \ldots + a_k + a_{k+1}$, and $\sum_{i=1}^{k} a_i = a_1 + a_2 + \ldots + a_k$.

* $A \Rightarrow B$ denotes an implication and is read "A implies B" (or "if A, then B").

Example 1. Prove that

$$1+3+5\ldots+(2n-1)=n^2,$$

or, shorter,

$$\sum_{i=1}^{n}(2i-1)=n^2. \tag{1}$$

○ (1) We have $\sum_{i=1}^{1}(2i-1)=1^2$. Consequently, the assertion is valid for $n=1$, that is, $A(1)$ is true.

(2) We prove that $A(k)\Rightarrow A(k+1)$. Let k be any natural number, and let assertion (1) be valid for $n=k$, that is, $\sum_{i=1}^{k}(2i-1)=k^2$.

We prove that then the assertion is also true for the next natural number $n=k+1$, that is, $\sum_{i=1}^{k+1}(2i-1)=(k+1)^2$. Indeed,

$$\sum_{i=1}^{k+1}(2i-1)=\sum_{i=1}^{k}(2i-1)+(2i-1)\Big|_{i=k+1}=k^2+(2k+1)=(k+1)^2.$$

Thus, $A(k)\Rightarrow A(k+1)$. Using the principle of mathematical induction, we conclude that the proposition $A(n)$ is true for any $n\in\mathbf{N}$. ●

Example 2. Prove that

$$1+x+x^2+x^3+\ldots+x^n=\frac{x^{n+1}-1}{x-1},$$

or, shorter,

$$1+\sum_{i=1}^{n}x^i=\frac{x^{n+1}-1}{x-1}, \tag{2}$$

where $x\neq 1$.

○ (1) For $n=1$ we get $1+x=\frac{x^2-1}{x-1}$, consequently, for $n=1$ the formula is correct.

(2) Let k be any natural number, and let formula (2) hold for $n=k$, that is,

$$1+\sum_{i=1}^{k}x^i=\frac{x^{k+1}-1}{x-1}. \tag{3}$$

Let us prove that then the following equality is satisfied:

$$1+\sum_{i=1}^{k+1}x^i=\frac{x^{k+2}-1}{x-1}.$$

Indeed,

$$1+\sum_{i=1}^{k+1} x^i = 1+\sum_{i=1}^{k} x^i + x^{k+1} = \frac{x^{k+1}-1}{x-1} + x^{k+1} = \frac{x^{k+2}-1}{x-1}$$

(we have used equality (3)). Thus, $A(k) \Rightarrow A(k+1)$. Based on the principle of mathematical induction, we conclude that formula (2) is valid for any natural n. ●

Example 3. Prove that if $n \geqslant 2$ and $x > 0$, then the inequality $(1+x)^n > 1 + nx$ (the **Bernoulli inequality**) holds.

○ (1) For $n = 2$ the inequality is true, since

$$(1+x)^2 = 1 + 2x + x^2 > 1 + 2x.$$

Hence, $A(2)$ is true.

(2) We prove that $A(k) \Rightarrow A(k+1)$ if $k \geqslant 2$. Suppose that $A(k)$ is true, that is, that the following inequality is valid:

$$(1+x)^k > 1 + kx. \tag{4}$$

We prove that then $A(k+1)$ is also true, that is, the inequality $(1+x)^{k+1} > 1 + (k+1)x$ is valid.

Indeed, multiplying both sides of inequality (4) by a positive number $1 + x$, we get

$$(1+x)^{k+1} > (1+kx)(1+x).$$

Consider the right-hand side of the last inequality; we have

$$(1+kx)(1+x) = 1 + (k+1)x + kx^2 > 1 + (k+1)x.$$

As a result, we get $(1+x)^{k+1} > 1 + (k+1)x$. Thus, $A(k) \Rightarrow A(k+1)$.

Based on the principle of mathematical induction, we may assert that the Bernoulli inequality is valid for any $n \geqslant 2$. ●

Example 4. Prove that $(11^{n+2} + 12^{2n+1}) \vdots 133$ (recall that the symbol $\vdots$ means "is divided without a remainder").

○ (1) Let $n = 1$. Then

$$11^3 + 12^3 = (11+12)(11^2 - 11 \cdot 12 + 12^2) = 23 \cdot 133.$$

But $(23 \cdot 133) \vdots 133$, hence, for $n = 1$ the assertion holds; $A(1)$ is true.

(2) Suppose that $(11^{k+2} + 12^{2k+1}) \vdots 133$. We prove that in this case $(11^{k+3} + 12^{2k+3}) \vdots 133$. Indeed

$$11^{k+3} + 12^{2k+3} = 11 \cdot 11^{k+2} + 12^2 \cdot 12^{2k+1} = 11 \cdot 11^{k+2} + 144 \cdot 12^{2k+1}$$
$$= 11(11^{k+2} + 12^{2k+1}) + 133 \cdot 12^{2k+1}.$$

The obtained sum is divisible by 133. Thus, $A(k) \Rightarrow A(k+1)$. By virtue of the principle of mathematical induction, the assertion has been proved. ●

Example 5. Prove that the number of diagonals of a convex n-gon is equal to $n(n-3)/2$.

○ (1) For $n = 3$ the assertion is valid, since in the triangle $D_3 = 3(3-3)/2 = 0$ diagonals, $A(3)$ is true.

(2) Suppose that in any convex k-gon there are $D_k = k(k-3)/2$ diagonals. We prove that then in a convex $(k+1)$-gon the number of diagonals $D_{k+1} = (k+1) \times (k-2)/2$.

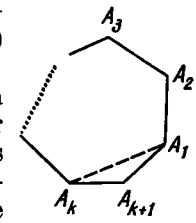


Fig. 165

Let $A_1A_2 \ldots A_kA_{k+1}$ be a convex $(k+1)$-gon (Fig. 165). We draw the diagonal A_1A_k. In order to determine the total number of diagonals in this $(k+1)$-gon, we have to find the number of diagonals in k-gon $A_1A_2 \ldots A_k$, to add $k-2$ to the number obtained, that is, the number of the diagonals of the $(k+1)$-gon emanating from the vertex A_{k+1}, and, besides, to take into account the diagonal A_1A_k. Thus,

$$D_{k+1} = D_k + (k-2) + 1 = \frac{k(k-3)}{2} + k - 1 = \frac{(k+1)(k-2)}{2}.$$

Hence, $A(k) \Rightarrow A(k+1)$. According to the principle of mathematical induction, the assertion is true for any convex n-gon. ●

Exercises

1. Prove that each even natural number n greater than 2 but less than 40 can be represented as a sum of two prime numbers.

2. Prove that for any value of x the inequality $x + |x| \geqslant 0$ holds. Explain how complete induction is used here.

3. Prove that the number which, when divided by 4, yields 3 as a remainder cannot serve as a value of the expression $x^2 + y^2$, where $x \in \mathbf{Z}$ and $y \in \mathbf{Z}$. Consider the cases when x and y are even numbers, odd numbers, one is even and the other is odd. Explain how complete induction is used here.

In Exercises 4 to 8, using the method of mathematical induction, prove that for any natural n the given equalities are satisfied:

4. $1+2+3+\ldots+n = \dfrac{n(n+1)}{2}$.

5. $1^2+2^2+3^2+\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

6. $1^3+2^3+3^3+\ldots+n^3 = \dfrac{n^2(n+1)^2}{4}$.

7. $1\cdot2+2\cdot3+3\cdot4+\ldots+n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

8. $-1+3-5+7-9+\ldots+(-1)^n(2n-1) = (-1)^n \cdot n$.

9. Using the method of incomplete induction, guess the value of the sum

$$\frac{1}{1\cdot2} + \frac{1}{3\cdot4} + \ldots + \frac{1}{n(n+1)}.$$

Prove the result by the method of mathematical induction.

10. Prove that for any natural n the following relationships are valid: (a) $(4^n - 1) \vdots 3$, (b) $(4^n + 15n - 1) \vdots 9$, (c) $(7^{n+2} + 8^{2n+1}) \vdots 57$.

11. Prove that for all $n \geqslant 3$ the inequality $2^n > 2n + 1$ is satisfied.

§ 7.2. Basic Notions Related to Sequences

1. Definition of a Sequence and Ways of Its Representation. In § 1.3, Item 5, we considered an example of a sequence and used this term without giving to it, however, an exact definition. For a given positive number a there was constructed a sequence of its decimal approximations by deficiency:

$$n,\ n.n_1,\ n.n_1n_2,\ n.n_1n_2n_3,\ \ldots,\ n.n_1n_2n_3\ \ldots\ n_k,\ \ldots.$$

This notation means that to every natural number denoting the step number in constructing a sequence, there corresponds a decimal approximation obtained at the step being considered:

$$1 \to n,\quad 2 \to n.n_1,\quad 3 \to n.n_1n_2,\quad 4 \to n.n_1n_2n_3,\ \ldots.$$

A numerical function specified on the set N of natural numbers is called a *numerical sequence.*

A sequence is regarded to be specified if every natural number n is associated with a number $f(n)$. This number is called the *nth term of the sequence.* Instead of $f(n)$, we frequently write y_n, and the sequence is denoted* by $y_1, y_2, \ldots, y_n, \ldots$ or by (y_n).

Here are some examples of numerical sequences:

the sequence of odd natural numbers:

$$1,\ 3,\ 5,\ 7,\ \ldots,\ 2n - 1,\ \ldots,$$

the sequence of squares of natural numbers:

$$1,\ 4,\ 9,\ 16,\ 25,\ \ldots,\ n^2,\ \ldots,$$

the sequence of prime numbers:

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ \ldots.$$

Sequences can be represented by the following two leading methods.

1. A sequence can be given **analytically**, that is, by the formula of the general (nth) term: $a_n = \varphi(n)$. Let, for instance, $a_n = 1/n$, $b_n = n^2$. Assigning the values 1, 2, 3, ..., to the argument n, we shall get the corresponding values of the terms of the sequence. Thus, the sequence (a_n) has the form 1, 1/2, 1/3, 1/4, ..., $1/n$, ..., and the sequence (b_n) the form 1, 4, 9, 16, 25, ..., n^2, ...

Sometimes, the nth term of a sequence is given by several expressions, for example,

$$y_n = \begin{cases} n^2 \text{ if } n \text{ is an even number,} \\ -n^2 \text{ if } n \text{ is an odd number.} \end{cases}$$

* Instead of y, any other letter of the Latin alphabet may be used.

Thereby the following sequence has been specified:

$$-1,\ 4,\ -9,\ 16,\ -25,\ 36,\ \ldots\ .$$

However, the same sequence can be given by a single analytical expression $y_n = (-1)^n n \cdot n^2$. But this is not always possible. For instance, the sequence

$$y_n = \begin{cases} n \text{ if } n \text{ is a prime number,} \\ n^2 \text{ if } n \text{ is a composite number,} \end{cases}$$

cannot be specified by a single analytical expression.

2. Any term of a sequence, beginning with any one, can be often expressed in terms of the previous (one or several) terms. For example, the sequence 1, 2, 3, 5, 8, 13, . . . can be represented in the following way:

$$a_1 = 1, \quad a_2 = 2, \quad a_{n+2} = a_n + a_{n+1}.$$

Indeed, $a_3 = a_1 + a_2 = 1 + 2 = 3$, $a_4 = a_2 + a_3 = 2 + 3 = 5$, ...

Such a way of representing a sequence is called the **recurrence formulae method** (from the Latin "recurrere" ("to return")). When making use of the recurrence formulae method, we usually indicate: (a) the first term of a sequence (or the first several terms), (b) the formula that enables one to determine any term of the sequence using the known previous terms.

Example 1. Write the first several terms of a sequence if it is known that $a_1 = 1$, $a_n = a_{n-1} + 2$ for $n > 1$.

○ We have $a_1 = 1$, $a_2 = a_1 + 2 = 1 + 2 = 3$, $a_3 = a_2 + 2 = 3 + 2 = 5$, ... Thus, we get the sequence 1, 3, 5, 7, 9, ...●

Example 2. Write several terms of a sequence if it is known that $a_1 = 1$, $a_n = a_{n-1}/2$ for $n > 1$.

○ We have $a_1 = 1$, $a_2 = a_1/2 = 1/2$, $a_3 = a_2/2 = 1/4$, $a_4 = a_3/2 = 1/8$, ... Thus, we get the sequence 1, 1/2, 1/4, 1/8, ...●

As one more example, let us consider the sequence (x_n) resulting from taking the square root of a positive number by the method of successive approximations.

Suppose we have to take the square root of a positive number a. We take an approximate value of this root by excess. Since for $a \geqslant 1$ we have $a \geqslant \sqrt{a}$, and for $0 < a < 1$ we have $1 \geqslant \sqrt{a}$, we may take as x_1 the greater of the numbers a and 1, that is, $x_1 = \max(a, 1)$. Then a/x_1 is an approximate value of the root by deficiency. Naturally, their arithmetic mean $\frac{1}{2}\left(x_1 + \frac{a}{x_1}\right)$ is a more accurate approximate value of the root. Since the arithmetic mean of two positive numbers is not less than their geometric mean (see § 5.1, Item I), we have

$$x_2 = \frac{1}{2}\left(x_1 + \frac{a}{x_1}\right) \geqslant \sqrt{x_1 \cdot \frac{a}{x_1}} = \sqrt{a}.$$

Hence, x_2 is an approximate value of the root by excess. Then a/x_2 is an approximate value of the root by deficiency. Reasoning as above, we find a more accurate

approximation to the root by excess $x_3 = \frac{1}{2}\left(x_2 + \frac{a}{x_2}\right)$. In general, if an approximate value x_n is already found for $\sqrt{a}$, then the next approximation is expressed by the formula $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$.

Thus, here we have to consider the sequence numerically according to recurrence relations

$$x_1 = \max(a, 1), \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Calculating, for instance, the number $\sqrt{3}$ by the method of successive approximations, we get $x_1 = 3, x_2 = 2, x_3 = 1.75, x_4 \approx 1.73214, x_5 \approx 1.73205 \ldots$. Each successive step of the process leads to still more accurate approximations to $\sqrt{3}$. The approximation process is stopped when the difference between x_{n+1} and x_n becomes less than the required accuracy of computations. For instance if it is required to compute $\sqrt{3}$ accurate to 0.001, then it suffices to take five approximations and set $\sqrt{3} = 1.732$.

Sequences can be specified not only analytically or recurrently, but also in a "verbal" way. For example, for the sequence of prime numbers 2, 3, 5, 7, 11, 13, ... or for the sequence 1, 1.4, 1.41, 1.414, 1.4142 ... of decimal approximations by deficiency to $\sqrt{2}$ there is neither an analytic formula nor a recurrence relation.

2. Bounded Sequences. A sequence (y) is said to be *bounded above* if there is a number M such that $y_n \leqslant M$ for any $n \in \mathbf{N}$ (this number M is called the *upper bound* of the sequence). A sequence (y_n) is said to be *bounded below* if there is a number m such that $y_n \geqslant m$ for any $n \in \mathbf{N}$ (this number m is called the *lower bound* of the sequence). Finally, a sequence (y_n) is said to be *bounded* if it is bounded both above and below.

If a sequence is bounded above, then it has infinitely many upper bounds. Indeed, if M is the upper bound of the sequence (y_n), that is, $y_n \leqslant M$ for all n, then any number $P > M$ is also an upper bound for (y_n). Analogously, a sequence bounded below has infinitely many lower bounds.

Example 1. The sequence 3, 2, 1, 0, —1, —2, —3, ..., is bounded above (for instance, by the number 3).

Example 2. The sequence 1, 2, 3, 4, 5, ..., n, ... is bounded below (for instance, by the number 1).

Example 3. The sequence 1, 1/2, 1/3, 1/4, ..., $1/n$, ... is bounded both above and below, that is, it is bounded.

3. Monotonic (Monotone) Sequences. A sequence (a_n) is said to be *increasing* (*nondecreasing*) if $a_n < a_{n+1}$ $(a_n \leqslant a_{n+1})$ for any n and *decreasing* (*nonincreasing*) if $a_n > a_{n+1}$ $(a_n \geqslant a_{n+1})$ for any n.

Example 1. The sequence 1/2, 2/3, 3/4, 4/5, ..., $n/(n+1)$, ... is increasing.

Example 2. The sequence 1, 1/2, 1/3, 1/4, . . ., $1/n$, . . . is decreasing.

Increasing and decreasing sequences are called *monotonic* (or *monotone*) *sequences*.

Exercises

1. (a) A sequence (x_n) is given by the formula $x_n = (2n + 1)/(n + 3)$. Find x_{10}, x_{25}, x_{n+1}, $x_{n+1} - x_n$.

(b) A sequence (b_n) is given by the formula $b_n = n/3^n$. Find b_3, b_5, b_{n+1}, b_{n+1}/b_n.

2. Write the first five terms of a sequence (x_n) and specify this sequence by the formula of the nth term if $x_1 = -10$, $x_{n+1} = x_n + 5$, $n \geqslant 1$.

3. For what terms of a sequence (y_n) is the indicated condition met: (a) $y_n > 200$ if $y_n = 2n - 5$, (b) $y_n \leqslant 30$ if $y_n = 3n - 100$?

4. Derive one of the possible formulas for the nth term of the sequence which begins as follows: (a) 1, 4, 9, 16, 25, . . ., (b) 2, −2.2, −2, . . ., (c) 5.0, 5.0, 5, . . ., (d) 0, 3, 8, 15, . . ., (e) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$, (f) $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$ (g) $\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \frac{1}{4\cdot 5}, \ldots$, (h) $\frac{10}{3}, \frac{20}{9}, \frac{30}{27}, \frac{40}{81}, \ldots$.

5. In two sequences whose nth terms are expressed by the formulas $a_n = n(n + 36) + 7$ and $b_n = n(5n + 9)$, find equal terms having the same number.

6. Represent the sequence (x_n) by points of the coordinate line: (a) $x_n = n^2 + 1$, (b) $x_n = 1 - \frac{1}{n}$, (c) $x_n = \frac{1 + (-1)^n}{n}$, (d) $x_n = (-1)^n \cdot \frac{1}{n}$. Is there a number to which this sequence "tends"?

7. A sequence (a_n) is given by the formula $a_n = (2n + 3)/n$. Find the distance from the point 2 to the point: (a) a_5, (b) a_{20}, (c) a_{100}, (d) a_{1000}.

8. A sequence (y_n) is given by the formula $y_n = -3 \cdot 5^{2n-1}$. Does the number −1875 belong to this sequence?

9. Given a sequence whose general term is expressed by the formula $z_n = (4n + 5)(n + 1)$. Prove that this is a decreasing sequence.

10. Is the sequence (x_n) decreasing or increasing if $x_n = (3n - 2)/n$?

11. Prove that the sequence (x_n) is bounded if: (a) $x_n = \left(-\frac{1}{2}\right)^n$, (b) $x_n = -2 + \frac{1}{n}$, (c) $x_n = \frac{2n + 7}{n}$.

12. Given a sequence (a_n). Write 3-6 terms of this sequence, represent them by points on the coordinate line, and answer the following questions: is this sequence increasing, decreasing, bounded above, bounded below, bounded? Is there any number to which the terms of the sequence approach without bound?

(a) $a_n = -2n$, (b) $a_n = (-1)^n \cdot n$, (c) $a_n = 5$, (d) $a_n = \frac{1}{n}$,

(e) $a_n = \frac{1 + (-1)^3}{2}$, (f) $a_n = \frac{1 + (-1)^n}{n}$, (g) $a_n = (-)^n + \frac{1}{n}$,

(h) $a_n = \frac{3n - 2}{n}$, (i) $a_n = 4 + \frac{(-1)^n}{n}$, (j) $a_n = \frac{1}{n - (-1)^n}$.

13. Give an example of a sequence which: (a) is bounded below but is unbounded above, (b) is bounded above but is not bounded below, (c) is bounded

both above and below, (d) is bounded neither above nor below, (e) has the greatest and least terms, (f) is bounded, has the greatest term and no least term, (g) is bounded, has the least term but no greatest term, (h) is bounded, but has neither greatest nor least term.

§ 7.3. Limit of a Sequence

1. Definition. Consider three sequences represented analytically:

$$x_n = \frac{1}{n}, \quad y_n = (-1)^n \cdot n^2, \quad z_n = \frac{n^2-2}{n^2}.$$

Write the initial terms in each of these sequences;

$$(x_n)\colon\ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots,$$

$$(y_n)\colon -1,\ 4,\ -9,\ 16,\ -25,\ 36,\ -49,\ 64,\ \ldots,$$

$$(z_n)\colon\ -1, \frac{2}{4}, \frac{7}{9}, \frac{14}{16}, \frac{23}{25}, \frac{34}{36}, \frac{47}{49}, \frac{62}{64}, \ldots.$$

Note that, as the number increases, the terms of the first and the third sequence approach a certain number more and more close: the terms of the sequence (x_n) approach the number 0, while those of the sequence (z_n) the number 1. The sequence (y_n) does not possess this property: its terms do not approach some number as their number increases. The sequences (x_n) and (z_n) are said to converge to the number 0 or 1, respectively, and the sequence (y_n) is said to diverge.

What do the words "a sequence converges to a number" mean from the point of view of mathematics? Consider the distance between x_n and 0: $\rho(x_n, 0) = |x_n - 0|$. In this case, it is equal to $1/n$. This distance can be made arbitrarily small. For example, $\rho(x_n, 0) < 1/1000$ for x_{1001} and all successive terms in the sequence, $\rho(x_n, 0) < 1/5000$ for x_{5001} and all subsequent terms of the sequence. The sequence (z_n) possesses a similar property. For instance,

$$\rho(z_{10}, 1) = \left|\frac{98}{100} - 1\right| = \frac{2}{100}, \quad \rho(z_{31}, 1) = \left|\frac{959}{961} - 1\right| = \frac{2}{961},$$

$$\rho(z_n, 1) = \left|\frac{n^2-2}{n^2} - 1\right| = \frac{2}{n^2}.$$

The greater n, the less is the distance from z_n to 1. Whatever positive number is taken (it is usually denoted by the Greek letter ε ("epsilon")), there is a number n_0 of a sequence term such that the term with this number and all subsequent terms are distant less than ε from 1. In other words, the inequality $n \geqslant n_0$ implies the inequality $|z_n - 1| < \varepsilon$. The definition of the limit of a numerical sequence is based on these considerations.

Definition. A sequence (x_n) is said to be *convergent to a number a* if the following conditions are satisfied: for any $\varepsilon > 0$ there is a

natural number n_0 such that from the inequality $n \geqslant n_0$ there follows the inequality $|x_n - a| < \varepsilon$ (in other words, the term with this number and all subsequent terms in the sequence are found at a distance less than ε from a). In this case, the number a is called the *limit of the sequence* (x_n) and is written as $\lim\limits_{n\to\infty} x_n = a$ or $x_n \to a$ as $n \to \infty$.

For example, for the sequences (x_n) and (z_n) considered above we have $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$, $\lim\limits_{n\to\infty} z_n = \lim\limits_{n\to\infty} \frac{n^2-2}{n^2} = 1$. Both of these sequences are convergent.

However, by far not all sequences have limits. If a sequence has no limit, then it is said to be *divergent*. The sequence (y_n): -1, 4, -9, 16, -25, 36, ... is a divergent sequence.

The terms of a convergent sequence (x_n) may approach the limit a in different ways: the sequence may converge to the number a either increasing or decreasing or, finally, oscillating about the number a.

For example, the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

increases and converges to the number 1. The sequence

$$2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots, \frac{n+1}{n}, \ldots$$

decreases and converges to the number 1. The terms of the sequence

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, \frac{(-1)^{n-1}}{n}, \ldots$$

approach zero, oscillating about this value. Finally, the terms of the sequence

$$0, 1, \frac{1}{2}, 1, \frac{2}{3}, 1, \frac{3}{4}, 1, \ldots$$

approach 1, infinitely many of them being equal to 1.

2. Geometrical Meaning of the Limit of a Sequence. The terms of a sequence (x_n) can be represented by points on the coordinate line. Let the nth term of the sequence satisfy the inequality $|x_n - a| < \varepsilon$, this means that the number x_n belongs to the interval $(a - \varepsilon, a + \varepsilon)$ with centre at the point a (Fig. 166). This interval is usually called the *ε-neighbourhood of the point a*, and the number ε the *radius* of this neighbourhood.

Let the inequality $|x_n - a| < \varepsilon$ be satisfied for all $n \geqslant n_0$, this means that the term of the sequence with number n_0 and all subsequent terms are found in the chosen ε-neighbourhood of the point a. Only a finite set of terms of the sequence are found outside this neighbourhood.

Henceforward, if a certain property is fulfilled for all the terms of a sequence except for, perhaps, a finite set of its terms, then we shall say that *almost all the sequence possesses the considered property.*

Now, we can give a **geometrical interpretation** of the limit of a sequence. *A number a is the limit of a sequence* (x_n) *if the following*

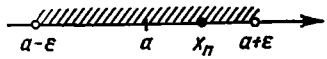


Fig. 166

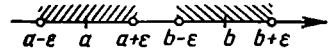


Fig. 167

condition is met: whatever neighbourhood of the point a is taken, almost the entire sequence is contained in this neighbourhood.

Hence it is clear that addition or elimination of a finite set of terms of a sequence does not affect the convergence of the sequence and the value of its limit.

3. Properties of Limits. Consider some properties of limits of sequences. As we shall see, these properties are readily proved using the geometrical interpretation of the limit.

Theorem 7.3.1. *If a sequence* (x_n) *converges, then only to a single limit.*

□ Suppose that the sequence (x_n) has a limit a. We take a number $b \neq a$, say, $a < b$, and show that b cannot be the limit of the given sequence.

Consider the nonintersecting ε-neighbourhoods of the points a and b (Fig. 167). Since $a = \lim\limits_{n\to\infty} x_n$, almost the entire sequence (x_n) is contained in the ε-neighbourhood of the point a. But then the ε-neighbourhood of the point b can contain only a finite number of

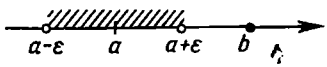


Fig. 168

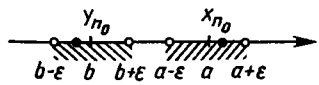


Fig. 169

terms of the sequence (x_n), and therefore the number b cannot be the limit of the given sequence. ■

Theorem 7.3.2. *If a sequence* (x_n) *has a number a as its limit, and if* $b > a$, *then almost all the terms of the sequence are less than b.*

□ Let us take the ε-neighbourhood of the point a not containing the point b (Fig. 168). Since $a = \lim\limits_{n\to\infty} x_n$, almost the entire sequence is contained in the ε-neighbourhood of the point a, that is, to the left of b, whence there follows the statement being proved. ■

The following property is proved in a similar way.

Theorem 7.3.3. *If a sequence has the number a as its limit, and if* $b < a$, *then almost all the terms of the sequence are greater than b.*

It follows from the last two theorems, in particular, that *if a sequence converges to a negative limit, then almost all of its terms are negative, and if a sequence converges to a positive limit, then almost all of its terms are positive.*

Lemma. *Let* $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$ *and* ε *be an arbitrary positive number. Then there exists a number* n_0 *such that for* $n \geqslant n_0$ *two inequalities* $|x_n - a| < \varepsilon$ *and* $|y_n - b| < \varepsilon$ *are satisfied simultaneously.*

□ Since $\lim\limits_{n\to\infty} x_n = a$, there is a number n_1, beginning with which (that is, for $n \geqslant n_1$), the inequality $|x_n - a| < \varepsilon$ is satisfied. Further, since $\lim\limits_{n\to\infty} y_n = b$, there is a number n_2, beginning with which (that is, for $n \geqslant n_2$), the inequality $|y_n - b| < \varepsilon$ is satisfied.

Let us choose the greatest of the two numbers n_1 and n_2 and denote it by n_0, that is, $n_0 = \max(n_1, n_2)$. Then both inequalities $|x_n - a| < \varepsilon$ and $|y_n - b| < \varepsilon$ will be satisfied. ■

This lemma can be extended to the case of any number of convergent sequences.

Let us use this lemma to prove the following property (it will be used repeatedly for other purposes).

Theorem 7.3.4. *Let there be given three sequences* (x_n), (y_n) *and* (z_n), *the inequality* $x_n \leqslant y_n \leqslant z_n$ *being satisfied for any natural number* n. *If the sequences* (x_n) *and* (z_n) *converge to one and the same limit*: $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = a$, *then the sequence* (y_n) *also converges to the same limit*: $\lim\limits_{n\to\infty} y_n = a$.

□ Let us take an arbitrary ε-neighbourhood of the point a. It follows from the hypothesis and the lemma that, beginning with a certain number n_0, all the terms of the sequences (x_n) and (z_n) are contained in this neighbourhood. Since $x_n \leqslant y_n \leqslant z_n$, beginning with number n_0, all the terms of the sequence (y_n) are also contained in the ε-neighbourhood of the point a. This implies that $\lim\limits_{n\to\infty} y_n = a$. ■

Theorem 7.3.5. *Let* $\lim\limits_{n\to\infty} x_n = a$ *and* $\lim\limits_{n\to\infty} y_n = b$, *the inequality* $x_n \leqslant y_n$ *being valid for any* n, *then the inequality* $a \leqslant b$ *is satisfied.*

□ Suppose the contrary, that is, $a > b$, and choose nonintersecting ε-neighbourhoods of the points a and b (Fig. 169). According to the lemma, there is a number n_0 such that x_{n_0} is contained in the ε-neighbourhood of the point a, and y_{n_0} in the ε-neighbourhood of the point b. Consequently, $x_{n_0} > y_{n_0}$, which contradicts the hypothesis. Our supposition is false, and, hence, $a \leqslant b$. ■

The statement proved is usually called the **theorem on passage to the limit in an inequality.** Note that passage to the limit is valid

only for nonstrict (weak) inequalities: $x_n < y_n$ does not imply $\lim\limits_{n\to\infty} x_n < \lim\limits_{n\to\infty} y_n$.

Consider, for instance, two sequences: $x_n = -1/n$ and $y_n = 1/n$. It is clear that $x_n < y_n$, but $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 0$. At the same time, this example does not contradict the theorem on passage to the limit under the sign of a **nonstrict inequality**: instead of $x_n < y_n$, we can write $x_n \leqslant y_n$, and, instead of $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$, $\lim\limits_{n\to\infty} x_n \leqslant \lim\limits_{n\to\infty} y_n$. We then get: $x_n \leqslant y_n \Rightarrow \lim\limits_{n\to\infty} x_n \leqslant \lim\limits_{n\to\infty} y_n$.

The following two properties directly follow from the theorem on passage to the limit in an inequality.

Theorem 7.3.6. *If all the terms of a convergent sequence are nonnegative, then the limit of the sequence is a nonnegative number.*

Theorem 7.3.7. *If all the terms of a convergent sequence are nonpositive, then the limit of the sequence is a nonpositive number.*

Note one more property of convergent sequences.

Theorem 7.3.8. *If a sequence converges, then it is bounded.*

□ Let $\lim\limits_{n\to\infty} x_n = a$. We take an arbitrary number $\varepsilon > 0$. Then almost the entire sequence (x_n) is contained in the ε-neighbourhood of the point a, that is, only a finite number of terms of the sequence $x_{n_1}, x_{n_2}, \ldots, x_{n_k}$ lie outside this neighbourhood (Fig. 170). If now we take a number M greater than any number from the finite set $\{x_{n_1}, x_{n_2}, \ldots, x_{n_k}, a + \varepsilon\}$, then it turns out that $x_n < M$ for all n. Hence, the sequence (x_n) is bounded above. The fact that the sequence is bounded below is proved in a similar way. ■

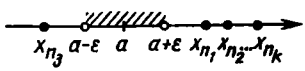


Fig. 170

4. Computing the Limit of a Sum, a Product, and a Quotient. To compute the limits of sequences, we often use the theorem on arithmetical operations on limits.

Theorem 7.3.9. *If* $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$, *then*:

(1) $\lim\limits_{n\to\infty} (x_n + y_n) = a + b$, (2) $\lim\limits_{n\to\infty} (x_n y_n) = ab$,

(3) $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \frac{a}{b}$ (*if* $b \neq 0$), (4) $\lim\limits_{n\to\infty} (cx_n) = ca$.

In other words: *the limit of a sum, a product, and a quotient is respectively equal to the sum, product, and quotient of the limits; a constant factor can be taken outside the limit sign.*

We shall prove only the first assertion of the theorem.

□ Let us set $z_n = x_n + y_n$ and take an arbitrary $\varepsilon > 0$. According to the lemma from Item 3, there is an n_0 such that for $n \geqslant n_0$ the following two inequalities are satisfied simultaneously: $|x_n - a| < \varepsilon/2$, $|y_n - b| < \varepsilon/2$.

Let us now consider $|z_n - (a+b)|$. We have

$$\begin{aligned} |z_n - (a+b)| &= |(x_n + y_n) - (a+b)| \\ &= |(x_n - a) + (y_n - b)| \\ &\leqslant |x_n - a| + |y_n - b| \end{aligned}$$

(we have used the fact that the modulus of a sum does not exceed the sum of moduli). Then $|z_n - (a+b)| < \varepsilon$ for $n \geqslant n_0$, and this means that $\lim\limits_{n\to\infty} z_n = a + b$, that is,

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n. \blacksquare$$

From this theorem it follows, in particular, that, since $\lim\limits_{n\to\infty}(1/n) = 0$, the following equalities hold:

$$\lim_{n\to\infty}\frac{1}{n^2} = \lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1}{n}\right) = 0\cdot 0 = 0,$$

$$\lim_{n\to\infty}\frac{1}{n^3} = 0, \ \ldots, \ \lim_{n\to\infty}\frac{1}{n^p} = 0 \ (p \in \mathbf{N}).$$

Example. Compute $\lim\limits_{n\to\infty}\dfrac{n^2-5n+4}{2n^2-3n+7}$.

○ Dividing termwise both the numerator and denominator of the nth term of the given sequence by n^2, we get

$$\frac{n^2-5n+4}{2n^2-3n+7} = \frac{1-\frac{5}{n}+\frac{n}{n^2}}{2-\frac{3}{n}+\frac{7}{n^2}}.$$

Hence

$$\begin{aligned} \lim_{n\to\infty}\frac{n^2-5n+4}{2n^2-3n+7} &= \lim_{n\to\infty}\frac{1-\frac{5}{n}+\frac{4}{n^2}}{2-\frac{3}{n}+\frac{7}{n^2}} \\ &= \frac{1-5\lim\limits_{n\to\infty}\frac{1}{n}+4\lim\limits_{n\to\infty}\frac{1}{n^2}}{2-3\lim\limits_{n\to\infty}\frac{1}{n}+7\lim\limits_{n\to\infty}\frac{1}{n^2}} \\ &= \frac{1-5\cdot 0+4\cdot 0}{2-3\cdot 0+7\cdot 0} = \frac{1}{2}. \ \bullet \end{aligned}$$

5. Tests for the Existence of the Limit of a Sequence. It was proved above that any convergent sequence is bounded (see Theorem 7.3.8). However, not any bounded sequence has a limit. For example, the bounded sequence

$$0, 1, 2, 3, 0, 1, 2, 3, 0, 1, \ldots$$

has no limit.

Monotonicity of a sequence is an additional condition ensuring the existence of a limit. For example, the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots,$$

is bounded and increasing. This sequence converges: $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$. The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$$

is bounded and decreases. This is also a convergent sequence $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

It turns out that the collection of two mentioned tests (boundedness and monotonicity) is a sufficient condition for a sequence to converge. Here, the following theorems are valid (they are given without proof).

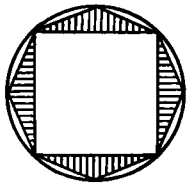


Fig. 171

Theorem 7.3.10. *If a sequence increases and is bounded above, then it converges.*

Theorem 7.3.11. *If a sequence decreases and is bounded below, then it converges.*

Let us use these theorems to get important results concerning the relation between real numbers and sequences of their decimal approximations by deficiency and by excess.

Let a be a positive number and (x_n) be the sequence of its decimal approximations by deficiency. This sequence increases (for any n we have $x_n \leqslant x_{n+1}$) and is bounded above (for instance, by the number a itself). Hence, there exists the limit $\lim\limits_{n\to\infty} x_n$. This limit is equal to the number a, since the difference $a - x_n$ does not exceed $1/10^n$ and can be made arbitrarily small. In exactly the same manner we prove that the limit of a decreasing sequence of decimal approximations to the number a by excess is equal to a. Similar results are valid for negative real numbers. Thus, *any real number is the limit of sequences of its decimal approximations both by deficiency and by excess.*

From the result obtained and from the theorem on taking a constant factor outside the limit sign we immediately deduce the assumption on non-terminating decimal fractions (see § 1.2, Item 6).

Let us now consider some more examples illustrating the application of Theorems 7.3.1 and 7.3.2.

Example 1. Let S_n be the area of a regular 2^{n+1}-gon, inscribed in a circle of radius R. Prove that the sequence (S_n) has a limit.

○ Figure 171 shows that when the number of sides is doubled, the area of the polygon increases (since 2^{n+1} triangles are added). Therefore the sequence (S_n) increases. All of its terms do not exceed

the number $4R^2$, the area of the circumscribed square. Hence, the sequence (S_n) is bounded above. According to Theorem 7.3.1, (S_n) has a limit which is equal to the area of a circle of radius R. ●

Example 2. Prove that if $|q|<1$, then $\lim\limits_{n\to\infty} q^n=0$, if $|q|>1$, then the sequence (q^n) diverges.

○ It is clear that if $q=0$, then $\lim\limits_{n\to\infty} q^n=0$. Let $0<q<1$. Consider the sequence $q, q^2, q^3, \ldots, q^n, \ldots$. It is decreasing: $q>q^2>q^3>\ldots>q^n>\ldots$ and bounded below (for instance, by zero). Hence, there exists the limit $\lim\limits_{n\to\infty} q^n=a$. Since $q^{n+1}=q\cdot q^n$, we have $a=\lim\limits_{n\to\infty} q^{n+1}=q\cdot\lim\limits_{n\to\infty} q^n=q\cdot a$. Thus, $a=qa$, whence $a=0$.

If $-1<q<0$, then $0<|q|<1$, and therefore $\lim\limits_{n\to\infty}|q|^n=0$. But then $\lim\limits_{n\to\infty} q^n=0$.

Thus, if $-1<q<1$, then $\lim\limits_{n\to\infty} q^n=0$.

And if $|q|>1$, then the sequence (q^n) diverges. Indeed, if it converged and had the number b as its limit, then b would be equal to 0 (this is established in the same way as it was done above for the number a). But this is impossible, since for $|q|>1$ the inequality $|q^n|>1$ holds, and therefore for $\varepsilon<1$ the inequality $|q^n-0|<\varepsilon$ cannot be satisfied. ●

Example 3. Let a sequence (x_n) be represented recurrently:

$$x_1=\max(a, 1),\quad x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right),\ \text{where } a>0. \tag{1}$$

Prove that the limit of this sequence exists and is equal to $\sqrt{a}$ (see § 7.2, Item 1).

○ Suppose that this sequence has a limit $\lim\limits_{n\to\infty} x_n=b$. Let us find its value.

Since the value of the limit is independent of adding or rejecting a finite number of terms of the sequence, $\lim\limits_{n\to\infty} x_{n+1}$ is also equal to b. Let us pass to the limit in the equality $x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)$ as $n\to\infty$:

$$\lim_{n\to\infty} x_{n+1}=\frac{1}{2}\lim_{n\to\infty} x_n+\frac{a}{2\lim\limits_{n\to\infty} x_n}.$$

Hence $b=\frac{1}{2}b+\frac{a}{2b}$, whence we find $b=\sqrt{a}$.

Thus, $\lim\limits_{n\to\infty} x_n=\sqrt{a}$, that is, if the limit exists, then it is equal to $\sqrt{a}$. Let us now prove that the limit actually exists.

First we assume that the equality $x_n=\sqrt{a}$ is satisfied at a certain step. Then from (1) we get that $x_{n+1}=\sqrt{a}$, $x_{n+2}=\sqrt{a}$ and so on. It is clear that in such a case $\lim\limits_{n\to\infty} x_n=\sqrt{a}$.

Let now the equality $x_n = \sqrt{a}$ be satisfied at no step. Then x_n and a/x_n are unequal positive numbers, and their arithmetic mean is greater than their geometric mean. Therefore for any n we have

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)>\sqrt{x_n\cdot\frac{a}{x_n}}=\sqrt{a}.$$

This means that the sequence (x_n) is bounded below, and $\sqrt{a}$ is one of its lower bounds.

Further, for any n we have

$$x_{n+1}-x_n=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)-x_n=\frac{a-x_n^2}{2x_n}.$$

Since $x_n>\sqrt{a}$, we have $a-x_n^2<0$, and therefore $x_{n+1}<x_n$. This means that the sequence (x_n) decreases.

Thus, we have proved that the sequence (x_n) is bounded below and decreases. According to Theorem 7.3.2, it has a limit, which was required to be proved. ●

It was noted above that relationship (1) underlies the method of computing square roots, that is, the method of successive approximations. The equality $\lim\limits_{n\to\infty} x_n = \sqrt{a}$ proved convinces us that this method is correct.

The tests for the existence of a limit of a sequence formulated in Theorems 7.3.1 and 7.3.2 are not always convenient to be used in practice, since using these tests, we cannot always say to what extent a given term of a sequence differs from the limit. The test formulated below suits more for these purposes.

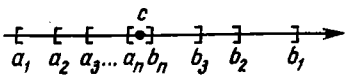


Fig. 172

Let there be given a sequence of closed intervals $[a_1, b_1]$, $[a_2, b_2]$, . . ., $[a_n, b_n]$, . . . such that each next interval is contained in the preceding one, and the sequence of lengths of these intervals tends to zero. Geometrical considerations (Fig. 172) suggest that in such a case there exists a single point c, common to all the intervals. The sequence of the left-hand end points of the intervals shrinks to this point; the sequence of the right-hand end points of the intervals also shrinks to this point, and therefore the given sequence of closed intervals is said to be *shrinking*. The result obtained is formulated in the form of the following theorem which is known as the **theorem on a shrinking system of closed intervals** (given without proof).

Theorem 7.3.12. *Let a sequence* (a_n) *increase and a sequence* (b_n) *decrease, and for any n we have* $a_n \leqslant b_n$, *and let* $\lim\limits_{n\to\infty}(b_n - a_n) = 0$. *Then both sequences* (a_n) *and* (b_n) *converge, and* $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

Let, for instance, there be given a circle of radius R, and (p_n) and (P_n) are, respectively, the sequences of the perimeters of inscribed and circumscribed regular 2^{n+1}-gons. These sequences satisfy the conditions of Theorem 7.3.12 and therefore have a common limit. This limit is equal to the circumference of the circle.

Exercises

1. Represent the terms of the sequence $y_n = (3n - 1)/n$ by points on the coordinate line. Is it correct that $\lim_{n\to\infty} (3n - 1)/n = 3$? Indicate the numbers of the terms of the sequence (y_n) which are at a distance: (a) less than 0.1; (b) less than 0.04 from the number 3.

2. Given a sequence whose general term is expressed by the formula $z_n = (6n - 1)/(n + 3)$.

(a) Prove that this is an increasing sequence.

(b) Prove that any term of this sequence is less than 6.

(c) Find the number of the term beginning with which the terms of the sequence differ from 6 by less than 0.19.

3. Using the definition of limit, prove that:

(a) $\lim_{n\to\infty} \frac{5}{n} = 0$, (b) $\lim_{n\to\infty} \frac{8n-5}{2n} = 4$.

4. A sequence (y_n) is given by the formula $y_n = -(5 + 6n)/(3n)$. For which terms of this sequence is the inequality $|-2 - y_n| < 0.3$ satisfied?

5. Using the limit theorems, find

(a) $\lim_{n\to\infty} \frac{6-7n}{3n}$, (b) $\lim_{n\to\infty} \frac{3n^2+4}{2n^2-1}$,

(c) $\lim_{n\to\infty} \frac{(2n+1)(3n-1)}{2n^2-3}$, (d) $\lim_{n\to\infty} \frac{3-n}{n^2+2n-1}$.

6. Are the following assertions true:

(a) "If a sequence is not bounded, then it has no limit",

(b) "If a sequence has a limit, then it is bounded",

(c) "If a sequence is not monotonic, then it has no limit",

(d) "If a sequence has a limit, then it is monotonic"?

7. Give examples of sequences whose limits are the numbers: (a) 5, (b) -3, (c) $4/7$, (d) 0.

§ 7.4. Progressions

1. Arithmetic Progression.

Definition. An *arithmetic progression* (or arithmetic sequence) is a sequence of numbers each term of which (beginning with the second) is equal to the sum of the preceding term and a constant number d. The number d is the *common difference*, or, simply, the *difference* of the arithmetic progression. Thus, an arithmetic progression is defined by the following two conditions: (1) $a_1 = a$, (2) $a_{n+1} = a_n + d$ for any $n \geqslant 1$.

For example, if $a_1 = 2$, $a_{n+1} = a_n + 3$, then $a_2 = 2 + 3 = 5$, $a_3 = 5 + 3 = 8$, . . ., that is, we obtain the sequence 2, 5, 8, 11, 14, 17, ...

The following are examples of arithmetic progressions:

(1) -1, 5, 11, 17, 23, 29, . . ., here $a_1 = -1$, $a_{n+1} = a_n + 6$, $d = 6$,

(2) 17, 14, 11, 8, 5, 2, -1, -4, . . ., here $a_1 = 17$, $a_{n+1} = a_n - 3$, $d = -3$.

(3) 8, 8, 8, 8, 8, . . ., here $a_1 = 8$, $a_{n+1} = a_n + 0$, $d = 0$.

In order to indicate that the sequence (a_n) is an arithmetic progression, we use the notation $a_1, a_2, a_3, \ldots, a_n, \ldots$.

If an arithmetic progression is truncated at some kth term, then we get a finite arithmetic progression $a_1, a_2, \ldots, a_{k-1}, a_k$.

It is obvious that if $d > 0$, then the arithmetic progression is an increasing sequence, if $d < 0$, then it is a decreasing sequence; and if $d = 0$, then $a_n = a$, and we have a constant sequence.

For example, 1, 4, 7, 10, 13, ... ($d = 3$) is an increasing progression, 12, 10, 8, 6, 4, ... ($d = -2$) is a decreasing progression; and 5, 5, 5, ... ($d = 0$) is a constant progression.

As a matter of fact, the above representation of the arithmetic progression is a recursive representation of a sequence. In many cases, it is inconvenient. Indeed, to find some term of an arithmetic progression having a large ordinal number (say, a_{1000}), in the case of recursive representation, it is necessary to know all previous terms (in the given case their number amounts to 999). This enormous computational work can be reduced by deriving the formula for the nth (or general) term of the arithmetic progression from the recurrence relation $a_{n+1} = a_n + d$.

Formula for the nth term of arithmetic progression. *Let (a_n) be an arithmetic progression. Then its nth term can be given by the following analytic expression:*

$$a_n = a_1 + d\,(n - 1). \tag{1}$$

□ Let us use the method of mathematical induction.

(1) For $n = 1$ equality (1) is valid. Indeed, $a_1 = a_1 + d\,(1 - 1)$ is a true equality. Thus, A (1) is true.

(2) Let us prove that $A\,(k) \Rightarrow A\,(k + 1)$. Suppose that $a_k = a_1 + d\,(k - 1)$, and let us prove that $a_{k+1} = a_1 + dk$.

Indeed, by the definition of arithmetic progression, $a_{k+1} = a_k + d$. Substituting the expression $a_1 + d\,(k - 1)$ for a_k into this equality, we get

$$a_{k+1} = (a_1 + d\,(k - 1)) + d = a_1 + dk,$$

which was required to be proved. Thus, $A\,(k) \Rightarrow A\,(k + 1)$. On the basis of the principle of mathematical induction, equality (1) holds for any natural number n. ■

For example, for the arithmetic progression 2, 5, 8, 11, 14, 17, ..., where $a_1 = 2$, $d = 3$, we have $a_{15} = a_1 + 14d = 2 + 14 \cdot 3 = 44$, $a_{74} = a_1 + 73d = 2 + 73 \cdot 3 = 221$, etc.

Characteristic property of arithmetic progression. *In order for a sequence (a_n) to be an arithmetic progression, it is necessary and sufficient that each of its terms, except the first one, be equal to the arithmetic mean of its neighbouring terms:*

$$a_{n+1} = \frac{a_n + a_{n+2}}{2}. \tag{2}$$

□ **Necessity.** Let (a_n) be an arithmetic progression. Then $a_{n+1} = a_n + d$, that is, $a_n = a_{n+1} - d$, $a_{n+2} = a_{n+1} + d$. Hence,

$$\frac{a_n + a_{n+2}}{2} = \frac{(a_{n+1} - d) + (a_{n+1} + d)}{2} = a_{n+1}.$$

The necessity of the condition has been proved.

Sufficiency. Let equality (2) hold in the sequence (a_n) for any n. It follows from this equality that $a_{n+1} - a_n = a_{n+2} - a_{n+1}$. Hence we get

$$\begin{aligned}
&\text{for } n = 1\text{: } a_2 - a_1 = a_3 - a_2,\\
&\text{for } n = 2\text{: } a_3 - a_2 = a_4 - a_3,\\
&\text{for } n = 3\text{: } a_4 - a_3 = a_5 - a_4,\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Thus, $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = a_5 - a_4 = \ldots$, that is, the difference between the subsequent and preceding terms of the sequence is the same. Hence, (a_n) is an arithmetic progression. ■

Note one more property of the arithmetic progression (a_n): *if* $n + m = k + l$, *then* $a_n + a_m = a_k + a_l$.

□ Let us use equality (1) representing the formula for the nth term of arithmetic progression. We have

$$\begin{aligned}
a_n &= a_1 + d(n-1), \quad a_m = a_1 + d(m-1),\\
a_k &= a_1 + d(k-1), \quad a_l = a_1 + d(l-1).
\end{aligned}$$

Hence,

$$\begin{aligned}
a_n + a_m &= (a_1 + d(n-1)) + (a_1 + d(m-1))\\
&= 2a_1 + d(n+m) - 2d,\\
a_k + a_l &= (a_1 + d(k-1)) + (a_1 + d(l-1))\\
&= 2a_1 + d(k+l) - 2d.
\end{aligned}$$

Since by the hypothesis $n + m = k + l$, we get

$$a_n + a_m = a_k + a_l. \quad ■$$

For instance, if (a_n) is an arithmetic progression, then

$$a_1 + a_{20} = a_2 + a_{19} = a_3 + a_{18} = \ldots = a_{10} + a_{11}$$

(the sum of the subscripts is constant).

Formula for the sum of the first k terms of arithmetic progression. *Let (a_n) be an arithmetic progression, and S_k the sum of its first k terms:*

$$S_k = a_1 + a_2 + a_3 + \ldots + a_k = \sum_{i=1}^{k} a_i.$$

Then the following equality is true:

$$S_k = \frac{a_1 + a_k}{2} \cdot k. \tag{3}$$

□ We have

$$S_k = \frac{1}{2} \cdot 2S_k = \frac{1}{2}((a_1 + a_2 + a_3 + \ldots + a_k) + (a_1 + a_2 + a_3 + \ldots + a_k))$$

$$= \frac{1}{2}((a_1 + a_2 + a_3 + \ldots + a_{k-2} + a_{k-1} + a_k) + (a_k + a_{k-1} + a_{k-2} + \ldots + a_3 + a_2 + a_1))$$

$$= \frac{1}{2}((a_1 + a_k) + (a_2 + a_{k-1}) + (a_3 + a_{k-2}) + \ldots + (a_{k-2} + a_3) + (a_{k-1} + a_2) + (a_k + a_1)).$$

Since

$$1 + k = 2 + (k - 1) = 3 + (k - 2) = \ldots = (k - 1) + 2 = k + 1,$$

according to the property proved, we have

$$a_1 + a_k = a_2 + a_{k-1} = a_3 + a_{k-2} = \ldots = a_{k-1} + a_2 = a_k + a_1.$$

Consequently,

$$S_k = \frac{1}{2} \cdot k(a_1 + a_k), \text{ that is, } S_k = \frac{a_1 + a_k}{2} \cdot k. \quad \blacksquare$$

If in equality (3), instead of a_k, we substitute its expression from equality (1), then we get

$$S_k = \frac{2a_1 + d(k-1)}{2} \cdot k. \qquad (4)$$

Example 1. In the arithmetic progression (a_n) $a_1 = 3$, and $d = 4$. Find S_{20}.

○ Using equality (4), we get

$$S_{20} = \frac{2a_1 + d \cdot 19}{2} \cdot 20 = \frac{6 + 4 \cdot 19}{2} \cdot 20 = 820. \quad \bullet$$

Example 2. Given a finite arithmetic progression (a_n). Knowing three numbers from the five (a_1, d, n, a_n, S_n), find the remaining two:

(1) given: $a_1 = 10$, $d = 4$, $a_n = 50$, find n and S_n,

(2) given: $a_1 = 10$, $d = 4$, $S_n = 330$, find n and a_n,

(3) given: $a_1 = 10$, $n = 11$, $S_n = 330$, find a_n and d,

(4) given: $d = 4$, $a_n = 50$, $S_n = 330$, find a_1 and n.

○ (1) From the formula $a_n = a_1 + d(n - 1)$ we get

$$n - 1 = \frac{a_n - a_1}{d}, \text{ that is,}$$

$$n = \frac{a_n - a_1}{d} + 1, \quad n = \frac{50 - 10}{4} + 1 = 11.$$

Further, we have $S_{11} = \frac{10 + 50}{2} \cdot 11 = 330$.

(2) Let us make use of formula (4). Substituting the given numbers, we have $330 = \frac{20+4(n-1)}{2} n$. We then solve the obtained quadratic equation with respect to n:

$$330 = 10n + 2n^2 - 2n, \quad 2n^2 + 8n - 330 = 0, \quad n^2 + 4n - 165 = 0,$$

$n = -2 \pm \sqrt{4+165}$, $n_1 = -15$ (does not suit the hypothesis), $n_2 = 11$. Knowing n, we find $a_n = a_1 + d(n-1) = 10 + 4 \cdot 10 = 50$.

(3) From the formula $S_n = \frac{2a_1 + d(n-1)}{2} n$ we find d:

$$2S_n = 2a_1 n + d(n-1)n, \quad d(n-1)n = 2S_n - 2a_1 n,$$

$$d = \frac{2S_n - 2a_1 n}{n(n-1)}, \quad d = \frac{2 \cdot 330 - 2 \cdot 10 \cdot 11}{10 \cdot 11} = \frac{62-22}{11} = 4.$$

Now, we can find $a_n = a_1 + d(n-1) = 10 + 4 \cdot 10 = 50$.

(4) From the formula $a_n = a_1 + d(n-1)$ we find $a_1 = a_n - d(n-1)$ and substitute the latter expression into the formula $S_n = \frac{a_1 + a_n}{2} n$; then $S_n = \frac{2a_n - d(n-1)}{2} n$. We now find n:

$$330 = \frac{2 \cdot 50 - 4(n-1)}{2} n, \quad 330 = 50n - 2n^2 + 2n,$$

$$2n^2 - 52n + 330 = 0, \quad n^2 - 26n + 165 = 0,$$

$$n = 13 \pm \sqrt{169 - 165} = 13 \pm 2, \quad n_1 = 11, \quad n_2 = 15.$$

If $n = 11$, then $a_1 = a_n - d(n-1) = 50 - 4 \cdot 10 = 10$.

If $n = 15$, then $a_1 = a_n - d(n-1) = 50 - 4 \cdot 14 = -6$. ●

Example 3. Find the sum of all even three-digit numbers multiple of three.

○ The least number satisfying the given hypothesis is 102, followed by the number 108, next the number 114, and so forth. We get the arithmetic progression 102, 108, 114, 120, 126, . . ., 996 being the greatest number satisfying the hypothesis.

Thus, for the arithmetic progression (a_n) in which $a_1 = 102$, $d = 6$, $a_k = 996$, we have to compute S_k.

We have $a_k = a_1 + d(k-1)$. Substituting the values of a_k, a_1, and d into this equality, we get $996 = 102 + 6(k-1)$, whence $k = 150$.

Further, we have $S_k = \frac{a_1 + a_k}{2} \cdot k$. Substituting the values of a_1, a_k, and k into this equality, we find

$$S_{150} = \frac{102 + 996}{2} \cdot 150 = 82\,350. \quad ●$$

Example 4. The sequence $a_1, a_2, \ldots, a_n, \ldots$ possesses the following property: the sum S_n of its first terms is equal to $2n^2 + 3n$. Prove that this sequence is an arithmetic progression.

○ Using the characteristic property of the arithmetic progression, we conclude that it suffices to prove the relationship $a_{k+1} = (a_k + a_{k+2})/2$, where $k = 1, 2, 3, \ldots$.

We note that $a_{k+1} = S_{k+1} - S_k$. Indeed,

$$S_{k+1} - S_k = (a_1 + a_2 + \ldots + a_k + a_{k+1}) - (a_1 + a_2 + \ldots + a_k) = a_{k+1}.$$

Hence,

$$a_{k+1} = (2(k+1)^2 + 3(k+1)) - (2k^2 + 3k) = 4k + 5.$$

Similarly, $a_k = S_k - S_{k-1}$ (if $k = 1$, then $a_1 = S_1 - S_0$, where $S_0 = 0$), whence

$$a_k = (2k^2 + 3k) - (2(k-1)^2 + 3(k-1)) = 4k + 1,$$

and, finally, $a_{k+2} = S_{k+2} - S_{k+1} = 4k + 9$. Since for the numbers $a_k = 4k + 1$, $a_{k+1} = 4k + 5$, $a_{k+2} = 4k + 9$ the relationship $2a_{k+1} = a_k + a_{k+2}$ holds, we have thereby proved that (a_n) is an arithmetic progression. ●

Example 5. The division of the 13th term of an arithmetic progression by the 3rd term yields 3 as a quotient, and the division of the 18th term by the 7th term yields 2 as a quotient and 8 as a remainder. Determine the difference and the first term of the progression.

○ It follows from the hypothesis that $a_{13} = 3a_3$ and $a_{18} = 2a_7 + 8$. Expressing a_3, a_7, a_{13}, and a_{18} in terms of a_1 and d, we get the system of two equations in two variables a_1 and d:

$$\begin{cases} a_1 + 12d = 3(a_1 + 2d), \\ a_1 + 17d = 2(a_1 + 6d) + 8. \end{cases}$$

Solving this system, we find $a_1 = 12$, $d = 4$. ●

2. Geometric Progression.

Definition. A sequence of numbers whose first term is other than zero and each term, beginning with the second, is equal to the preceding term multiplied by the same nonzero number q is called a *geometric progression.* The number q is called the *common ratio* (or, simply, *ratio*).

Thus, a geometric progression is determined by the following conditions: (1) $b_1 = b$ $(b \neq 0)$, (2) $b_{n+1} = b_n q$ $(q \neq 0)$.

For example, if $b_1 = 1$, $b_{n+1} = b_n \cdot 2$, then $b_2 = 1 \cdot 2 = 2$, $b_3 = b_2 \cdot 2 = 4$, . . ., that is, we get the sequence 1, 2, 4, 8, 16, ...

In the definition, we set $b \neq 0$, $q \neq 0$ in order to avoid the trivial case of the sequence consisting only of zeros.

Let us give a few examples of geometric progressions:

2, 8, 32, 128, 512, . . ., here $b_1 = 2$, $b_{n+1} = b_n \cdot 4$, $q = 4$,

2, −8.32, −128, 512, . . ., here $b_1 = 2$, $b_{n+1} = b_n (-4)$, $q = -4$,

27, 9, 3, 1, 1/3, 1/9, . . ., here $b_1 = 27$, $b_{n+1} = b_n \cdot (1/3)$, $q = 1/3$,

8, 8, 8, 8, 8, . . ., here $b_1 = 8$, $b_{n+1} = b_n \cdot 1$, $q = 1$.

In order to indicate that the sequence (b_n) is a geometric progression, we use the notation $b_1, b_2, b_3, \ldots, b_n, \ldots$.

If a geometric progression is truncated at some kth term, then we obtain a finite (or terminating) geometric progression of the form $b_1, b_2, b_3, \ldots, b_{k-1}, b_k$.

It is obvious that if $q > 1$ and $b_1 > 0$, then the geometric progression is an increasing sequence; if $0 < q < 1$ and $b_1 > 0$, then it is a decreasing sequence; and if $q = 1$, then we have a constant sequence. For $q < 0$ the geometric progression is not monotonic.

Here are some examples of geometric progressions:

(1) 1, 3, 9, 27, . . . ($q = 3$, $b_1 > 0$) is an increasing sequence,

(2) 1, 1/2, 1/4, 1/8, . . . ($q = 1/2$, $b_1 > 0$) is a decreasing sequence,

(3) 2, 2, 2, 2, . . . ($q = 1$) is a constant sequence,

(4) 1, —2, 4, —8, . . . ($q = -2$) is a nonmonotonic sequence.

The above represented geometric progression is a recursive representation of a sequence. However, it turns out that in many cases it is inconvenient. Indeed, in order to find some term of a geometric progression with a sufficiently large ordinal number, for a recursively defined sequence it is necessary to know all its previous terms. This enormous amount of computations can be reduced by deriving the formula for the nth (or general) term of the geometric progression from the recurrence relation $b_{n+1} = b_n q$.

Formula for the nth term of geometric progression. *Let (b_n) be a geometric progression. Then its nth term can be represented by the following analytic expression:*

$$b_n = b_1 q^{n-1}. \tag{5}$$

□ We make use of the method of mathematical induction.

(1) For $n = 1$ equality (5) holds. Indeed, $b_1 = b_1 q^{1-1}$ which is a correct equality. Thus, A (1) is true.

(2) Let us prove that $A(k) \Rightarrow A(k+1)$. We assume that $b_k = b_1 q^{k-1}$ and prove that $b_{k+1} = b_1 q^k$.

Indeed, by definition of a geometric progression

$$b_{k+1} = b_k q = (b_1 q^{k-1})\, q = b_1 q^k.$$

Thus, $A(k) \Rightarrow A(k+1)$. By virtue of the principle of mathematical induction, equality (5) is valid for any natural n. ■

For instance, for the geometric progression 2, 8, 32, 128, 512, ..., where $b_1 = 2$, $q = 4$, we have $b_5 = b_1 q^4 = 2 \cdot 4^4 = 512$, $b_8 = b_1 q^7 = 2 \cdot 4^7 = 32\,768$, etc.

Characteristic property of geometric progression. *For a sequence (b_n) to be a geometric progression, it is necessary and sufficient that*

each of its terms, except the first one, be equal in absolute value to the geometric mean of its neighbouring terms:

$$|b_{n+1}| = \sqrt{b_n b_{n+2}}. \tag{6}$$

□ **Necessity.** Let (b_n) be a geometric progression, then $b_{n+1} = b_n q$, that is, $b_n = b_{n+1}/q$; $b_{n+2} = b_{n+1}q$. Hence,

$$\sqrt{b_n b_{n+2}} = \sqrt{\frac{b_{n+1}}{p} \cdot b_{n+1} q} = \sqrt{b_{n+1}^2} = |b_{n+1}|.$$

Sufficiency. Let in the sequence (b_n) equality (6) holds for any n. It follows from this equality that

$$b_{n+1}^2 = b_n b_{n+2}, \tag{7}$$

that is, $b_{n+1}/b_n = b_{n+2}/b_{n+1}$. Hence we get

$$\begin{aligned} &\text{for } n = 1\text{: } b_2/b_1 = b_3/b_2,\\ &\text{for } n = 2\text{: } b_3/b_2 = b_4/b_3,\\ &\text{for } n = 3\text{: } b_4/b_3 = b_3/b_4,\\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Thus, $b_2/b_1 = b_3/b_2 = b_4/b_3 = b_5/b_4 = \ldots$, that is, the ratio of each term of the sequence to the preceding term is the same. This means that (b_n) is a geometric progression. ■

Remark. In practice, it is more convenient to write the characteristic property of the geometric progression in the form of equality (7) rather than in the form of equality (6).

Formula for k terms of geometric progression. *Let (b_n) be a geometric progression (its ratio q being different from unity), and S_k the sum of its first k terms: $S_k = b_1 + b_2 + b_3 + \ldots + b_k$. Then the following equality holds**

$$S_n = \frac{b_1(q^k - 1)}{q - 1}. \tag{8}$$

□ We have

$$\begin{aligned} S_k &= b_1 + b_2 + b_3 + \ldots + b_k = b_1 + b_1 q + b_1 q^2 + \ldots + b_1 q^{k-1}\\ &= b_1 (1 + q + q^2 + \ldots + q^{k-1}). \end{aligned}$$

In § 7.1$_3$ we proved that

$$1 + \sum_{i=1}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}, \quad x \neq 1.$$

* If $q = 1$, then the geometric progression has the form $b_1, b_1, b_1, \ldots, b_1, \ldots$. In this case $S_k = kb_1$.

Setting $x=q$, $n=k-1$, we get

$$1+\sum_{i=1}^{k-1} q^i=\frac{q^k-1}{q-1}, \text{ that is, } 1+q+q^2+\ldots+q^{k-1}=\frac{q^k-1}{q-1}.$$

Consequently, $S_k=\frac{b_1(q^k-1)}{q-1}$. ■

Example 1. Find the sum of the first ten terms of the geometric progression in which $b_2+b_6=34$, $b_3+b_7=68$.

○ Since $b_2=b_1q$, $b_6=b_1q^5$, $b_3=b_1q^2$, $b_7=b_1q^6$, using the above mentioned conditions, we get the system of two equations in two variables

$$\begin{cases} b_1q+b_1q^5=34, \\ b_1q^2+b_1q^6=68, \end{cases} \text{ or } \begin{cases} b_1q(1+q^4)=34, \\ b_1q^2(1+q^4)=68, \end{cases}$$

Rewriting the second equation of the system in the form $q\cdot b_1q(1+q^4)=68$ and taking advantage of the fact that $b_1q(1+q^4)=34$ (from the first equation), we get $q\cdot 34=68$, that is, $q=2$. Substituting the value $q=2$ into the first equation, we get $b_1\cdot 2(1+2^4)=34$, whence $b_1=1$. Thus, $b_1=1$, $q=2$. Consequently

$$S_{10}=\frac{b_1(q^{10}-1)}{q-1}=\frac{1\cdot(2^{10}-1)}{2-1}=1023. \bullet$$

Example 2. Three numbers form a geometric progression. If the second number is increased by 2, then the numbers obtained form an arithmetic progression. If then the third number is increased by 9, then a geometric progression is obtained anew. Find these numbers.

○ Let us rewrite the conditions of the problem in the following way:

$$b_1,\ b_2,\ b_3, \qquad b_1,\ b_2+2,\ b_3, \qquad b_1,\ b_2+2,\ b_3+9.$$

Using the characteristic properties of the progressions, we get

$$\begin{cases} 2(b_2+2)=b_1+b_3, \\ (b_2+2)^2=b_1(b_3+9). \end{cases}$$

Expressing now b_2 and b_3 in terms of b_1 and q, we get the system of two equations in two variables:

$$\begin{cases} 2(b_1q+2)=b_1+b_1q^2, \\ (b_1q+2)^2=b_1(b_1q^2+9). \end{cases}$$

Solving this system, we find $b_1=4$, $q=2$ or $b_1=4/25$, $q=-4$. Thus, we get 4, 8, 16 or 4/25, −16/25, 64/25. ●

Given below is a table illustrating the properties of arithmetic and geometric progressions:

Arithmetic progression	Geometric progression
$a_1=a,\ a_{n+1}=a_n+d$	$b_1=b\ (b\neq 0),\ b_{n+1}=b_nq\ (q\neq 0)$
$a_n=a_1+d\,(n-1)$	$b_n=b_1q^{n-1}$
$a_{n+1}=\dfrac{a_n+a_{n+2}}{2}$	$b_{n+1}^2=b_nb_{n+2}$
$S_n=\dfrac{a_1+a_n}{2}\,n=\dfrac{2a_1+d\,(n-1)}{2}\,n$	$S_n=\dfrac{b_1\,(q^n-1)}{q-1},\ q\neq 1$

3. Sum of a Geometric Progression in Which $|q|<1$. Let us take a closed interval [0, 1] and divide it into two equal parts. Taking then the right-hand half of the interval, that is, the closed interval [1/2, 1], we subdivide it again into two equal parts. Then, we subdivide the right-hand part of the interval [1/2, 1], that is, the interval [3/4, 1] again into two equal parts, and so forth. Continuing this process ad infinitum, we get the partition of the taken interval into infinitely many subintervals [0, 1/2], [1/2, 3/4], [3/4, 7/8], . . . (Fig. 173). It is natural to hold that the "sum" of the lengths of all the subintervals into which the interval [0, 1] is divided is usually equal to the length of the closed interval under partition, that is, to unity. In other words, it is natural to hold the following "equality" as a correct one:

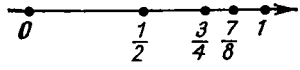


Fig. 173

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}+\ldots=1. \tag{9}$$

The left-hand side of "equality" (9) represents "the sum of an infinite set of terms". We use the quotation marks here because this expression cannot be understood literally. The case in point is not a usual sum (the sum of a finite number of terms), but something which should be correctly understood and explained.

When first dividing the closed interval [0, 1] into two equal parts, we took the subinterval [0, 1/2], its length $S_1 = 1/2$. At the second step, we added to the latter the subinterval [1/2, 3/4], whose length is 1/4. As a result, two subintervals [0, 1/2] and [1/2, 3/4] formed the interval [0, 3/4] with length S_2 equal to $1/2 + 1/4 = 3/4$. At the third step, a third subinterval [3/4, 7/8] was added to these

two, as a result the interval [0, 7/8] with length S_3 equal to $1/2 + 1/4 + 1/8 = 7/8$ was formed.

In general, at the nth step n subintervals will form a closed interval of length S_n, where

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n}.$$

It is clear that the greater n, the closer S_n approaches unity, that is, the length of the closed interval [0, 1], that is, $\lim\limits_{n\to\infty} S_n = 1$. This is what equality (9) means.

The terms on the left-hand side of equality (9) form the sequence $1/2, 1/4, 1/8, \ldots, 1/2^n, \ldots$, which is a geometric progression. The above reasoning enables us to speak not only of the sum S_n of the first n terms of this progression, but also of the sum S of all the terms of the progression understanding this sum as the number $\lim\limits_{n\to\infty} S_n$.

It turns out that the notion of the sum can be introduced for any geometric progression (b_n) in which $|q| < 1$, where q is the common ratio of the progression. The thing is that for such a progression there exists $\lim\limits_{n\to\infty} S_n$ (and this limit can be, naturally, called the sum of the progression).

Indeed, let (b_n) be a geometric progression, and $|q| < 1$. Then

$$S_n = \frac{b_1(q^n - 1)}{q - 1}.$$

Let us pass in this equality to the limit as $n \to \infty$. Using the theorems on arithmetic operations on limits and the fact that $\lim\limits_{n\to\infty} q^n = 0$, if $|q| < 1$ (see § 7.3, Item 5), we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{b_1(q^n - 1)}{q - 1} = \frac{b_1}{q-1} \lim_{n\to\infty} (q^n - 1) = \frac{b_1}{q-1}(0 - 1) = \frac{b_1}{1-q}.$$

Thus,

$$S = \lim_{n\to\infty} S_n = \frac{b_2}{1-q}.$$

Let us sum up the foregoing. If we are given a geometric progression (b_n) whose ratio q satisfies the inequality $|q| < 1$, then we can introduce the notion of the sum of a progression:

$$b_1 + b_2 + b_3 + \ldots + b_n + \ldots = S.$$

Definition. By the *sum S of a geometric progression* $b_1, b_2, \ldots, b_n$ in which $|q| < 1$ (such a progression is sometimes called *infinitely decreasing*) we understand the limit of the sequence (S_n), where $S_1 = b_1$, $S_2 = b_1 + b_2$, $S_3 = b_1 + b_2 + b_3$, $\ldots$, $S_n = b_1 + b_2 + b_3 + \ldots + b_n$, that is, $S = \lim\limits_{n\to\infty} S_n$. In this case, the following

equality holds:

$$S=\frac{b_1}{1-q}. \tag{10}$$

Example 1. Find the sum of the geometric progression

$$15,\ 6,\ 12/5,\ 24/25,\ 48/125,\ \ldots .$$

○ Since $b_1 = 15$, $b_2 = 6$, we have $q = b_2/b_1 = 6/15 = 2/5$, that is, $|q| < 1$. Using formula (10), we find

$$S=\frac{b_1}{1-q}=\frac{15}{1-2/5}=25. \ \bullet$$

Example 2. The sum of a geometric progression in which $|q| < 1$ is equal to 9, and the sum of squares of its terms amounts to 40.5. Find the first term and the ratio of the progression.

○ Let us denote the first term of the desired progression by b_1 and its ratio by q. Using the formula for the sum of a geometric progression, we get $\frac{b_1}{1-q} = 9$.

Consider the sequence $b_1^2,\ b_2^2,\ b_3^2,\ \ldots,\ b_n^2,\ \ldots$. We have $b_2 = b_1q$, that is, $b_2^2 = b_1^2q^2$, $b_3 = b_2q$, that is, $b_3^2 = b_2^2q^2$, $b_4 = b_3q$, that is, $b_4^2 = b_3^2q^2$, etc. We notice that each term of the sequence (b_n^2) is obtained from the preceding one by multiplying it by q^2. This means that (b_n^2) is a geometric progression whose first term is equal to b_1^2, its ratio being q^2. It follows from the condition $|q| < 1$ that $q^2 < 1$. Then the sum S of the progression (b_n^2) can be computed by the formula $S = \frac{b_1^2}{1-q^2}$, and, by hypothesis, this sum is equal to 40.5.

Thus, the problem is reduced to solving the system of equations

$$\begin{cases} \dfrac{b_1}{1-q}=9, \\ \dfrac{b_1^2}{1-q^2}=40.5, \end{cases}$$

from which we find $b_1 = 6$, $q = 1/3$. ●

Exercises

1. Which of the given sequences are arithmetic progressions, and which are geometric progressions:

(a) $1,\ 4,\ 7,\ 10,\ 13,\ \ldots$, (b) $3,\ 0,\ -3,\ -6,\ -9,\ \ldots$,

(c) $2,\ 4,\ 8,\ 16,\ 32,\ \ldots$, (d) $4,\ 9,\ 16,\ 25,\ \ldots$,

(e) $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots$, (f) $1,\ 8,\ 27,\ 64,\ \ldots$,

(g) $1,\ \frac{1}{5},\ \frac{1}{25},\ \frac{1}{125},\ \ldots$, (h) $1,\ \frac{1}{4},\ \frac{1}{9},\ \frac{1}{16},\ \ldots$,

(i) 5, 5, 5, 5, ..., (j) −2, 2, −2, 2, ...?

2. Find out whether the sequence (x_n) is an arithmetic or a geometric progression if: (a) $x_1 = 5$ and $x_{n+1} = 4x_n$, (b) $x_1 = -8$ and $x_{n+1} = 0.5 + x_n$?

3. Show that the given sequences of numbers are arithmetic progressions. Find a_1 and d: (a) $a_n = 4n + 3$, (b) $a_n = 2n - 1$, (c) $a_n = (3n + 2)/5$, (d) $a_n = (7n - 2)/3$, (e) $a_n = (8n + 1)/5$.

4. It is known that the sizes of the angles of a triangle form an arithmetic progression. Prove that the size of one of the angles of the triangle is equal to 60°.

In Exercises 5 and 6, compute the unknown quantities in each row of the table using three other quantities which are given:

5.

	a_1	d	n	a_n	S_n
(a)	110	−10	11		
(b)	5		26	105	
(c)		3	12		210
(d)		2	15	−10	

6.

	a_1	d	n	a_n	S_n
(a)	−9	0.5			−75
(b)	−28		9		0
(c)	0.2			5.2	137.7
(d)			30	15.75	146.25

7. Find the first term and the difference of the arithmetic progression in which:

$$\text{(a)} \begin{cases} a_2 + a_8 = 10, \\ a_3 + a_{14} = 31, \end{cases} \qquad \text{(b)} \begin{cases} S_5 - S_3 - a_5 = 0.1, \\ S_4 + a_7 = 0.1, \end{cases}$$

$$\text{(c)} \begin{cases} S_4 = 9, \\ S_6 = 22.5, \end{cases} \qquad \text{(d)} \begin{cases} a_3 + a_5 + a_8 = 18, \\ a_4 + a_2 = -2, \end{cases}$$

8. Given: the sum S_n of the first n terms of an arithmetic progression (a_n). Find the first four terms of the progression if $S_n = n^2/4 - n$.

9. A body in a free fall covers the distance of 4.9 m during the first second, and by 9.8 m more during each consecutive second. Find: (a) the distance covered by the body during the 11th second, (b) the distance covered by the body after eleven seconds have elapsed, (c) the time during which the body will fall freely from an altitude of 4410 m.

10. Find the sum of: (a) two-digit numbers, (b) three-digit numbers, (c) even numbers less than 1000, (d) three-digit numbers multiple of 5.

11. The division of the 10th term of an arithmetic progression by the second one yields 6 as a quotient, and the division of the 8th term of this progression by the 4th yields 2 as a quotient and 2 as a remainder. Find the sum of the fifteen terms of the progression.

12. In each row of the table, compute the unknown quantities using the three given quantities if (b_n) is a geometric progression:

	b_1	q	n	b_n	S_n
(a)	1	3	10		
(b)		1/2	8	2	
(c)	2		7	1458	
(d)		3		567	847
(e)	1/2			1/128	127/128
(f)	1/3	1/3		1/6561	
(g)		—3	4		30

13. Prove that for a finite geometric progression (b_n) the equality $b_3 b_{n-2} = b_1 b_n$ holds true.

14. Find the first term, the common ratio, and the number of terms of the geometric progression (b_n) in which $b_7 - b_5 = 48$, $b_6 + b_5 = 48$, $S_n = 1023$.

15. Find the four numbers forming a geometric progression in which the third term exceeds the first one by 9, and the second term is greater than the fourth by 18.

16. Find the first and fifth terms of a geometric progression if it is known that its common ratio is equal to 3, the sum of the first six terms being 1820.

17. Find the number of terms of a finite geometric progression if its first, second, and last terms are equal to 3, 12, and 3072, respectively.

18. In a geometric progression, the eighth term is equal to 13, and the common ratio to —3. Find its tenth term and the sum of the first ten terms.

19. Between the numbers 1 and 16 place three numbers such that, together with the given numbers, they form a geometric progression.

20. For what values of n do the numbers $\sqrt{n-5}$, $\sqrt[4]{10n+4}$, and $\sqrt{n+2}$ form a geometric progression?

21. For what values of n do the numbers $\sqrt{2n+9}$, $\sqrt{5n}$, and $\sqrt{8n+9}$ form an arithmetic progression?

22. The sum of three numbers forming an arithmetic progression is equal to 30. If 2 is subtracted from the second term of this progression, and the remaining numbers are left unchanged, then a geometric progression is obtained. Find these numbers.

23. Find the four natural numbers the first three of which form a geometric progression, and the last three an arithmetic progression if it is known that the sum of the extremes is equal to 21 and the sum of the means is equal to 18.

24. Three numbers, the third of which is 12, form a decreasing geometric progression. If 12 is replaced by 9, then the three numbers will form an arithmetic progression. Find the first two numbers.

25. The sum of an infinitely decreasing geometric progression is equal to 2/3, and the third term of the progression to 1/4. Find the eighth term of the progression.

26. Write the first five terms of an infinitely decreasing geometric progression whose sum is equal to 3 if the sum of cubes of its terms is equal to $61\frac{5}{7}$.

Chapter 8

ELEMENTS OF DIFFERENTIAL AND INTEGRAL CALCULUS

§ 8.1. Continuous Functions

1. A Function Continuous at a Point. Let a load be suspended on a thread above a table. Under this load the thread extends, therefore the distance l from the load (supposed to be a material particle) to the point of thread suspension is a function of mass m of the load, that is, $l = f(m)$, $m \geqslant 0$.

If the mass of the load is somewhat changed, then the distance l will change but little. Thus, to small changes of m there correspond small changes in l. However, if the mass of the load approaches the tensile strength m_0 of the thread, then a small increase in the mass of the load may cause a break in the thread, and, as a result, the distance l will increase jumpwise and become equal to the distance L from the suspension point to the surface of the table. The graph of the function $l = f(m)$ is schematically shown in Fig. 174. On the interval $[0, m_0)$ this graph is a continuous line, and at the point m_0 it suffers a discontinuity. As a result, we get a graph consisting of two branches. At all points, except m_0, the function $l = f(m)$ is said to be continuous, but at the point m_0 it has a discontinuity.

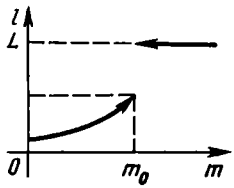


Fig. 174

Such a smooth continuous change in a quantity followed by a jump occurs in many processes.

In mathematics, to the difference between a smooth and jumpwise change in quantities there corresponds a difference between continuous and discontinuous functions. To be more precise, since the same quantity can change both smoothly and jumpwise, we should distinguish between the points of continuity and the points of discontinuity of the function. Points of continuity are characterized by that for small changes of the argument the value of the function changes but little, and points of discontinuity by that in them for small changes in the argument the function can change considerably. If we speak of an indirect measurement of y by measuring the quantity x, then the words "for a small change in x the quantity y changes but little" mean: "for a small error in measuring x the error in the value of y is small". More precisely, the error in the value of y can

be made arbitrarily small if the value of x is measured sufficiently precise.

The value of the error of measurement is estimated by the greatest permissible error or, as we say, by measurement accuracy. If a quantity x is measured with an accuracy to $\delta > 0$ (δ "delta", a letter of the Greek alphabet), then this means that the deviation of the obtained value* of x from the exact value of a is less than δ, that is, $|x - a| < \delta$. The words "the error in the value of $y = f(x)$ can be made arbitrarily small" mean the following: whatever the accuracy of $\varepsilon > 0$ is preassigned, we can always make the deviation of $f(x)$ from $f(a)$ less than ε, that is, $|f(x) - f(a)| < \varepsilon$. Such an accuracy is possible to achieve by choosing x sufficiently close to a.

Thus, the words "with a sufficiently small error in measuring x, the error in the value of y is arbitrarily small" mean the following: whatever positive number ε (the permissible error for y) is preassigned, there is a positive number δ (the permissible error for x) such that if x is measured with an accuracy to δ, then y will be obtained with the preassigned accuracy ε. In other words, the inequality $|x - a| < \delta$ implies the inequality $|f(x) - f(a)| < \varepsilon$.

Let us return to the example with a hanging load. Suppose that the mass of the load is far from the tensile strength of the thread. Then, if we assign some value for the thread elongation, we can always choose an additional load such that the thread elongation does not exceed the indicated value. For example, it may appear that in order for the thread not to be stretched by more than 1 mm, the load should not be more than 200 g heavier, and for the elongation not to exceed 0.1 mm, we must not increase the load by more than 25 g, and so on. For any preassigned elongation δ, it is possible to choose a value of δ such that if the mass of the additional load is less than δ, the thread will stretch by less than δ. Similar reasoning applies for the case of shortening the thread by decreasing the mass of the load. Thus, in the case under consideration, the inequality $|x - a| < \delta$ implies the inequality $|f(x) - f(a)| < \varepsilon$.

Definition. A function $y = f(x)$ is said to be *continuous* at a point a if the following two conditions are satisfied:

(1) the function is defined in some neighbourhood of the point a,

(2) for any $\varepsilon > 0$ there is $\delta > 0$ such that $|x - a| < \delta$ yields $|f(x) - f(a)| < \varepsilon$ (that is, if the deviation of x from a is less than δ, then the deviation of $f(x)$ from $f(a)$ is less than ε).

Let us give a geometric interpretation of this definition. Suppose that $M(a, f(a))$ is a point on the graph of the function $y = f(x)$. Let us project this point on the coordinate axes and consider the ε-neighbourhood of the point $f(a)$ on the y-axis. If the function $y = f(x)$ is continuous at the point a, then there is a δ-neighbourhood

* That is, the distance from x to a.

of the point a on the x-axis which possesses the following property: *whatever point x is taken in the δ-neighbourhood, the corresponding point on the y-axis lies in the given ε-neighbourhood of the point $f(a)$* (Fig. 175).

Example. Prove the continuity of the function $y = f(x)$ at any point $x = a$ if: (1) $f(x) = C$, (2) $f(x) = kx + b$, $k \neq 0$, (3) $f(x) = \cos x$, (4) $f(x) = \sin x$.

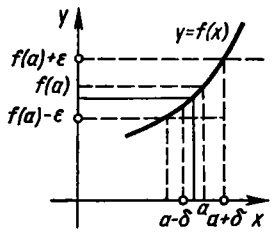


Fig. 175

○ (1) We have $|f(x) - f(a)| = |C - C| = 0$. Hence, $|f(x) - f(a)| < \varepsilon$ in any neighbourhood of the point a. This means that *a constant function is continuous at any point a.*

(2) The graph of a linear function is a straight line, therefore the continuity of a linear function at any point a is geometrically obvious. Let us carry out a rigorous proof of this assertion.

Let us take an arbitrary number $\varepsilon > 0$ and find a $\delta > 0$ such that the inequality $|x - a| < \delta$ implies the inequality $|f(x) - f(a)| < \varepsilon$. We have

$$|f(x) - f(a)| = |(kx + b) - (ka + b)| = |k| \cdot |x - a|.$$

Therefore the inequality $|f(x) - f(a)| < \varepsilon$ is equivalent to the inequality $|x - a| < \varepsilon/|k|$, that is, $|x - a| < \varepsilon/|k|$ implies $|f(x) - f(a)| < \varepsilon$. Hence, if we set $\delta = \varepsilon/|k|$, then from the inequality $|x - a| < \delta$ there follows the inequality $|f(x) - f(a)| < \varepsilon$, and this means that a *linear function is continuous at any point a.*

(3) Let us take an arbitrary number $\varepsilon > 0$ and find $\delta > 0$ such that from the inequality $|x - a| < \delta$ there follows the inequality $|\cos x - \cos a| < \varepsilon$. We have

$$|\cos x - \cos a| = \left| -2 \sin \frac{x-a}{2} \sin \frac{x+a}{2} \right| = 2 \left| \sin \frac{x-a}{2} \right| \cdot \left| \sin \frac{x+a}{2} \right|.$$

In § 6.1, Item 10, we proved that $|\sin x| \leqslant |x|$. Hence, $\left|\sin \frac{x-a}{2}\right| \leqslant \left|\frac{x-a}{2}\right|$. Since, in addition, $\left|\sin \frac{x-a}{2}\right| \leqslant 1$, we get

$$|\cos x - \cos a| = 2 \left| \sin \frac{x-a}{2} \right| \cdot \left| \sin \frac{x+a}{2} \right| \leqslant 2 \cdot \left| \frac{x-a}{2} \right| \cdot 1 = |x - a|.$$

Thus, $|\cos x - \cos a| \leqslant |x - a|$.

From this inequality it follows that in this case it is possible to set $\delta = \varepsilon$. Indeed, if $|x - a| < \delta$, that is, $|x - a| < \varepsilon$, then, by virtue of the fact that $|\cos x - \cos a| \leqslant |x - a|$, we get $|\cos x - \cos a| < \varepsilon$. This means that *the function $y = \cos x$ is continuous at any point a.*

(4) Similarly, we can prove the continuity of the function $y = \sin x$ at any point a. ●

2. Discontinuity Points. Let us first introduce the notion of a "deleted" neighbourhood of a point. The union of two intervals: $(a - \delta, a) \cup (a, a + \delta)$ will be *a deleted* δ*-neighbourhood* of the point a. In other words, this is the interval $(a - \delta, a + \delta)$ with the point a (the centre of the interval) deleted.

Let a function $y = f(x)$ be defined in some deleted neighbourhood of the point a. If at the point a the function is not defined or is defined but not continuous, then the point a will be called a *discontinuity point* of the function $f(x)$.

Consider, for example, the function $y = (x - 1)^0$. If $x \neq 1$, then $(x - 1)^0 = 1$, if $x = 1$, then the expression $(x - 1)^0$ is not defined.

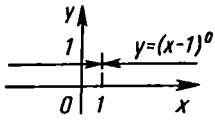


Fig. 176

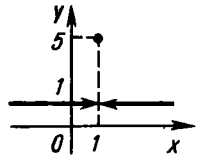


Fig. 177

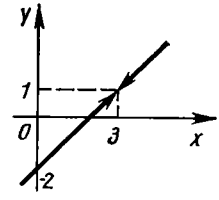


Fig. 178

The graph of the function is shown in Fig. 176. This is a straight line parallel to the axis of abscissas with one point "deleted". The function is continuous everywhere, except for the point $x = 1$, at which the function is discontinuous. Note, however, that if we set $y(1) = 1$, then we get a new function which is defined and continuous for all x.

The same property is inherent in the function

$$y = \begin{cases} 1 \text{ for } x \neq 1, \\ 5 \text{ for } x = 1. \end{cases}$$

The graph of this function is given in Fig. 177. It is discontinuous at the point $x = 1$, but if we change its value at this point by setting $y(1) = 1$, then the function obtained will become continuous at the point $x = 1$.

If a function $y = f(x)$, $x \in X$ has a discontinuity at the point a, and if there is a function $F(x)$ which coincides with $f(x)$ for all $x \in X$, except $x = a$, and is continuous at the point a, then we shall say that the function $y = f(x)$ has a *removable discontinuity* at the point a.

Example. Consider the function $y = \dfrac{x^2 - 5x + 6}{x - 3}$. If $x \neq 3$, then $\dfrac{x^2 - 5x + 6}{x - 3} = \dfrac{(x - 2)(x - 3)}{x - 3} = x - 2$. The graph of the

function is represented in Fig. 178 (the straight line $y = x - 2$ with the point $x = 3$ deleted). Here $x = 3$ is a point of removable discontinuity: the discontinuous function $y = x - 2$ coincides with the given function at all points except $x = 3$.

One should not hold that the discontinuity of a function can be removed in any case. Thus, the function $y = |x|/x$, whose graph is shown in Fig. 179, has an unremovable discontinuity at the point $x = 0$ (a jump). Each of the following functions also has an unremovable discontinuity at the indicated point: the function $y = 1/x^2$ at the point $x = 0$ (see Fig. 67), the function $y = (x - 2)/(x + 2)$ at the point $x = -2$ (see Fig. 94), the function $y = \tan x$ at points $\pi/2 + \pi n$, $n \in Z$ (see Fig. 121).

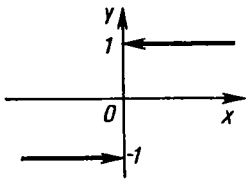


Fig. 179

3. Arithmetic Operations on Continuous Functions. In this subsection we shall show that the sum, product, and quotient of continuous functions are again continuous functions.

Lemma. *Let the functions $u = f(x)$ and $v = g(x)$ be continuous at the point a, and $\varepsilon > 0$ be any positive number. Then there is $\delta > 0$ such that the inequality $|x - a| < \delta$ yields simultaneously two inequalities $|f(x) - f(a)| < \varepsilon$ and $|g(x) - g(a)| < \varepsilon$.*

□ Since the function $u = f(x)$ is continuous at the point a, there exists $\delta_1 > 0$ such that $|x - a| < \delta_1$, implies $|f(x) - f(a)| < \varepsilon$. Similarly, since the function $v = g(x)$ is continuous at the point a, there is $\delta_2 > 0$ such that $|x - a| < \delta_2$ implies $|g(x) - g(a)| < \varepsilon$.

Let δ denote the smaller of the numbers δ_1, δ_2. Then from $|x - a| < \delta$ there follow the inequalities $|x - a| < \delta_1$ and $|x - a| < \delta_2$, wherefrom it follows, in turn, that $|f(x) - f(a)| < \varepsilon$ and $|g(x) - g(a)| < \varepsilon$. ■

It is clear that this lemma can be generalized for the case of any finite number of functions continuous at the point a.

Theorem 8.1.1. *If functions $u = f(x)$ and $v = g(x)$ are continuous at a point a, then the function $y = f(x) + g(x)$ is continuous at the point a.*

In short this theorem can be formulated as follows: *the sum of continuous functions is a continuous function.*

□ Let us take an arbitrary $\varepsilon > 0$. According to the lemma, there is $\delta > 0$ such that for $|x - a| < \delta$ the following inequalities are satisfied:

$$|f(x) - f(a)| < \varepsilon/2 \quad \text{and} \quad |g(x) - g(a)| < \varepsilon/2,$$

that is,

$$|u - f(a)| < \varepsilon/2 \quad \text{and} \quad |v - g(a)| < \varepsilon/2.$$

Then

$$|(u+v)-(f(a)+g(a))| \leqslant |u-f(a)|+|v-g(a)| < \varepsilon/2+\varepsilon/2=\varepsilon.$$

Thus, for any $\varepsilon>0$ there is $\delta>0$ such that $|x-a|<\delta$ implies the inequality $|(f(x)+g(x))-(f(a)+g(a))|<\varepsilon$. Therefore the function $y=f(x)+g(x)$ is continuous at the point a. ■

The theorem proved is almost obvious: if u and v undergo a small change with a small change in x, then $u+v$ also slightly change with a small change in x, and this means that the function $y=u+v$ is continuous.

Theorem 8.1.2. *If functions $u=f(x)$ and $v=g(x)$ are continuous at a point a, then the function $y=f(x)\,g(x)$ is also continuous at this point.*

This theorem can be shortly formulated as follows: *the product of continuous functions is a continuous function.*

The proof of this theorem is similar to that of Theorem 8.1.1 but somewhat more complicated technically; therefore it is not given here (the same refers to Theorem 8.1.3 given below).

This assertion is readily extended to the product of any finite number of continuous functions. Hence, in particular, there follows the continuity of the function $y=x^n$ (n is a natural number) for any x. Indeed, the function $y=x^n$ can be represented as a product of n continuous linear functions $y=x$:

$$x^n=\underbrace{x\cdot x\ldots x}_{n\text{ times}}.$$

Since the constant function $v=C$ is continuous at any point, the following corollary results from Theorem 8.1.2:

Corollary 1. *If a function $u=f(x)$ is continuous at a point a, and C is a real number, then the function $y=Cf(x)$ is also continuous at the point a.*

Let us note two more corollaries to Theorem 8.1.1 and 8.1.2.

Corollary 2. *If the functions $u_1=f_1(x)$, $u_2=f_2(x)$, ..., $u_n=f_n(x)$ are continuous at a point a, then the function $y=C_1f_1(x)+C_2f_2(x)+\ldots+C_nf_n(x)$, where C_1, C_2, ..., C_n are arbitrary real numbers, is also continuous at this point.*

Corollary 3. *If the functions $u=f(x)$ and $v=g(x)$ are continuous at a point a, then their difference $y=f(x)-g(x)$ is also continuous at this point.*

Theorem 8.1.3. *If the functions $u=f(x)$ and $v=g(x)$ are continuous at a point a and $g(a)\neq 0$, then the function $y=\frac{f(x)}{g(x)}$ is also continuous at this point.*

In other words, the *quotient of continuous functions is continuous at a point where the denominator is other than zero.*

The obtained results imply the continuity of the polynomial $y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$ at any point x (see the note to Theorem 8.1.2 and Corollary 2), the continuity of rational function $y = P(x)/Q(x)$ (where $P(x)$ and $Q(x)$ are polynomials) at any point x such that $Q(x) \neq 0$, and the continuity of the functions $\tan x$ and $\cot x$ at all points except those of the form $x = \pi/2 + \pi n$ and $x = \pi n$, respectively.

The main conclusion can be briefly formulated as follows: *all rational and trigonometric functions are continuous at every point where they are defined.*

4. Composite Function and Its Continuity. Consider the function $y = \sin x^2$. To find the value of this function at a fixed point x, we have to: (1) compute x^2; (2) determine the sine of the obtained value. In other words, first we have to find the value of the function $g(x) = x^2$, and then to find $\sin g(x)$ of the obtained value of the function $g(x)$. In such cases we say that a composite function $f(g(x))$ or a *composition of functions is given.* Here $g(x) = x^2$, and $f(x) = \sin x$.

Example 1. Form a composite function $f(g(x))$ from the functions $g(x) = x^2 + x + 5$ and $f(x) = \sqrt[3]{x}$.

○ We have $f(g(x)) = \sqrt[3]{g(x)} = \sqrt[3]{x^2+x+5}$. ●

Example 2. What functions form the composite function $y = \tan^5(2x+1)$?

○ This function consists of three functions: $g(x) = 2x + 1$, $h(x) = \tan x$, and $f(x) = x^5$. Indeed, $f(h(g(x))) = (h(g(x)))^5 = (\tan g(x))^5 = \tan^5(2x + 1)$. ●

Consider a composite function $y = f(u)$, where $u = g(x)$. Note that if y is a continuous function of u, then y undergoes a slight change with a small change in u; if, in turn, u is a continuous function of x, then u slightly changes with a small change in x. As a result, we get that y undergoes a small change with a small change in x, and this means that y is continuous as a composite function of x. The following theorem holds (we give it without proof).

Theorem 8.1.4. *If a function $g(x)$ is continuous at a point x_0, and a function $f(u)$ is continuous at the corresponding point $u_0 = g(x_0)$, then the composite function $y = f(g(x))$ is continuous at the point x_0.*

Thus, not only arithmetic operations on rational and trigonometric functions, but also the operation of forming composite functions from them lead to a function continuous in its domain of definition.

Example 3. Test the function $y = \sin^3 x + \dfrac{3x-1}{x^2-4}$ for continuity.

○ The given function is not defined only at two points: $x = 2$ and $x = -2$. Since it is a sum of a rational and a composite function

formed from a power and a trigonometric function, the given function is continuous at all the points of the number line except for the points $x = 2$ and $x = -2$. At these points the function is discontinuous. ●

5. Continuity of an Inverse Function. Consider the graph of a function $y = f(x)$ consisting of one branch. Suppose that there is an inverse of this function: $y = f^{-1}(x)$. The graph of the inverse function $y = f^{-1}(x)$ is obtained from the graph of the function $y = f(x)$ by interchanging the x- and y-axes, that is, under symmetry transformation with respect to the bisector of the first quadrant (Fig. 180). Since the graph of the function $y = f(x)$ consists of one branch, and this property is obviously retained under symmetry transformation, the graph of the function $y = f^{-1}(x)$ also consists of one branch. This means that if the function $y = f(x)$ is continuous and has an inverse, then the inverse is also continuous.

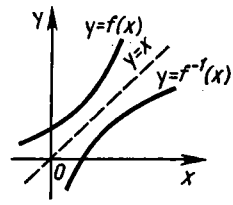


Fig. 180

The following theorem holds true (given without proof).

Theorem 8.1.5. *Let a function $y = f(x)$ be continuous on some interval X, increasing on this interval, and Y be its range. Then there exists an inverse function $y = f^{-1}(x)$ which is defined, continuous and increasing on the interval Y.*

A similar theorem holds for a decreasing continuous function.

For instance, the function $y = x^n$ is increasing and continuous on the ray $[0, +\infty)$, and its range is the ray $[0, +\infty)$. Consequently, there exists an inverse function, which, as we know, is denoted as $y = \sqrt[n]{x}$, continuous and increasing on the ray $[0, +\infty)$. The function $y = \sin x$ is increasing and continuous on the closed interval $[-\pi/2, \pi/2]$, the closed interval $[-1, 1]$ being its range. Hence, there exists an inverse function which, as we know, is denoted as $y = \arcsin x$, continuous and increasing on the interval $[-1, 1]$. The range of its values is the interval $[-\pi/2, \pi/2]$. The other inverse trigonometric functions are defined (this has already been done above) and their continuity is ascertained in a similar way.

At the end of Item 3, we concluded that all rational and trigonometric functions are continuous at every point where they are defined. Now we can give a wider description of the class of known functions: *all rational, trigonometric, irrational (that is, those containing the variable under the radical sign or in a fractional power), and inverse trigonometric functions, as well as functions obtained from the above-mentioned functions by arithmetic operations (in a finite number) and by the operation of forming a composite function are continuous at every point where they are defined.*

6. Properties of a Continuous Function on an Interval. Let a function $y = f(x)$ be defined in an open interval (a, b) and continuous at every point of this interval. Then it is said to be *continuous in the*

interval (a, b). Similarly, the notion of continuity of a function is defined on intervals of the form: $(-\infty, b)$, $(a, +\infty)$, and $(-\infty, +\infty)$.

Using the results obtained in the preceding subsection, we may now say that the functions $y = C$, $y = kx + b$, $y = \sin x$, and $y = \cos x$ are continuous throughout the number line $(-\infty, +\infty)$, while the function $y = 1/x$ is continuous on the intervals $(-\infty, 0)$ and $(0, +\infty)$.

Let now a function $y = f(x)$ be defined on a closed interval $[a, b]$. We are going to introduce the notion of continuity of a function at an end point of a closed interval. Here we speak of the so-called one-sided continuity. As a matter of fact, remaining on the interval $[a, b]$, we can approach the point a only from the right, and the point b only from the left. We shall say that the function $y = f(x)$ is *continuous from the right at the point* a if the following condition is met: for any $\varepsilon > 0$ there is a $\delta > 0$ such that from the inequality $0 \leqslant x - a < \delta$ there follows the inequality $|f(x) - f(a)| < \varepsilon$ (the double inequality $0 \leqslant x - a < \delta$ means that the point x is taken to the right of the point a). We shall also say that the function $y = f(x)$ is continuous from the left at the point b if the following condition is satisfied: for any $\varepsilon > 0$ there is a $\delta > 0$ such that the inequality $0 \leqslant b - x < \delta$ implies the inequality $|f(x) - f(b)| < \varepsilon$.

A function is said to be *continuous on a closed interval* $[a, b]$ if it is continuous at all interior points of this interval, continuous from the right at the point a, and continuous from the left at the point b.

Consider the graph of some function $y = f(x)$ shown in Fig. 181. It consists of one branch. If we take any point $(0, y_0)$ on the axis of ordinates, lying between the points $(0, f(a))$ and $(0, f(b))$, and draw a horizontal line through it, then this straight line will intersect the graph at some point. Let the abscissa of this point be c; then $f(c) = y_0$. Thus, for any number y_0, lying between $f(a)$ and $f(b)$, there is c such that $a \leqslant c \leqslant b$ and $f(c) = y_0$. In other words, if the graph of a function consists of one branch, then this function takes on all the "intermediate" values lying between the two given values of the function.

If a function is continuous, then its graph consists of one branch. Hence, the assertion on an intermediate value holds true for any continuous function. For a discontinuous function this assertion may not be fulfilled. Figure 182 shows the graph of a discontinuous function. This graph consists of two branches. Despite the fact that the number y_0 lies between $f(a)$ and $f(b)$, there is no point c between the points a and b such that $f(c) = y_0$.

The following theorem holds (given without proof).

Theorem 8.1.6 (intermediate value theorem). *If a function* $y = f(x)$ *is continuous on a closed interval* $[a, b]$, *then on this interval it*

takes on any value M between f (a) and f (b), that is, there is a point c, $a < c < b$, such that f (c) = M.

Corollary 1. *If a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and monotonic on it, then the range of its values is the closed interval with end points f (a) and f (b).*

Note that in formulating Theorem 8.1.6 and Corollary 1 we may take an arbitrary interval X instead of the interval $[a, b]$, then a more general assertion holds true: *if a function is continuous and monotonic on an interval X, then the range of its values is an integral interval.*

Corollary 2 (zero-point theorem of a continuous function). *If a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and has values of opposite signs at the end points of this interval, then inside the interval there is a point c, $a < c < b$, such that f (c) = 0* (Fig. 183).

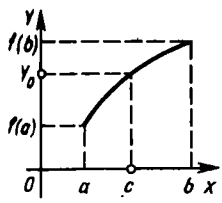


Fig. 181

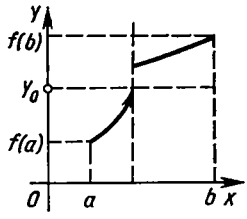


Fig. 182

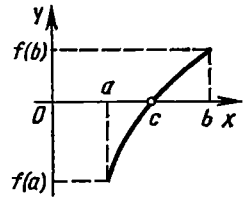


Fig. 183

Note that if a function $y = f(x)$ is monotonic, then the point c mentioned in this corollary is not only existent, but also unique. Indeed, by virtue of monotonicity of the function, from the equalities $f(c_1) = 0$ and $f(c_2) = 0$ there must follow that $c_1 = c_2$. Thus, *the graph of a continuous and monotonic on a closed interval (a, b) function $y = f(x)$, which takes on values of unlike signs at the end points of this interval intersects the axis of abscissas only at one point.*

The above corollary can be used for approximate computation of roots of equations. Let there be required to solve the equation $f(x) = 0$. If there is a closed interval $[a, b]$ on which the function is continuous, monotonic and takes on values of unlike signs at the end points of this interval, then exactly one root of the equation is contained inside the interval. In order to define more exactly the value of the desired root, we divide the interval $[a, b]$ into two equal parts, take that half of the interval at the end points of which the function takes on values of unlike signs, and so on. In such a way we get the value of the root of the equation $f(x) = 0$ with any preassigned accuracy.

We often proceed as follows: we seek for a closed interval of the form $[n, n + 1]$ in which the root of the equation $f(x) = 0$ lies, divide this interval into ten equal parts, and so forth. This process enables us to find one by one the decimal digits of the required root.

Example 1. Prove that the equation $x^3 + 4x + 1 = 0$ has a root on the closed interval $[-1, 0]$ and find the approximate value of this root accurate to 0.01.

○ The function $y = x^3 + 4x + 1$ is continuous on the interval $[-1, 0]$, increasing (as the sum of two increasing functions $y = x^3$ and $y = 4x + 1$), and takes on values of opposite signs at the end points of the interval: $f(-1) = -4 < 0$, $f(0) = 1 > 0$. Hence, the given equation has one root on the interval $[-1, 0]$.

To compute this root, we apply the method of successive division into ten parts. Dividing the interval $[-1, 0]$ into ten equal parts and computing the values of the function at the division points, we have: $f(0) = 1$, $f(-0.1) = 0.599$, $f(-0.2) = 0.192$, $f(-0.3) = 0.227$. Since $f(-0.3) < 0$ and $f(-0.2) > 0$, the required root lies on the closed interval $[-0.3, -0.2]$. Dividing this interval into ten equal parts and computing the values of the function at the division points, we have: $f(-0.21) \approx 0.151$, . . ., $f(-0.24) \approx 0.026$, $f(-0.25) \approx -0.016$. Hence, the required root lies in the interval $[-0.25, -0.24]$, that is, to within 0.01 it is equal to -0.25. ●

Corollary 2 also plays an important part in solving inequalities. Let there be required to solve an inequality $f(x) > 0$, where $y = f(x)$ is a function continuous throughout the number line. We first solve the equation $f(x) = 0$. Let $x_1, x_2, \ldots, x_n, \ldots$ be roots of the equation (the set of these roots may be both finite and infinite). These points divide the number line into intervals such that at the end points of each of them the function $y = f(x)$ vanishes, while inside it it is different from zero. Then within each of the intervals the sign of the function remains unchanged. Indeed, suppose that the points a and b lie inside one of the intervals, and the signs of $f(a)$ and $f(b)$ are opposite. Then the continuous function $y = f(x)$ will vanish between the points a and b, which is impossible.

Thus, we have proved that the roots of the equation $f(x) = 0$ divide the number line into intervals within which the function $y = f(x)$ retains the same sign. Now, it suffices to take a "sampling" point on each of the obtained intervals and find out what sign the function has at this point. The same sign will be preserved by the function on the entire interval. This will enable us to choose the intervals on which the inequality $f(x) > 0$ is satisfied.

If a function $y = f(x)$ has discontinuity points, then when solving the inequality $f(x) > 0$, we have to take not only the roots of the equation $f(x) = 0$, but also the discontinuity points of the function and find out what sign the function has on each of the obtained intervals.

This method of solving inequalities is called the **method of intervals.** We have already used this method for solving rational inequalities.

Prior to considering other properties of a continuous function on an interval, we introduce some new notions.

Definition 1. A function $y = f(x)$, defined on an interval X, is said to be *bounded above* (*below*) on this interval if there is a number P such that for all x from X the inequality $f(x) \leqslant P$ ($f(x) \geqslant P$) holds. The function $y = f(x)$ is called *bounded* on the interval X if it is bounded both above and below.

From the geometrical point of view boundedness of a function means that its graph lies entirely between two horizontal lines.

Definition 2. Let a function $y = f(x)$ be defined on an interval X and let there exist in this interval a point x_0 such that for all x

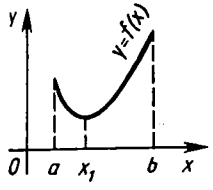


Fig. 184

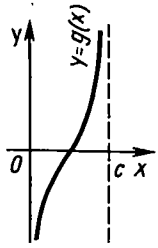


Fig. 185

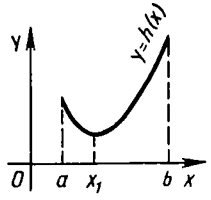


Fig. 186

from X other than x_0 the inequality $f(x) < f(x_0)$ ($f(x) > f(x_0)$) is satisfied. Then the function is said to *attain its greatest* (*least*) *value* on this interval (written: $f(x_0) = y_{\text{great}}$ ($f(x_0) = y_{\text{least}}$)).

As examples, consider the functions whose graphs are shown in Figs. 184-187. The function $y = f(x)$ is continuous on the closed interval $[a, b]$ and attains in this interval its greatest and least values: $y_{\text{great}} = f(b)$, $y_{\text{least}} = f(x_1)$. It is bounded on this interval since for all x's from $[a, b]$ the inequality $f(x_1) \leqslant f(x) \leqslant f(b)$ is satisfied (Fig. 184).

The function $y = g(x)$ is continuous on the open interval $(0, c)$, unbounded both above and below, and attains neither the greatest nor the least value (Fig. 185).

The function $y = h(x)$ is continuous on the open interval (a, b), bounded on it, attains the least value $y_{\text{least}} = h(x_1)$, but does not attain the greatest value (Fig. 186).

The function $y = s(x)$ is continuous on the half-interval $(0, c]$, bounded on it, attains the greatest value $y_{\text{great}} = f(c)$, but does not attain the least value (Fig. 187).

Note that on an open or half-open interval a continuous function can be both bounded and unbounded, either attaining or not attaining its greatest and least values. However, on a closed interval a continuous function is always bounded and always attains its great-

est and least values—this is proved in the course of mathematical analysis (here, we have obtained only a geometrical illustration of this fact).

Thus, the following theorem holds.

Theorem 8.1.7. *A continuous function is bounded on a closed interval and attains its greatest and least values on this interval.*

Example 2. Find the greatest and least values of the function $y = 4x - x^2$ on the closed interval $[-1, 4]$.

○ The problem has a solution since the given function is continuous on a closed interval. We construct the parabola $y = 4x - x^2$. To this end, we note that $y = x(4 - x)$ and, consequently, $y = 0$

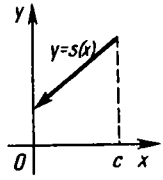


Fig. 187

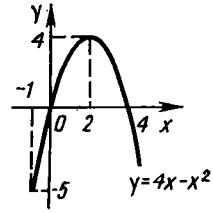


Fig. 188

for $x_1 = 0$ and $x_2 = 4$, that is, the parabola intersects the axis of abscissas at two points: (0, 0) and (4, 0). The vertex of the parabola has the abscissa $x = 2$ and ordinate $f(2) = 4 \cdot 2 - 2^2 = 4$. We draw the parabola through the three points (0,0), (2,4), and (4,0) and take its portion belonging to the closed interval $[-1, 4]$ (Fig. 188). From the graph, we find $y_{\text{great}} = f(2) = 4$, $y_{\text{least}} = f(-1) = -5$. ●

Remark. Below (see page 345), we shall consider the technique of finding the greatest and least values of a function without graphing this function.

Exercises

1. It is required to make a square metal plate whose side $x_0 = 10$ cm. Within what limits is it possible to vary the side x of the plate if its area $y = x^2$ may differ from the theoretical area $y_0 = 100$ cm^2 by no more than: (a) ± 1 cm^2, (b) ± 0.1 cm^2, (c) ± 0.01 cm^2, (d) $\pm \varepsilon$ cm^2?

2. To what accuracy should blocks 2 cm on a side be made if the error of the volume must not exceed 0.1 cm^3?

3. Three cylinders stand one upon the other on a horizontal plane P. The radii of their bases are 3 m, 2 m, and 1 m, their altitudes being equal to 5 m each. Express the cross-sectional area of the solid thus obtained as a function of the distance from the cross-section to the plane P. Is this function continuous? Construct its graph.

4. Prove that the function $y = 3/x$ is continuous for all the values of x other than zero.

5. Test the function

$$y=\begin{cases} 0 & \text{for } x<0, \\ x & \text{for } 0\leqslant x<1, \\ -x^2+4x-2 & \text{for } 1\leqslant x<3, \\ 4-x & \text{for } x\geqslant 3 \end{cases}$$

for continuity. Construct its graph.

6. For what value of a is the function

$$f(x)=\begin{cases} x+1 & \text{for } x\leqslant 1, \\ 3-ax^2 & \text{for } x>1 \end{cases}$$

continuous everywhere? Sketch its graph.

7. Test the following functions for continuity and indicate the discontinuity points:

(a) $f(x)=\begin{cases} \dfrac{|x+1|}{x+1}\cdot(x-1) & \text{for } x\neq -1, \\ 1 & \text{for } x=-1, \end{cases}$

(b) $f(x)=\dfrac{\sin x}{x}$, (c) $f(x)=\sin x+x^3-3x^2+1$, (d) $f(x)=\cos(x^2-5)$.

8. Prove that the equation $x^5+3x-5=0$ has a root on the interval $[1, 2]$. Find this root to within 0.1.

In Exercises 9 to 11, find out whether the given function attains the greatest and the least value on the indicated interval; find y_{great} and y_{least} (if any).

9. $y=-x^2+5x-7$, $x\in[-7, -2]$.

10. $y=(x+5)(x-3)$, $x\in[-2, 4)$.

11. $y=\dfrac{x-5}{x-1}$, (a) $x\in(-\infty, -2)$, (b) $x\in[-3, 0]$, (c) $x\in[0, 1)$.

§ 8.2. Limit of a Function

1. Limit of a Function at a Point. Consider the functions $y=f(x)$, $y=g(x)$, and $y=h(x)$ graphed in Fig. 189. These functions differ from one another by their behaviour at the point $x=a$:

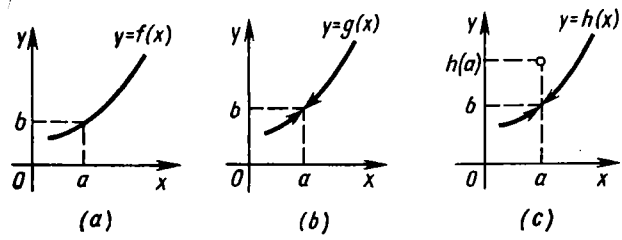


Fig. 189

the function $f(x)$ is continuous at the point a, the other two functions being discontinuous at this point. The function $g(x)$ is not defined at the point a, whereas the function $h(x)$ is defined at this

point. If $x \neq a$, then $f(x) = g(x) = h(x)$. In all the three cases, we note that the closer x to a, the smaller is the difference between the value of the function $f(x)$ or $g(x)$ or $h(x)$ and the number b; this difference being characterized by the expression $|f(x) - b|$, $|g(x) - b|$, $|h(x) - b|$, respectively. We say that for any of the functions under consideration its limit is equal to b as x tends to a and we write $\lim\limits_{x \to a} f(x) = b$, $\lim\limits_{x \to a} g(x) = b$, $\lim\limits_{x \to a} h(x) = b$. The behaviour of the function at the point a itself is not taken into account here.

A more correct definition of the limit can be formulated as follows: the number b is said to be the *limit of the function* $y = f(x)$ as x *tends to* a if, whatever number $\varepsilon > 0$ is chosen, for all the values of $x \neq a$ lying sufficiently close to a the inequality $|f(x) - b| < \varepsilon$ will be satisfied. The equality $\lim\limits_{x \to a} f(x) = b$ means that $f(x) \approx b$ if x is taken sufficiently close to a (that is, x belongs to some deleted neighbourhood of the point a).

Note that for the continuous function $y = f(x)$ graphed in Fig. 189*a* the equality $b = f(a)$ is satisfied, that is, $\lim\limits_{x \to a} f(x) = f(a)$. Thus, *the limit of a continuous function* $f(x)$, *as* $x \to a$, *is equal to the value of the function at this point.*

Example 1. Compute $\lim\limits_{x \to 4} \dfrac{\sqrt{x} + x^2}{2x + 1}$.

○ The function $f(x) = \dfrac{\sqrt{x} + x^2}{2x + 1}$ is defined at the point $x = 4$, and, hence, it is continuous at this point (see the preceding section). We have $f(4) = \dfrac{\sqrt{4} + 4^2}{2 \cdot 4 + 1} = 2$. Hence, $\lim\limits_{x \to 4} \dfrac{\sqrt{x} + x^2}{2x + 1} = f(4) = 2$. ●

Example 2. Compute $\lim\limits_{x \to \pi} \dfrac{\sin x}{\cos^2 x + \tan x}$.

○ The function $f(x) = \dfrac{\sin x}{\cos^2 x + \tan x}$ is defined at the point $x = \pi$, and, hence, it is continuous at this point. Therefore

$$\lim_{x \to \pi} \frac{\sin x}{\cos^2 x + \tan x} = f(\pi) = \frac{\sin \pi}{\cos^2 \pi + \tan \pi} = \frac{0}{1 + 0} = 0. \quad \bullet$$

Example 3. Compute $\lim\limits_{x \to 3} \dfrac{2x + 1}{x - 3}$.

○ The function $f(x) = \dfrac{2x + 1}{x - 3}$ is not defined at the point $x = 3$ since at this point the denominator of the fraction vanishes. Therefore, here it is impossible to reason as in Examples 1 and 2.

If $f(x) = P(x)/Q(x)$, and at the point $x = a$ the denominator

vanishes and the numerator is different from zero, then we write $\lim_{x \to a} f(x) = \infty$. Hence $\lim_{x \to a} \frac{2x+1}{x-3} = \infty$. ●

Example 4. Compute $\lim_{x \to 2} \frac{x^2-4}{x^2-6x+8}$.

○ Here, as distinct from the preceding example, both the numerator and the denominator of the fraction vanish for $x = 2$. In such cases, to compute the limit, we have to carry out certain identical transformations of the analytic expression of the function. We have

$$\frac{x^2-4}{x^2-6x+8} = \frac{(x-2)(x+2)}{(x-2)(x-4)}.$$

Since, as $x \to 2$, the behaviour of the function at the point $x = 2$ itself is of no importance, we may reduce the fraction by $x - 2$, then we get $\frac{x+2}{x-4}$. Thus,

$$\lim_{x \to 2} \frac{x^2-4}{x^2-6x+8} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)(x-4)} = \lim_{x \to 2} \frac{x+2}{x-4} = \frac{2+2}{2-4} = -2. \bullet$$

Example 5. Compute $\lim_{x \to -2} \frac{x+2}{\sqrt{7-x}-3}$.

○ For $x = -2$ both the numerator and denominator vanish. Let us perform the following transformations of the given expression:

$$\frac{x+2}{\sqrt{7-x}-3} = \frac{(x+2)(\sqrt{7-x}+3)}{(\sqrt{7-x}-3)(\sqrt{7-x}+3)} = \frac{(x+2)(\sqrt{7-x}+3)}{(\sqrt{7-x})^2-3^2}$$

$$= \frac{(x+2)(\sqrt{7-x}+3)}{7-x-9} = -\frac{(x+2)(\sqrt{7-x}+3)}{x+2} = -(\sqrt{7-x}+3)$$

(we have multiplied the numerator and denominator of the given fraction by the expression conjugate to the denominator). Thus,

$$\lim_{x \to -2} \frac{x+2}{\sqrt{7-x}-3} = \lim_{x \to -2} (-\sqrt{7-x}-3) = (-\sqrt{7+2}-3) = -6. \bullet$$

Let us give (without proof) two important theorems on limits.

Theorem 8.2.1 (on arithmetical operations on limits). *If* $\lim_{x \to 0} f(x) = b$, *and* $\lim_{x \to 0} g(x) = c$, *then*:

(1) $\lim_{x \to a} (f(x) + g(x)) = b + c$, (2) $\lim_{x \to a} kf(x) = kb$,

(3) $\lim_{x \to a} f(x)\, g(x) = bc$, (4) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{b}{c}$ (for $c \neq 0$).

Theorem 8.2.2 (on the limit of an intermediate function). *If* $\lim_{x \to a} f(x) = b$, $\lim_{x \to a} h(x) = b$ *and in some deleted neighbourhood of* a

point a the inequality $f(x) \leqslant g(x) \leqslant h(x)$ *is satisfied, then* $\lim\limits_{x \to a} g(x) = b$.

Geometrically, this theorem is illustrated in Fig. 190.

2. The First Remarkable Limit. In § 6.1, we proved that for $0 < x < \pi/2$ the following inequality holds:

$$\sin x < x < \tan x. \tag{1}$$

And if $-\pi/2 < x < 0$, then $0 < -x < \pi/2$ and, consequently, $\sin(-x) < -x < \tan(-x)$. Multiplying both sides of this double inequality by -1, we get

$$\tan x < x < \sin x. \tag{2}$$

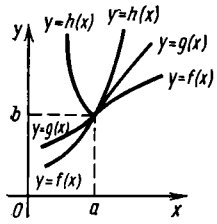


Fig. 190

Dividing all members of double inequality (1) by $\sin x$ and taking into consideration that $\sin x > 0$ for $0 < x < \pi/2$, we get

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}. \tag{3}$$

Similarly, dividing the double inequality (2) by $\sin x$ and taking into account that for $-\pi/2 < x < 0$ for $\sin x < 0$, we get

$$\frac{1}{\cos x} > \frac{x}{\sin x} > 1,$$

that is, the same inequality (3).

Thus, in the deleted neighbourhood of the point $x = 0$ contained in the interval $(-\pi/2, \pi/2)$ we have $\frac{1}{\cos x} > \frac{x}{\sin x} > 1$, whence $\cos x < \frac{\sin x}{x} < 1$.

Let us make x tend to zero. Since $y = \cos x$ is a function continuous everywhere, $\lim\limits_{x \to 0} \cos x = \cos 0 = 1$. On the other hand, the limit of a constant function is equal to the value of this constant, that is, $\lim\limits_{x \to 0} 1 = 1$. Then, by the theorem on the limit of an intermediate function, we get

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{4}$$

Relationship (4) is the **first remarkable limit.**

Note that the following more general equality holds:

$$\lim_{x \to 0} \frac{\sin kx}{kx} = 1, \text{ where } k \neq 0. \tag{5}$$

Using this equality and the theorem on arithmetical operations on limits, we compute limits of many trigonometric functions.

Example 1. Compute $\lim\limits_{x\to 0}\dfrac{\sin 5x}{2x}$.

○ We have

$$\lim_{x\to 0}\frac{\sin 5x}{2x}=\lim_{x\to 0}\frac{\sin 5x\cdot 5}{5x\cdot 2}=\frac{5}{2}\lim_{x\to 0}\frac{\sin 5x}{5x}=\frac{5}{2}\cdot 1=\frac{5}{2}.$$ ●

Example 2. Compute $\lim\limits_{x\to 0}\dfrac{\tan x}{\sin 3x}$.

○ We find

$$\lim_{x\to 0}\frac{\tan x}{\sin 3x}=\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\cdot\frac{1}{\dfrac{3\sin 3x}{3x}}\right).$$

Since $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$, $\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}=1$, $\lim\limits_{x\to 0}\cos x=1$, we have

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\cdot\frac{1}{\dfrac{3\sin 3x}{3x}}\right)=\frac{1}{3}.$$ ●

Example 3. Compute $\lim\limits_{x\to 0}\dfrac{\cos 5x-\cos 7x}{x^2}$.

○ Since $\cos 5x-\cos 7x=2\sin x\sin 6x$, we have

$$\lim_{x\to 0}\frac{\cos 5x-\cos 7x}{x^2}=\lim_{x\to 0}\frac{2\sin x\sin 6x}{x^2}=12\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{\sin 6x}{6x}\right)=12.$$ ●

3. Limit of a Function at Infinity. Consider the function $y=\dfrac{3x-1}{x+2}$. It can be rewritten as follows:

$$y=3-\frac{7}{x+2}.$$

It is clear that with an increase in x the denominator of the fraction $\dfrac{7}{x+2}$ increases and, consequently, the value of this fraction becomes arbitrarily small. Therefore, for sufficiently large values of x the values of the function $y=\dfrac{3x-1}{x+2}$ differ arbitrarily little from the number 3. This function is said to tend to 3 as x tends to plus infinity $\left(\text{written: } \lim\limits_{x\to+\infty}\dfrac{3x-1}{x+2}=3\right)$.

In general, the notation $\lim\limits_{x\to+\infty} f(x)=b$ means the following: *for any* $\varepsilon>0$ *there is* c *such that for* $x\geqslant c$ *the inequality* $|f(x)-b|<\varepsilon$ *is satisfied.*

In other words, the deviation of the values of the function $y=f(x)$ from the number b becomes less than ε if $x\geqslant c$.

If $\lim\limits_{x\to+\infty} f(x) = b$; then the number b is said to be the *limit of the function* $y = f(x)$ as $x \to +\infty$.

The analogy between the notions "the limit of a function $y = f(x)$ as $x \to +\infty$" and "the limit of a sequence (x_n) as $n \to \infty$" is obvious. The only difference consists in that for a sequence the inequality $|x_n - b| < \varepsilon$ is satisfied for **natural** values of n greater than, or equal to, n_0, while for a function the inequality $|f(x) - b| < \varepsilon$ is fulfilled for **any real values** of x greater than, or equal to, the real number c. This analogy is explained by the fact that the sequence is a particular case of function, namely, this is a function given on the set of natural numbers.

The following therem similar to the theorem on arithmetical operations on the limits of sequences (see § 7.3) holds.

Theorem 8.2.3. *If* $\lim\limits_{x\to+\infty} f(x) = a$, $\lim\limits_{x\to+\infty} g(x) = b$, *then*:

(1) $\lim\limits_{x\to+\infty} (f(x) + g(x)) = a + b$, (2) $\lim\limits_{x\to+\infty} k \cdot f(x) = ka$,

(3) $\lim\limits_{x\to+\infty} f(x)\, g(x) = ab$, (4) $\lim\limits_{x\to+\infty} \dfrac{f(x)}{g(x)} = \dfrac{a}{b}$ (for $b \neq 0$).

Along with the limit of a function as $x \to +\infty$, we also consider the limit of a function as $x \to -\infty$: the number b is said to be the *limit of the function* $y = f(x)$ as $x \to -\infty$ if for any $\varepsilon > 0$ there is c such that for $x \leqslant c$ the inequality $|f(x) - b| < \varepsilon$ is satisfied. In this case, we write $\lim\limits_{x\to-\infty} f(x) = b$.

Example. Compute $\lim\limits_{x\to+\infty} \dfrac{2x^2-5x+3}{x^2+4}$.

○ Dividing termwise the numerator and denominator of the fraction by x^2, we get

$$\lim_{x\to+\infty} \frac{2x^2-5x+3}{x^2+4} = \lim_{x\to+\infty} \frac{2-\frac{5}{x}+\frac{3}{x^2}}{1+\frac{4}{x^2}}.$$

It is clear that $\lim\limits_{x\to+\infty} \dfrac{5}{x} = \lim\limits_{x\to+\infty} \dfrac{3}{x} = \lim\limits_{x\to+\infty} \dfrac{4}{x^2} = 0$.

Then, by the rules for computing the limit of a quotient and the limit of a sum, we get

$$\lim_{x\to+\infty} \frac{2-\frac{5}{x}+\frac{3}{x^2}}{1+\frac{4}{x^2}} = \frac{2-0+0}{1+0} = 2. \quad \bullet$$

Sometimes, it is convenient to consider the so-called infinite limits. Consider, for example, the function $y = x^2$. If the argument x is assigned ever increasing positive values, then the function x^2 will also take on ever increasing values. To be more exact, we can

make the values of the function greater than any preassigned number M if the argument is taken greater than a certain number P, namely, $x^2 > M \Leftrightarrow x > \sqrt{M}$.

Hence, if $x > \sqrt{M}$ ($P = \sqrt{M}$), then $x^2 > M$. In such cases, a *function is said to have* $+\infty$ *as its limit* as $x \to +\infty$. In general, we say that $f(x) \to +\infty$ as $x \to +\infty$ and we write $\lim\limits_{x \to +\infty} f(x) = +\infty$ if for any positive number M there is a positive number P such that $x > P \Rightarrow f(x) > M$.

Similarly, the notation $\lim\limits_{x \to +\infty} f(x) = -\infty$ means that for any $M > 0$ there is a positive number P such that $x > P \Rightarrow f(x) < -M$.

Infinite limits as $x \to -\infty$ are defined in a similar manner.

4. Asymptotes. A horizontal or an oblique straight line is called an *asymptote* to the graph of a function $y = f(x)$ as $x \to +\infty$ if, as

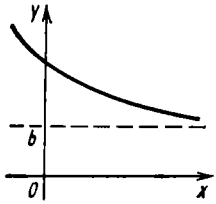


Fig. 191

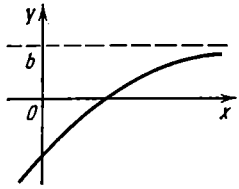


Fig. 192

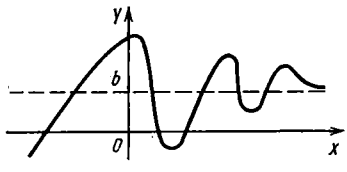


Fig. 193

$x \to +\infty$, the distance from the point on the graph to the point with the same abscissa lying on the straight line tends to zero.

Finding *horizontal asymptotes* involves no difficulties. In principle, this problem is equivalent to finding the limit of a function as $x \to +\infty$. Indeed, let $y = b$ be the equation of a horizontal asymptote, as $x \to +\infty$, to the graph of the function $y = f(x)$. By the definition of the asymptote, this means that, as $x \to +\infty$, the distance from a point on the graph to the corresponding point on the straight line $y = b$ tends to zero. But this distance is nothing but the deviation of $f(x)$ from b, that is, $|f(x) - b|$, and $|f(x) - b|$ tends to zero if and only if $f(x)$ tends to b.

Thus, the *horizontal asymptote has the equation* $y = b$, *where* $b = \lim\limits_{x \to +\infty} f(x)$ (Fig. 191).

The graph of a function may occupy different positions with respect to a horizontal asymptote. Figure 191 depicts the graph of a function approaching its asymptote from above, and Fig. 192 from below. The graph of a function may intersect its asymptote, and the set of points of intersection may be both finite (as in Fig. 193) and infinite.

We now show how to find *oblique asymptotes* to the graphs of rational functions. Let there be given a function $y = \dfrac{2x^2 - 5x + 2}{x - 3}$.

We divide the numerator by the denominator:

$$\begin{array}{r|l} x-3 & \begin{array}{l} 2x+1 \\ \hline 2x^2-5x+2 \\ 2x^2-6x \\ \hline x+2 \\ x-3 \\ \hline 5 \end{array} \end{array}$$

Consequently,

$$\frac{2x^2-5x+2}{x-3} = 2x+1+\frac{5}{x-3}.$$

It is clear that $\lim\limits_{x\to+\infty} \frac{5}{x-3} = 0$. Therefore the graph of the function considered (see Fig. 194) almost merges with the straight line $y = 2x + 1$ for large values of x. This line is an oblique asymptote to the graph.

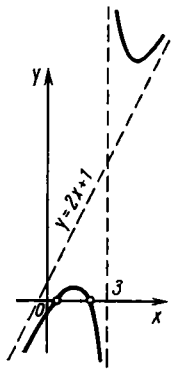


Fig. 194

In general, if the degree of the numerator of a rational function exceeds the degree of the denominator by unity, then the graph of this function has an oblique asymptote. To find this asymptote, we have to single out the integral part of the fraction.

Above we spoke of a horizontal and an oblique asymptotes to the graph of the function $y = f(x)$ as $x \to +\infty$. Similar results are obtained if $x \to -\infty$. For example, $y = 2x + 1$ is an oblique asymptote to the graph of the function $y = \frac{2x^2-5x+2}{x-3}$ as $x \to -\infty$ (Fig. 194).

Consider the function $y = \frac{2x^2-5x+2}{x-3}$ once again. If the variable x is assigned values ever approaching the number 3, then the denominator $x - 3$ will tend to zero, while the numerator of the fraction to the number $2 \cdot 3^2 - 5 \cdot 3 + 2 = 5$. Then the fraction will take on values arbitrarily large in absolute value. In this case, as it was noted above (see page 313), we write $\lim\limits_{x\to 3} \frac{2x^2-5x+2}{x-3} = \infty$. Here, if $x > 3$, then $y > 0$, and if $x < 3$ (but sufficiently close to 3), then $y < 0$.

Hence, the graph of the function considered in the vicinity of the straight line $x = 3$ behaves as shown in Fig. 194. The straight line $x = 3$ is said to be a *vertical asymptote* to the graph of the considered function.

In general, *if* $\lim\limits_{x\to a} f(x) = \infty$, *then a straight line* $x = a$ *is a vertical asymptote to the graph of the function* $y = f(x)$.

Exercises

In Exercises 1 to 5, compute the given limits.

1. (a) $\lim\limits_{x\to -2}(2x^3-3x^2+6)$, (b) $\lim\limits_{x\to 8}\dfrac{\sqrt[3]{x}+\sin(x-8)}{2x-15}$.

2. (a) $\lim\limits_{x\to 1}\dfrac{x^2-1}{4x-4}$, (b) $\lim\limits_{x\to -2}\dfrac{x^2-x-6}{3x^2-12}$, (c) $\lim\limits_{x\to 9}\dfrac{9-x}{3-\sqrt{x}}$.

3. (a) $\lim\limits_{x\to 3}\dfrac{\sqrt{x+1}-2}{x-3}$, (b) $\lim\limits_{x\to 0}\dfrac{x^2}{\sqrt{2+x^2}-\sqrt{2}}$.

4. (a) $\lim\limits_{x\to 0}\dfrac{\sin 7x}{5x}$, (b) $\lim\limits_{x\to 0}\dfrac{\sin 5x-\sin 3x}{x}$.

5. (a) $\lim\limits_{x\to +\infty}\dfrac{3x^2-5x+1}{x^2+5x+10}$,

   (b) $\lim\limits_{x\to +\infty}\dfrac{x^2-1}{x^3+x+2}$, (c) $\lim\limits_{x\to -\infty}\dfrac{3x-1}{5x+7}$.

In Exercises 6 and 7, find the asymptotes to the graphs of the given functions:

6. (a) $y=\dfrac{3}{x}$, (b) $y=\dfrac{x-2}{x-3}$, (c) $y=\dfrac{1}{x^2-4x+4}$.

7. (a) $y=\dfrac{2x^2-3x+1}{x-2}$, (b) $y=\dfrac{x^3+x-5}{x^2-1}$.

§ 8.3. Derivatives

1. Increment in the Argument. Increment of a Function. Let a function $y = f(x)$ be defined at points x and x_1. The difference $x_1 - x$ is called the *increment of the argument* and is denoted by Δx; the difference $f(x_1) - f(x)$ is called the *increment of the function* and is denoted by Δf or Δy. Hence, $\Delta x = x_1 - x$, that is, $x_1 = x + \Delta x$, $\Delta f = f(x_1) - f(x) = f(x + \Delta x) - f(x)$.

Example 1. Find the increment of the function $y = x^3$ when changing from the point x to the point $x + \Delta x$.

○ We have $f(x) = x^3$, $f(x + \Delta x) = (x + \Delta x)^3$. Hence

$$\begin{aligned}\Delta f &= f(x+\Delta x) - f(x) = (x+\Delta x)^3 - x^3\\ &= (x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3) - x^3\\ &= 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.\end{aligned}$$

Thus

$$\Delta f = (3x^2 + 3x\Delta x + (\Delta x)^2)\cdot\Delta x. \quad \bullet$$

Using this formula, we can compute the value of Δf for any given x and Δx. For instance, for $x = 2$, $\Delta x = 0.1$ we have

$$\Delta f = f(2.1) - f(2) = (3\cdot 2^2 + 3\cdot 2\cdot 0.1 + (0.1)^2)\cdot 0.1 = 1.261,$$

for $x = 1$, $\Delta x = -0.2$ we get

$$\Delta f = f(0.8) - f(1) = (3 \cdot 1^2 + 3 \cdot 1 \cdot (-0.2) + (-0.2)^2)(-0.2) = -0.488.$$

Example 2. Prove that for a linear function $y = kx + b$ the equality $\frac{\Delta y}{\Delta x} = k$ is valid.

○ We have $f(x) = kx + b$, $f(x + \Delta x) = k(x + \Delta x) + b$. Hence

$$\Delta y = f(x + \Delta x) - f(x) = (k(x + \Delta x) + b) - (kx + b) = k \cdot \Delta x,$$

whence we get $\frac{\Delta y}{\Delta x} = k$. The geometrical meaning of this equality is illustrated in Fig. 195. ●

Theorem 8.3.1. *A function $y = f(x)$ is continuous at a point $x = a$ if and only if* $\lim\limits_{\Delta x \to 0} \Delta f = 0$, where $\Delta x = x - a$, $\Delta f = f(x) - f(a)$.

□ Indeed, the function $y = f(x)$ is continuous at the point $x = a$ if and only if $\lim\limits_{x \to a} f(x) = f(a)$ (see page 311) or, which is the same, if $\lim\limits_{x-a \to 0} (f(x) - f(a)) = 0$, that is, if $\lim\limits_{\Delta x \to 0} \Delta f = 0$. ■

2. Problem on Drawing a Tangent Line to the Graph of a Function. We first explain what we shall understand by a tangent line to an

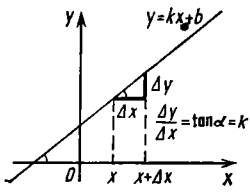


Fig. 195

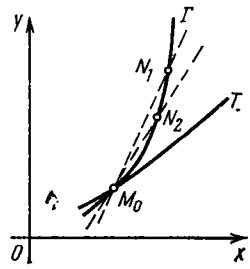


Fig. 196

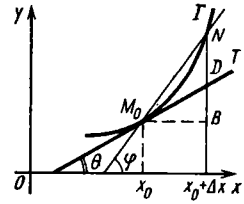


Fig. 197

arbitrary plane curve. A tangent line cannot be defined as a straight line having only one common point with the curve. Indeed, the axis of the parabola $y = x^2$ has only one point in common with the curve but it does not touch the parabola (see Fig. 32). At the same time the straight line $y = 1$ has infinitely many common points with the sinusoid $y = \sin x$ (see Fig. 116) but touches the sinusoid at each of these points. The requirement that the curve be located on one side of the straight line—the axis of abscissas touches the curve $y = x$ at the point (0, 0) although at this point the curve intersects the axis

of abscissas (see Fig. 62)—cannot be taken as the definition of a tangent line.

In order to define a tangent line, we have to use the notion of the limit. Let Γ be an arc of some curve, and M_0 a point of this curve. We draw a secant M_0N through the point M_0. If the point N, moving in the curve, approaches the point M_0, then the secant M_0N turns about the point M_0. It may so happen that as the point N approaches M_0, the secant tends to a certain limit position M_0T. Then M_0T is referred to as the secant to the curve Γ at the point M_0 (Fig. 196).

Thus, the *tangent line to the curve* Γ *at the point* M_0 *is defined as the limit position of the secant* M_0N *as* $N \to M_0$. The limit position is independent of the direction from where the point N approaches M_0.

It may so happen that the secant may have no limit position. Then we say that it is impossible to draw a tangent line to the curve Γ at the point M_0. In most cases it is due to the fact that M_0 is a cusp point, a break, a self-intersection point, etc. for the curve Γ. Practically for most curves a tangent line can be drawn almost at all points.

Let us find the slope of the tangent line for the case when the curve Γ is the graph of a certain function $y = f(x)$. Let M_0 be a point of the graph with abscissa x_0 and ordinate $y_0 = f(x_0)$. Suppose that the tangent line to the curve Γ at the point M_0 does exist. We take one more point $N(x_0 + \Delta x, y_0 + \Delta y)$ on this curve (Fig. 197) and draw a straight line through the points M_0 and N; let φ be the slope of this secant to the positive direction of the x-axis. Then $|BN| = \Delta y$, $|M_0B| = \Delta x$ and

$$\tan\varphi = \frac{|BN|}{|M_0B|} = \frac{\Delta y}{\Delta x}, \quad k_{\tan} = \lim_{N \to M_0} \tan\varphi = \lim_{\Delta x \to 0} \tan\varphi.$$

The slope of the tangent line to the axis of abscissas will be denoted by θ. Then

$$k_{\tan} = \tan\theta = \lim_{\Delta x \to 0} \tan\varphi = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Thus, *in order to draw a nonvertical tangent line to the graph of the function* $y = f(x)$ *at a point with abscissa* x_0, *it is necessary and sufficient that at this point the limit* $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ *be existent, this limit being equal to the slope of the tangent line.*

3. Problem on Instantaneous Velocity of Rectilinear Motion. Let a point move according to the law $s = s(t)$ in a straight line on which a reference point, the unit of measurement, and the direction are chosen ($s(t)$ being the position of the point on the straight line at instant t). The *average velocity* of motion during a time interval is understood in physics as the ratio of the net displacement to the elapsed

time, that is, the average velocity during the time interval from t_1 to t_2 is expressed by the equality

$$v_{av} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

Setting $t_1 = t$, $t_2 + \Delta t$, we get

$$v_{av} = \frac{s(t + \Delta t) - s(t)}{\Delta t} = \frac{\Delta s}{\Delta t}.$$

Suppose that a particle is moving in such a way that its average velocity, measured for a number of different time intervals, does not turn out to be constant. This particle is said to move with varying velocity. Then we must seek to determine the velocity of the particle at any given instant of time, called the instantaneous velocity.

The *instantaneous velocity* at instant t (to be more exact, the numerical value of the velocity at instant t) is defined as the limit of the average velocity of motion during the time interval $[t, t + \Delta t]$ provided that $\Delta t \to 0$. Thus,

$$v_{inst} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

4. Derivative. Two different problems considered in the previous subsections led, in the process of solution, to the same result, that is, to the limit of the ratio of the increment of the function caused by the increment in the argument provided that the increment in the argument tends to zero. It turns out that many problems lead to the necessity of computing the same limit, therefore it is worth while studying it.

Definition. If for a function $y = f(x)$, at a fixed point x, there exists the limit of the ratio of the increment of the function Δf to the increment in the argument Δx provided that $\Delta x \to 0$, then the function $y = f(x)$ is said to be *differentiable at the point* x, and this limit is called the *derivative of the function* $y = f(x)$ at the point x and is denoted by $f'(x)$ or y'. Thus,

$$y' = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Note that $f'(x)$ is a new function defined at all such points x at which the indicated limit exists; this function is called the *derivative of the function* $y = f(x)$.

Thus, for the linear function $y = kx + b$ we have (see Example 2 from Item 1):

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} k = k.$$

Thus,

$$(kx + b)' = k.$$

In particular, for the constant function $y = C$ (that is, for $k = 0$) we get

$$(C)' = 0,$$

and for the function $y = x$ (that is, for $k = 1$) we find

$$(x') = 1.$$

Using the definition, we can recommend the following plan of finding the derivative for the function $y = f(x)$.

Algorithms for computing the derivative of a function $y = f(x)$:

1^0. On fixing the value of x, find $f(x)$.

2^0. Assigning an increment Δx to the argument x (so as not to fall outside the limits of the domain of definition of the function $f(x)$ find $f(x + \Delta x)$.

3^0. Compute the increment of the function $\Delta y = f(x + \Delta x) - f(x)$.

4^0. Form the ratio $\frac{\Delta y}{\Delta x}$.

5^0. Find the limit of the ratio $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$.

Example 1. Find the derivative of the function $y = x^2$.

○ We have:

1^0. $f(x) = x^2$.

2^0. $f(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$.

3^0. $\Delta y = f(x + \Delta x) - f(x) = (x^2 + 2x\Delta x + (\Delta x)^2) - x^2 = 2x\Delta x + (\Delta x)^2 = \Delta x(2x + \Delta x)$.

4^0. $\frac{\Delta y}{\Delta x} = \frac{\Delta x(2x + \Delta x)}{\Delta x} = 2x + \Delta x$.

5^0. $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} (2x + \Delta x) = 2x$.

Thus, $(x^2)' = 2x$. ●

Example 2. Find the derivative of the function $y = x^3$.

○ We have (see Item 1):

3^0. $\Delta y = \Delta x(3x^2 + 3x\Delta x + (\Delta x)^2)$.

4^0. $\frac{\Delta y}{\Delta x} = \frac{\Delta x(3x^2 + 3x\Delta x + (\Delta x)^2)}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2$.

5^0. $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2 + 3x \cdot 0 + 0^2 = 3x^2$.

Thus, $(x^3)' = 3x^2$. ●

Example 3. Find the derivative of the function $y = 1/x$ at a point $x \neq 0$.

○ Without loss of generality, we shall assume that $x > 0$ and $x + \Delta x > 0$. We have:

1°. $f(x) = \frac{1}{x}$.

2°. $f(x+\Delta x) = \frac{1}{x+\Delta x}$.

3°. $\Delta y = f(x+\Delta x) - f(x) = \frac{1}{x+\Delta x} - \frac{1}{x} = -\frac{\Delta x}{x(x+\Delta x)}$.

4°. $\frac{\Delta y}{\Delta x} = -\frac{\Delta x}{x(x+\Delta x)\Delta x} = -\frac{1}{x^2+x\Delta x}$.

5°. $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\frac{1}{\lim\limits_{\Delta x \to 0}(x^2+x\Delta x)} = -\frac{1}{x^2}$.

Thus, $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$. ●

Using the obtained results, we compile the following table of derivatives:

y	C	x	$kx+b$	x^2	x^3	$\frac{1}{x}$
y'	0	1	k	$2x$	$3x^2$	$-\frac{1}{x^2}$

The process of finding the derivative of a function is called *differentiation.*

We established above that if a tangent line can be drawn to the graph of a function $y = f(x)$ at a point $M_0(x_0, f(x_0))$, then the slope of the tangent line is equal to the limit $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, and this limit is the value of the derivative of the function $y = f(x)$ at the point x_0. Thus, *the slope of the tangent line to the graph of a function* $y = f(x)$ *is equal to the value of the derivative at the point of tangency:* $k_{\tan} = f'(x_0)$. This is the **geometrical meaning of the derivative.**

From the problem considered in the preceding subsection it follows that in rectilinear motion the *numerical value of the velocity at instant* t *is equal to the value of the derivative of the function which represents the law of motion:* $v = s'(t)$. This is the **mechanical meaning of the derivative.**

The notion of the derivative enables us to determine not only the instantaneous velocity of rectilinear motion but also the instantaneous rate of other physical processes. Consider, for example, rotation about an axis. It is given by the function $y = \varphi(t)$ which shows the angle through which a solid is rotated during the time interval $[0, t]$. If this rotation is uniform, then its angular velocity is defined as the ratio of the angle of turn to the time interval: $\omega = \frac{\varphi(t)}{t}$. If

rotation is nonuniform, then the average angular velocity of this rotation during the time interval $[t, t + \Delta t]$ is expressed by the ratio $\frac{\varphi(t+\Delta t)-\varphi(t)}{\Delta t}$, that is, $\omega_{av}=\frac{\Delta\varphi}{\Delta t}$. The instantaneous angular velocity at instant t is defined as the limit of the average angular velocity for the time interval $[t, t + \Delta t]$ provided that $\Delta t \to 0$:

$$\omega_{inst}=\lim_{\Delta t\to 0}\omega_{av}=\lim_{\Delta t\to 0}\frac{\Delta\varphi}{\Delta t}.$$

Thus, $\omega = \varphi'(t)$.

In similar fashion, we can say that the instantaneous rate of radioactive decay is expressed by the formula $v = m'(t)$, where $m(t)$ is the mass of the substance at instant t.

Generalizing the aforesaid, we introduce the notion of an average rate of change of a function on an interval $[x, x + \Delta x]$ which is understood as the ratio $\frac{f(x+\Delta x)-f(x)}{\Delta x}$, that is, $\frac{\Delta y}{\Delta x}$. Then $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=y'$ is the *rate of change of y with respect to x at a given point* ("instantaneous rate" of change of a function).

5. Approximate Computation of Values of Functions. Given the value of a function $f(x)$ at a point x_0, suppose we have to find its values in a sufficiently small neighbourhood of this point. If the function is differentiable at the point x_0, then $f'(x_0)=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$. Hence,

$$\frac{\Delta y}{\Delta x}\approx f'(x_0), \text{ that is, } \Delta y\approx f'(x_0)\,\Delta x.$$

Since $\Delta y = f(x_0 + \Delta x) - f(x_0)$, we have $f(x_0 + \Delta x) = f(x_0) + \Delta y$, and we get the approximate formula

$$f(x_0 + \Delta x)\approx f(x_0) + f'(x_0)\cdot\Delta x. \tag{1}$$

Example. Compute 2.008^3.

○ Here, it is convenient to hold that $f(x) = x^3$, $x_0 = 2$, $\Delta x = 0.008$. Using equality (1) applicable to the function x^3 and taking into consideration that $(x^3)' = 3x^2$, we get $(x_0 + \Delta x^3)\approx x_0^3 + 3x_0^2\Delta x$. Hence,

$$2.008^3\approx 2^3 + 3\cdot 2^2\cdot 0.008 = 8 + 12\cdot 0.008 = 8.096.$$

Estimating the accuracy of this value, we have

$$(x_0 + \Delta x)^2 = x^3 + 3x^2\Delta x + 3x_0(\Delta x)^2 + (\Delta x)^3.$$

Thus, the exact value differs from the found value by $3x_0(\Delta x)^2 + (\Delta x)^3$ which in this case amounts to $3\cdot 2\cdot 0.008^2 + 0.008^3$. Since $3\cdot 2\cdot 0.008^2+0.008^3 < 6\cdot 0.01^2 + 0.01^3 < 0.001$, we have computed 2.008^3 to within 0.001. ●

6. Equation of a Tangent Line. Let a function $y = f(x)$ be differentiable at a point x_0, and $M_0(x_0, y_0)$ be a point on the graph of

this function; here $y_0 = f(x_0)$. We set up an equation for the tangent line to the graph of the function at the given point.

We shall seek this equation in the form $y = kx + b$. Since the required straight line passes through the point $M_0(x_0, y_0)$, the equality $y_0 = kx_0 + b$ is satisfied, whence $b = y_0 - kx_0$. Therefore, the equation of the tangent line has the form $y = kx + (y_0 - kx_0)$ or $y = y_0 + k(x - x_0)$. Since, further, the slope k of the tangent line is equal to $f'(x_0)$, and $y_0 = f(x_0)$, the *equation of the tangent line to the graph of the function* $y = f(x)$ *at the point* $M_0(x_0, y_0)$ *has the form*

$$y = f(x_0) + f'(x_0)(x - x_0). \tag{2}$$

Comparing the right-hand sides of formulas (1) and (2), we arrive at the conclusion that from the geometrical point of view, differentiability means the possibility of approximately replacing the graph by a straight line (the tangent line) in a sufficiently small neighbourhood of the point being considered. If the graph of a function has a break, a cusp or, the more so, a discontinuity at some point, then the function is not differentiable at that point. For instance, the continuous function $y = |x|$ is not differentiable at the point $x = 0$ (see Fig. 35).

Thus, there are continuous but not differentiable functions. On the other hand, the following theorem is almost obvious.

Theorem 8.3.2. *If a function* $y = f(x)$ *is differentiable at a point* x, *then it is continuous at this point.*

□ Indeed, from the equality $\Delta y = \frac{\Delta y}{\Delta x} \cdot \Delta x$ we get

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta x = y' \cdot 0 = 0,$$

and this means that the function is continuous at the point x (see p. 320). ■

Example 1. Write the equation of the tangent line to the parabola $y = x^2$ at the point having the abscissa 1.

○ We have $f'(x) = (x^2)' = 2x$; hence, $f'(1) = 2$. In addition, $f(1) = 1^2 = 1$. Substituting the values $x_0 = 1$, $f(x_0) = 1$, $f'(x_0) = 2$ into equation (3), we get $y = 1 + 2(x - 1)$, that is, $y = 2x - 1$. This is the desired equation of the tangent line (Fig. 198). ●

Example 2. Find the equation of the tangent line to the cubic parabola $y = x^3$ at the origin.

○ We have $f'(x) = (x^3)' = 3x^2$; hence, $f'(0) = 0$. In addition, $f(0) = 0$. Substituting the values $x_0 = 0$, $f(x_0) = 0$, and $f'(x_0) = 0$ into the equation (2), we get $y = 0$. This means that the sought-for tangent line is the axis of abscissas, that is, the graph of the function $y = x^3$ touches the axis of abscissas at the origin (see Fig. 62). As we see, the curve passes from the upper half-plane to the lower half-plane. ●

7. Some More Examples of Applying the Derivative. Reasoning as when determining the instantaneous velocity of a moving point may also be applied to solving other problems. To illustrate this, we shall consider three different problems: the problem on the current strength, the problem on line density of a rod, and the problem on the derivative of the area of a curvilinear trapezoid.

Problem 1. Let $q = q(t)$ be the quantity of electricity passing through any cross-sectional area of the conductor in time t. Find the strength of current at a given instant of time t.

○ If Δt is the time interval, and Δq the quantity of electricity passing through the cross-sectional area of the conductor in time

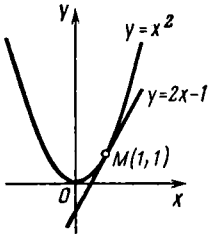


Fig. 198

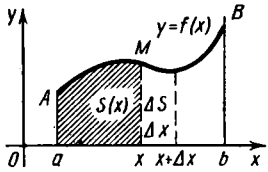


Fig. 199

Δt, then $\frac{\Delta q}{\Delta t}$ is the average current during the time interval Δt, that is $I_{av} = \frac{\Delta q}{\Delta t}$. Then the strength of current at instant t is defined as $\lim\limits_{\Delta t \to 0} I_{av}$. Thus, $I = q'(t)$. ●

Problem 2. Let there be given a nonhomogeneous rod of length 1, and let $m = m(x)$ be the mass of a portion of the rod of length x (one of the ends of the rod is taken as a reference point). Find the line density of the rod at a given point x.

○ If Δm is the mass of the portion of the rod between the points located at distances of x and $x + \Delta x$ from the fixed end, then $\frac{\Delta m}{\Delta x}$ is the average line density of the rod on the section under consideration, and $\lim\limits_{\Delta x \to 0} \frac{\Delta m}{\Delta x}$ the required line density ρ. Thus, $\rho = m'(x)$. ●

Problem 3. Let a function $y = f(x)$ be continuous on a closed interval $[a, b]$ and take on only nonnegative values on this interval; $S(x)$ the area of the curvilinear trapezoid $aAMx$ (Fig. 199). Find the derivative of the function $S = S(x)$.

Prior to solving this problem, note that here we use a graphic example of the area of a plane figure; the determination of area is considered in more detail in the course of geometry.

○ We assign an increment Δx to the abscissa x (for definiteness, we set $\Delta x > 0$), then ΔS will be the increment in the area. Let m denote the least value of the function $y = f(x)$ on the closed interval $[x, x + \Delta x]$ and M the greatest value of the same function on the same interval. It is clear that $m\Delta x \leqslant \Delta S \leqslant M\Delta x$, and hence, $m \leqslant \frac{\Delta S}{\Delta x} \leqslant M$. If $\Delta x \to 0$, then, by virtue of continuity of the function $y = f(x)$, we get $\lim\limits_{\Delta x \to 0} m = \lim\limits_{\Delta x \to 0} M = f(x)$. Then, by the theorem on the limit of an intermediate function (see page 313) $\lim\limits_{\Delta x \to 0} \frac{\Delta S}{\Delta x}$ also exists, and is equal to $f(x)$. Thus, $S'(x) = f(x)$. This equality means that the derivative of the area $S(x)$ of a curvilinear trapezoid is equal to the current ordinate (in Fig. 199 the current ordinate is the length of the line segment xM). ●

Exercises

1. Write the increment of the function at the point x corresponding to the increment in the argument Δx: (a) $y = 5 - 3x$, (b) $y = 2\sqrt{x}$, (c) $y = \sin x$; (d) $y = 2x - x^2$.

2. Find $f(x+\Delta x)$, $f(x+\Delta x) - f(x)$, $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ if: (a) $f(x) = 3x - 5$, (b) $f(x) = x^2 - 3$, (c) $f(x) = \cos x$, (d) $f(x) = ax^2 + bx + c$.

3. The argument x of the function $y = \sqrt{x+1}$ was assigned the increment $\Delta x = 0.17$ and took on the value $x = -0.19$. Find the increment in the function.

4. Using the definition of the derivative, find the values of the derivatives of the following functions: (a) $y = 4 - 2x$, $y'(-1) = ?$ $y'(\sqrt{2}) = ?$ (b) $y = x^2 - 7$, $y'(0) = ?$, $y'(-2/3) = ?$

5. Find the derivative of the function $y = x^2 + 7x$.

6. Compute: (a) 1.015^2, (b) 3.995^2, (c) 1.004^3 using formula (1).

7. Write the equation of the tangent line to the graph of the function $y = f(x)$ at the given point x_0: (a) $f(x) = x^2$, $x_0 = 0$, (b) $f(x) = x^2$, $x_0 = -1$, (c) $f(x) = x^3$, $x_0 = 1$, (d) $f(x) = x^3$, $x_0 = -2$, (e) $f(x) = 1/x$, $x_0 = 1$, (f) $f(x) = 1/x$, $x_0 = -1/2$.

8. A body of 2-kg mass performs rectilinear motion according to the law $s(t) = 13t + 2t^2$ (s in metres, t in seconds). Find: (a) the velocity v of motion of a particle at instant t, (b) the acceleration a (that is, the rate of change of velocity: $a = v'$) of motion of the particle at instant t, (c) the force $F = ma$ acting on the particle at instant $t = 4$.

§ 8.4. Techniques of Differentiation

1. Differentiating a Sum, a Product, and a Quotient. In what follows we shall show how, knowing the derivatives of two functions $u = u(x)$ and $v = v(x)$, to find the derivatives of the sum, product, and quotient of these functions. Henceforward, all the given functions are assumed to be differentiable.

Theorem 8.4.1. *The derivative of the sum of two functions is equal to the sum of their derivatives*:

$$(u + v)' = u' + v'. \tag{1}$$

□ Consider the function $y = u(x) + v(x)$. We take some value of the argument x (from the domain of definition of the function) and assign an increment Δx to it (so that to remain within the limits of the domain of its definition). Then $u(x)$ and $v(x)$ will receive the increments Δu and Δv and take on the values $u + \Delta u$ and $v + \Delta v$: $u + \Delta u = u(x + \Delta x)$, $v + \Delta v = v(x + \Delta x)$. Their sum $y = u + v$ will also receive an increment Δy and take the value $y + \Delta y$:

$$y + \Delta y = (u + \Delta u) + (v + \Delta v) = (u + v) + (\Delta u + \Delta v).$$

Since $y = u + v$, we have $\Delta y = \Delta u + \Delta v$. We divide both sides of this equality by Δx:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

and pass to the limit as $\Delta x \to 0$. Since the limit of a sum is equal to the sum of the limits, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \right) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}.$$

But

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y', \quad \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = u', \quad \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = v'.$$

Hence, $y' = u' + v'$, that is $(u + v)' = u' + v'$. ■

Theorem 8.4.1 may be extended to any number of terms, that is, the following equality occurs:

$$(u_1 + u_2 + \ldots + u_n)' = u_1' + u_2' + \ldots + u_n'. \tag{2}$$

Example 1. Find the derivative of the function $y = x^3 + x^2$.

○ We know that $(x^3)' = 3x^2$, $(x^2)' = 2x$. Using the theorem on the derivative of a sum, we get

$$(x^3 + x^2)' = (x^3)' + (x^2)' = 3x^2 + 2x. \bullet$$

Theorem 8.4.2. *A constant factor may be taken outside the derivative sign*:

$$(Cu)' = Cu'. \tag{3}$$

□ Let there be given a function $y = Cu(x)$. We take some value of the argument x and assign an increment Δx to it. Then the function $u(x)$ will gain an increment Δu and take on the value $u + \Delta u = u(x + \Delta x)$. The function $y = Cu(x)$ will also receive an increment Δy and take on the value $y + \Delta y = Cu(x + \Delta x) =$

$C(u + \Delta u) = Cu + C\Delta u$. Since $y = Cu$, we have $\Delta y = C\Delta u$. We divide both sides of this equality by Δx and pass to the limit as $\Delta x \to 0$. Since a constant factor can be taken outside the limit sign, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{C\Delta u}{\Delta x} = C \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}.$$

But $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y'$, and $\lim_{\Delta x \to 0} = u'$, that is $y' = Cu'$. ■

Example 2. Find the derivative of the function $y = 5x^3$.

○ Since $(x^3)' = 3x^2$, and a constant factor may be taken outside the derivative sign, we have

$$(5x^3)' = 5(x^3)' = 5 \cdot 3x^2 = 15x^2. \bullet$$

It follows from Theorems 8.4.1 and 8.4.2 that *if* C_1 *and* C_2 *are certain numbers, then*

$$(C_1u_1 + C_2u_2)' = C_1u_1' + C_2u_2', \tag{4}$$

□ Indeed, by Theorem 8.4.1, we have

$$(C_1u_1 + C_2u_2)' = (C_1u_1)' + (C_2u_2)'.$$

But, by Theorem 8.4.2, we have $(C_1u_1)' = C_1u_1'$, $(C_2u_2)' = C_2u_2'$, and therefore formula (4) holds true. ■

A similar formula is valid for any number of terms:

$$(C_1u_1 + C_2u_2 + \ldots + C_nu_n)' = C_1u_1' + C_2u_2' + \ldots + C_nu_n'. \tag{5}$$

The expression $C_1u_1 + C_2u_2 + \ldots + C_nu_n$ is called a *linear combination* of functions $u_1, u_2, \ldots, u_n$. Therefore formula (5) gives the rule for differentiating a linear combination of functions.

Example 3. Find the derivative of the function $y = 2x^3 - \frac{1}{3}x^2 + \frac{1}{x}$.

○ By the rule for differentiating a linear combination, we have

$$y' = \left(2x^3 - \frac{1}{3}x^2 + \frac{1}{x}\right)' = 2(x^3)' - \frac{1}{3}(x^2)' + \left(\frac{1}{x}\right)'.$$

Since $(x^3)' = 3x^2$, $(x^2)' = 2x$, $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$, we have

$$\left(2x^3 - \frac{1}{3}x^2 + \frac{1}{x}\right)' = 6x^2 - \frac{2}{3}x - \frac{1}{x^2}. \bullet$$

Theorem 8.4.3. *The derivative of the product of two differentiable functions is computed by the following rule:*

$$(uv)' = u'v + uv'. \tag{6}$$

□ Consider the function $y = u(x) \cdot v(x)$. We take some value of the argument x (from the domain of definition of the function) and

assign an increment Δx to it (so as to remain within the limits of the domain of definition of the functions). Then $u(x)$ and $v(x)$ will gain, respectively, increments Δu and Δv and take the values $u + \Delta u = u(x + \Delta x)$, $v + \Delta v = v(x + \Delta x)$. Their product $y = uv$ will receive the increment

$$\Delta y = u(x + \Delta x)\, v(x + \Delta x) - u(x)\, v(x) = (u + \Delta u)(v + \Delta v) - uv,$$

that is,

$$\Delta y = \Delta u \cdot v + u \cdot \Delta v + \Delta u \cdot \Delta v. \tag{7}$$

We divide both sides of equality (7) by Δx:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} \cdot v + u \cdot \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta x} \cdot \Delta v$$

and pass to the limit as $\Delta x \to 0$:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} \cdot v \right) + \lim_{\Delta x \to 0} \left(u \cdot \frac{\Delta v}{\Delta x} \right) + \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} \cdot \Delta v \right)$$

$$= v \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + u \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta v.$$

Since $\lim\limits_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = u'$, $\lim\limits_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = v'$, and $\lim\limits_{\Delta x \to 0} \Delta v = 0$ (this follows from the continuity of the differentiable function $v(x)$), we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = u'v + uv' + u' \cdot 0, \text{ that is, } y' = u'v + uv'. \blacksquare$$

Note that the above rule for differentiating the function $y = Cu(x)$ is a particular case of formula (6). Indeed, since C is a constant, $C' = 0$, and therefore

$$(Cu)' = C'u + Cu' = 0 \cdot u + Cu' = Cu'.$$

We have obtained the rule for differentiating the product of two functions. The product of a greater number of functions is differentiated in a similar way. For example, for three factors we have

$$(uvw)' = ((uv)w)' = (uv)'w + uv \cdot w'$$

$$= (u'v + uv')w + uv \cdot w' = u'vw + uv'w + uvw'.$$

Hence, in order to find the derivative of the product of three functions, we have to find the derivatives of all the factors, to multiply each of them by the product of other factors, and to add together the expressions obtained.

Example 4. Find the derivative of the function $y = x^2(5x - 4)$ and compute $f'(1)$.

○ By formula (6), we get

$$y' = (x^2)' \cdot (5x - 4) + x^2 \cdot (5x - 4)'.$$

Since $(x^2)' = 2x$, and $(5x - 4)' = 5$, we have

$$y' = 2x \cdot (5x - 4) + x^2 \cdot 5 = 15x^2 - 8x.$$

The same result will be obtained if first we remove the parentheses and then carry out differentiation. Indeed, we have $y = 5x^3 - 4x^2$, and, consequently,

$$y' = (5x^3 - 4x^2)' = 5(x^3)' - 4(x^2)' = 15x^2 - 8x.$$

Substituting the value 1 for x, we find

$$f'(1) = 15 \cdot 1^2 - 8 \cdot 1 = 7. \bullet$$

Theorem 8.4.4. *The derivative of the function $y = 1/v(x)$ is equal to the derivative of the denominator taken with an opposite sign and divided by the square of the denominator:*

$$\left(\frac{1}{v}\right)' = -\frac{v'}{v^2}. \tag{8}$$

□ We take some value of x, and assign an increment Δx to it. Then the function $y = 1/(v(x)$ will gain the increment

$$\Delta y = \frac{1}{v(x+\Delta x)} - \frac{1}{v(x)},$$

that is,

$$\Delta y = -\frac{v(x+\Delta x) - v(x)}{v(x+\Delta x)\, v(x)} = -\frac{\Delta v}{v(x+\Delta x)\, v(x)}. \tag{9}$$

Dividing both sides of equality (9) by Δx, we get

$$\frac{\Delta y}{\Delta x} = -\frac{\Delta v}{\Delta x} \cdot \frac{1}{v(x+\Delta x)\, v(x)}.$$

We pass in this equality to the limit as $\Delta x \to 0$. Since we agreed to consider only differentiable functions, the function $v(x)$ is differentiable and, hence, continuous at the point x in question. Therefore $\lim\limits_{\Delta x \to 0} v(x + \Delta x) = v(x)$. In addition, $\lim\limits_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = v'(x)$. As a result, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -v' \cdot \frac{1}{v^2}, \text{ that is, } y' = -\frac{v'}{v^2}. \blacksquare$$

Example 5. Find the derivative of the function $y = \frac{1}{x^3+5}$.

○ We have

$$y' = \left(\frac{1}{x^3+5}\right)' = -\frac{(x^3+5)'}{(x^3+5)^2} = -\frac{3x^2}{(x^3+5)^2}. \bullet$$

Theorem 8.4.4. that we have proved above enables us to easily obtain the rule for differentiating the quotient. Note that in both

Theorems 8.4.4 and 8.4.5 (considered below) we assume that quite an ordinary constraint $v(x) \neq 0$ is satisfied.

Theorem 8.4.5. *The derivative of the quotient of two functions is computed according to the following rule*:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}. \tag{10}$$

□ Since $\frac{u}{v} = u \cdot \frac{1}{v}$, using the rule for differentiating a product and formula (8), we get

$$\left(\frac{u}{v}\right)' = \left(u \cdot \frac{1}{v}\right)' = u' \cdot \frac{1}{v} + u \cdot \left(\frac{1}{v}\right)' = \frac{u'}{v} - \frac{uv'}{v^2} = \frac{u'v - uv'}{v^2}. \quad \blacksquare$$

Consider particular cases of differentiating the quotient. If $u = C$, then taking advantage of the fact that a constant factor can be taken outside the derivative sign and of formula (8), we get

$$\left(\frac{C}{v}\right)' = C \cdot \left(\frac{1}{v}\right)' = -\frac{Cv'}{v^2}.$$

And if $v = C$, then it is preferable to use the rule for taking a constant factor outside the derivative sign:

$$\left(\frac{u}{C}\right)' = \left(\frac{1}{C} \cdot u\right)' = \frac{1}{C} u' = \frac{u'}{C}.$$

Example 6. Find the derivative of the function $y = \frac{3x^2 - 2x + 1}{7x + 1}$.

○ We have

$$y' = \left(\frac{3x^2 - 2x + 1}{7x + 1}\right)' = \frac{(3x^2 - 2x + 1)'(7x + 1) - (3x^2 - 2x + 1)(7x + 1)'}{(7x + 1)^2}$$

$$= \frac{(6x - 2)(7x + 1) - (3x^2 - 2x + 1) \cdot 7}{(7x + 1)^2} = \frac{21x^2 + 6x - 9}{(7x + 1)^2}. \quad \bullet$$

Example 7. Write the equation of the tangent line to the graph of the function $y = \frac{x}{1 + x^2}$ at the point with abscissa 2.

○ The equation of the tangent line should be sought in the form

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Given the value $x_0 = 2$, we find $f(x_0) = \frac{2}{1 + 2^2} = \frac{2}{5}$. By the rule for differentiating the quotient, we find the derivative

$$f'(x) = \left(\frac{x}{1 + x^2}\right)' = \frac{x'(1 + x^2) - x(1 + x^2)'}{(1 + x^2)^2} = \frac{1 \cdot (1 + x^2) - x \cdot 2x}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2}.$$

Hence, $f'(x_0) = f(2) = \frac{1-2^2}{(1+2^2)^2} = -\frac{3}{25}$. Substituting the found values into the general equation of the tangent line, we get

$$y = \frac{2}{5} - \frac{3}{25}(x-2), \text{ that is, } y = -\frac{3}{25}x + \frac{16}{25}.$$

This is the required equation.

2. Differentiating a Power Function with a Natural Exponent. We know how to differentiate the functions $y = x$, $y = x^2$, $y = x^3$. These are particular cases of the power function $y = x^n$ for $n = 1$, 2, 3. We derive the formula for differentiating the power function for any natural exponent. The formulas $(x)' = 1$, $(x^2)' = 2x$, and $(x^3)' = 3x^2$ suggest that the *formula for differentiating the power function has the form*

$$(x^n)' = nx^{n-1}, \quad n \in \mathbf{N}. \tag{1}$$

Indeed, this formula turns out to be true.

□ To prove formula (1), we use the method of mathematical induction. As we saw, for $n = 1$ the formula is true. Suppose that it is true for $n = k$, that is,

$$(x^k)' = kx^{k-1}.$$

We prove that then formula (1) is also true for $n = k + 1$, that is,

$$(x^{k+1})' = (k+1)\,x^k.$$

Indeed, we have

$$(x^{k+1})' = (x \cdot x^k)' = (x)' \cdot x^k + x \cdot (x^k)'$$
$$= 1 \cdot x^k + x \cdot kx^{k-1} = x^k + kx^k = (k+1)\,x^k$$

(here, we have used the rule for differentiating the product of two functions).

Thus, by virtue of the principle of mathematical induction, we conclude that formula (1) holds for any natural number n. ■

From this formula we obtain, for instance, that $(x^{10})' = 10x^9$, $(x^{97})' = 97x^{96}$, etc. Using formula (1) and the rule for differentiating a linear combination, we can differentiate any polynomial.

Example. Find the derivative of the polynomial

$$P(x) = 5x^5 - 3x^4 + 2x^3 - 7x^2 - 3x - 8.$$

○ We have

$$P'(x) = 5(x^5)' - 3(x^4)' + 2(x^3)' - 7(x^2)' - 3(x') - (8)'$$
$$= 5 \cdot 5x^4 - 3 \cdot 4x^3 + 2 \cdot 3x^2 - 7 \cdot 2x - 3 \cdot 1 - 0$$
$$= 25x^4 - 12x^3 + 6x^2 - 14x - 3. \quad ●$$

3. Differentiating Power Functions with Any Rational Exponent. In the preceding subsection, we proved formula (1):

$$(x^n)' = nx^{n-1}, \quad n \in \mathbf{N}.$$

Making use of this formula, we find the derivative of the function $y = 1/x^n$, where $x \neq 0$ and n is a natural number. We have

$$y' = \left(\frac{1}{x^n}\right)' = -\frac{(x^n)'}{(x^n)^2} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}.$$

Thus,

$$(x^{-n})' = -nx^{-n-1}. \tag{2}$$

Note that formulas (1) and (2) can be combined:

$$(x^m)' = mx^{m-1}, \quad m \in \mathbf{Z} \tag{3}$$

(this equality is also satisfied for $m = 0$).

In mathematics, it often happens that some formulas derived under certain constraints turn out to be true under more general conditions. The same can be said about formula (3) which was obtained for integral exponents and, in fact, holds for any rational exponent:

$$(x^r)' = rx^{r-1}, \quad r \in \mathbf{Q}. \tag{4}$$

It is self-evident that this formula is used only for the values of x for which the expressions x^r and x^{r-1} are defined.

Formula (4) will be proved later (see § 9.2, Item 8).

Example 1. $\left(\frac{1}{x^{100}}\right)' = (x^{-100})' = -100 \cdot x^{-100-1} = -100x^{-101} = \frac{100}{x^{101}}$.

Example 2. $(x^{3/5})' = \frac{3}{5}\, x^{3/5-1} = \frac{3}{5}\, x^{-2/5}$.

Example 3. $(\sqrt[3]{x})' = (x^{1/3})' = \frac{1}{3}\, x^{1/3-1} = \frac{1}{3}\, x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$.

Example 4. $\left(\frac{1}{\sqrt[7]{x^4}}\right)' = (x^{-4/7})' = -\frac{4}{7}\, x^{-4/7-1} = -\frac{4}{7}\, x^{-11/7} = -\frac{4}{7x\sqrt[7]{x^4}}$.

4. Differentiating Trigonometric Functions. We find the derivative of the function $y = \sin x$. Assigning an increment Δx to x, we have $\Delta y = \sin(x + \Delta x) - \sin x$. Making use of the formula for transforming the difference of sines into a product, we get

$$\Delta y = 2 \sin \frac{\Delta x}{2} \cos\left(x + \frac{\Delta x}{2}\right).$$

We divide both sides of this equality by Δx and pass to the limit as $\Delta x \to 0$:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{2 \sin \frac{\Delta x}{2} \cos\left(x + \frac{\Delta x}{2}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(\Delta x/2)}{\Delta x/2} \cdot \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right).$$

Since $\lim\limits_{h \to 0} \frac{\sin h}{h} = 1$, we have $\lim\limits_{\Delta x \to 0} \frac{\sin(\Delta x/2)}{\Delta x/2} = 1$. Further, since the function $y = \cos x$ is continuous at any point x, we have $\lim\limits_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) = \cos x$. Finally, we get $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 1 \cdot \cos x = \cos x$. Thus,

$$(\sin x)' = \cos x. \tag{5}$$

The formula

$$(\cos x)' = -\sin x \tag{6}$$

is derived in much the same way.

Consider now the function $y = \tan x$. To compute the derivative of this function, we make use of the rule for differentiating the quotient

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

Thus,

$$(\tan x)' = \frac{1}{\cos^2 x}. \tag{7}$$

This formula is valid for $x \neq \pi/2 + \pi n$, $n \in \mathbf{Z}$.

Analogously, we derive the formula

$$(\cot x)' = -\frac{1}{\sin^2 x}, \tag{8}$$

which is valid for $x \neq \pi n$, $n \in \mathbf{Z}$.

Example 1. Find the angle formed by the graph of the function $y = \sin x$ with the x-axis at the origin.

○ We have to find the slope of the tangent line to the graph of the function $y = \sin x$ at $x = 0$. We have $y = (\sin x)' = \cos x$. Hence, $f'(0) = \cos 0 = 1$, that is, the slope of the tangent line to the curve $y = \sin x$ at the point $(0, 0)$ is equal to 1. The required angle is equal to $\pi/4$. ●

Example 2. Find the angle formed by the graph of the function $y = \tan x$ with the x-axis at the origin.

○ We have $y' = (\tan x)' = 1/\cos^2 x$. Hence, $f'(0) = 1$, that is, the desired angle is equal to $\pi/4$. ●

The results obtained in Examples 1 and 2 must be taken into consideration when constructing the graphs of the functions $y = \sin x$ and $y = \tan x$. The graphs of these functions are shown in Fig. 200. Consider the behaviour of the graphs at the origin: they touch the straight line $y = x$.

5. Differentiating Composite Functions. To compute the derivative of the function to be differentiated, the latter is often represented as

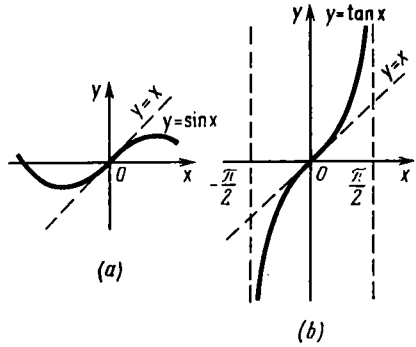


Fig. 200

a composition of two (or more) functions whose derivatives are known. Let, for instance, $y = f(x)$, where $f(x) = \varphi(u)$ and $u = \psi(x)$, that is, $y = \varphi(\psi(x))$. Then to every value x_0 from the domain of the function f there correspond values $u_0 = \psi(x_0)$ and $y_0 = \varphi(u_0)$ (here also we take only those values for which all the functions under consideration are differentiable). We have noted already that the value of the derivative of a function at a point is defined as the rate of change of the function at the point being considered. Hence, $\psi'(x_0)$ indicates the instantaneous rate of change of u with respect to x at the point x_0, and $\varphi'(u_0)$ the rate of change of y with respect to u at the point u_0. Therefore, to find $f'(x_0)$, that is, the rate of change of y with respect to x at the point x_0, we have to multiply $\psi'(x_0)$ by $\varphi'(u_0)$ (if, for instance, u changes two times quicker than x, and y three times quicker than u, then y changes $2 \cdot 3 = 6$ times quicker than x). Thus, $f'(x_0) = \varphi'(u_0) \cdot \psi'(x_0)$. Since the point x_0 was chosen arbitrarily, this equality may be written in more general form:

$$f'(x) = \varphi'(u) \cdot \psi'(x) = \varphi'(\psi(x)) \cdot \psi'(x),$$

where $f(x) = \varphi(\psi(x))$.

The variable u is usually called the *intermediate argument*, and the rule obtained for differentiating composite functions is stated by the following theorem (the proof is left to the reader).

Theorem 8.4.6. *The derivative of a composite function is equal to the derivative with respect to the intermediate argument multiplied by the derivative of the intermediate argument with respect to the independent variable*:

$$y'_x = y'_u \cdot u'_x$$

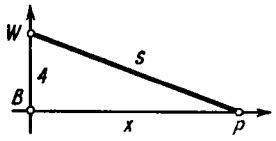


Fig. 201

(here y'_x denotes the derivative of y with respect to x, y'_u the derivative of y with respect to the intermediate argument u, and u'_x the derivative of the intermediate argument u with respect to the independent variable x).

Example 1. Find the derivative of the function $y = \left(x^2 - \frac{1}{2} + 5\right)^3$.

○ We set $u = x^2 - \frac{1}{x} + 5$. Then the given function may be considered as a composition of two functions $y = u^3$ and $u = x^2 - \frac{1}{x} + 5$. Making use of the rule for differentiating compositions of two functions, we get

$$y'_x = y'_u \cdot u'_x = (u^3)' \cdot \left(x^2 - \frac{1}{x} + 5\right)' = 3u^2 \cdot \left(2x + \frac{1}{x^2}\right).$$

Substituting for u its expression, we get

$$y' = 3\left(x^2 - \frac{1}{x} + 5\right)^2 \left(2x + \frac{1}{x^2}\right). \bullet$$

Example 2. Find the derivative of the function $y = \sqrt{x^2 + 16}$.

○ Since $y = u^{1/2}$, where $u = x^2 + 16$, we have

$$y'_x = y'_u \cdot u'_x = (u^{1/2})' \cdot (x^2 + 16)'.$$

But $(u^{1/2})' = 1/(2\sqrt{u})$, $(x^2 + 16)' = 2x$ and, consequently,

$$y' = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{x}{\sqrt{u}} = \frac{x}{\sqrt{x^2 + 16}}. \bullet$$

Example 3. A raft is pulled to the bank of a river with the help of a rope wound onto the windlass at a speed of 3 m/min. Determine the speed of the raft at the instant when its distance from the river bank is 25 m if it is known that the windlass is 4 m above the water level.

○ Let $s = PW$ be the length of the rope between the windlass and the raft, $x = PB$ the distance of the raft from the river bank, and $WB = 4$ (Fig. 201); consequently, $s = \sqrt{x^2 + 16}$. Here x is a func-

tion of time t, that is, $x = x(t)$. We have to find the speed of the raft, that is, x'_t. We have $s'_t = s'_x \cdot x'_t$. But in the preceding example we established that $s'_x = (\sqrt{x^2 + 16})' = x/\sqrt{x^2 + 16}$. Hence,

$$s'_t = \frac{x}{\sqrt{x^2+16}} \cdot x'_t = \frac{x}{s} \cdot x'_t,$$

whence $x'_t = (s/x) \cdot s'_t$.

By the hypothesis, $s'_t = 3$, $x = 25$, and, consequently, $s = \sqrt{25^2 + 4^2} = \sqrt{641}$. Then $x'_t = (\sqrt{641}/25) \cdot 3 \approx 3.03$. Thus, the required speed is 3.03 m/min. ●

Example 4. Find the derivative of the function $y = \sin^4 3x$.

○ We first find the derivative of the function $u = \sin 3x$. Since $u = \sin t$, $t = 3x$, we have

$$u'_x = u'_t \cdot t'_x = (\sin t)' \cdot (3x)' = \cos t \cdot 3 = 3 \cos 3x.$$

We now find the derivative of the function $y = \sin^4 3x$. Since $y = u^4$, $u = \sin 3x$, we have

$$y'_x = y'_u \cdot u'_x = (u^4)' \cdot (\sin 3x)' = 4u^3 \cdot 3 \cos 3x = 12 \sin^3 3x \cos 3x. \quad ●$$

Exercises

In Exercises 1 to 6, find the derivative of each of the given functions:

1. (a) $y = 2x^2 - 5x + 3$, (b) $y = \frac{1}{3} x^3 - \frac{3}{2} x^2 + \sqrt{5}$, (c) $y = \frac{4}{x} - x^3 - 1$.

2. (a) $y = (x^2 - 7x - 1)^2$, (b) $y = \left(\frac{3}{x}\right)^2$.

3. (a) $y = \frac{1}{x^3}$, (b) $y = x^4$, (c) $y = x^6$.

4. $y = \left(\frac{1}{x} + 5\right)(x^3 + 2x^2)$.

5. (a) $y = \frac{2x^3 + 3x - 1}{7}$, (b) $y = \frac{7}{2x^3 + 3x - 1}$.

6. (a) $y = \frac{x+2}{x-3}$, (b) $y = \frac{0.5x^2 - 2x + 1}{2x + 4}$.

7. Compute: (a) $f'(0)$ if $f(x) = 2x^3 - 3x - 4$, (b) $f'(1)$ if $f(x) = (x^3 + 5x - 1)^2$, (c) $f'(-1)$ if $f(x) = (2x+3)\left(x^3 + \frac{1}{x} + 3\right)$, (d) $f'(2)$ if $f(x) = \frac{3x+4}{x^2+1}$.

8. Write the equation of the tangent line to the graph of the function $y = f(x)$ at the given point x_0: (a) $f(x) = x^2 - 5x + 6$, $x_0 = 0$, (b) $f(x) = x^2(x^2 - 2x - 1)$, $x_0 = 1$, (c) $f(x) = \frac{1 + x^2}{x^3}$, $x_0 = -1$.

9. At what points do the tangent lines to the graph of the function $y = \frac{1}{6} x^3 + \frac{1}{4} x^2 - 2$ form an angle of 45° with the positive ray of the x-axis? Write the equation of these tangent lines.

10. A body of 2-kg mass performs a rectilinear motion according to the law $s = t^3 + 2t^2 + 5t - 1$ (s in metres, t in seconds). For the time instant $t = 2$, find: (a) the velocity v of the body; (b) the acceleration with which the body moves ($a = v'$); (c) the force acting upon the body ($F = ma$).

11. A body launched at the initial velocity $v_0 = 60$ m/s moves according to the law $s(t) = v_0 t - \frac{gt^2}{2}$ (s in metres, t in seconds). Find the instant at which the velocity of the body will be one third of the initial velocity assuming that $g = 10$ m/s.

In Exercises 12 to 16, find the derivative of the given function:

12. $y = 2x^{15} - \frac{3}{x^4}$. **13.** $y = x^{2/3} \sin x$. **14.** $y = \frac{\sqrt{x}}{\cos x}$.

15. $y = 3 \tan x - \frac{1}{2} \cot x$. **16.** $y = \left(\frac{3}{\sqrt[5]{x^2}} - 1\right)\left(\frac{2}{x^5} + 3\sqrt[3]{x}\right)$.

17. Set up the equation of the tangent line to the graph of the given function at the indicated point x_0: (a) $f(x) = 0.5 \cot x$, $x_0 = -\pi/4$, (b) $f(x) = \sqrt[3]{x}\left(\frac{x}{2} - 3\right)$, $x_0 = 8$.

18. At what points does the tangent line to the graph of the function $y = \frac{2}{3} \sin 3x$ make an angle of 60^0 with the positive ray of the x-axis?

In Exercises 19 to 33, find the derivatives of the given functions:

19. (a) $y = \sin^2 x$. (b) $y = \cos^3 x$.

20. (a) $y = \sqrt{\tan x}$, (b) $y = 1/\cot^5 x$.

21. (a) $y = \sqrt[3]{\sin 5x}$, (b) $y = \frac{1}{3} \cos^3 (6x - 5)$.

22. (a) $y = \cos x \sin (2x + 5)$, (b) $y = \sqrt{x^2 - x} + \frac{x^3}{\sin x}$.

23. Compute $f'(\pi/6)$ if $f(x) = \sin^5 x$.

§ 8.5. Applying Differential Calculus to Function Investigation

1. Increasing and Decreasing Functions. The coordinate of a point moving along the axis in the positive direction may serve as an example of an increasing function (function of time). In such a motion, the coordinate of the point increases with time. It is obvious that in this case the velocity of the point is positive all the time and may vanish only at some instants of time. It cannot be negative. Since the velocity is equal to the derivative of the coordinate with respect to time, we arrive at the following conclusion.

Theorem 8.5.1. *If a function $y = f(x)$ increases in an interval X, then its derivative is nonnegative in this interval* (the proof is left to the reader).

The same result can be obtained when considering the graph of an increasing function (Fig. 202). It is obvious that at all the points of the graph the tangent line forms either an acute angle φ with the positive ray of the axis of abscissas or is parallel to the axis of abscissas, and therefore $\tan \varphi \geqslant 0$. But, as is known, the slope of the

tangent line, that is, tan φ, is equal to the value of the derivative at the point of tangency. Thus, the figure illustrates the fact that the derivative of an increasing function is nonnegative.

A similar theorem occurs for a decreasing function.

Theorem 8.5.2. *If a function $y = f(x)$ decreases in an interval X, then its derivative is nonpositive in this interval* (Fig. 203).

For example, the function $y = x^3$ increases throughout the number line (see Fig. 62), and its derivative $y' = 3x^2$ is nonnegative everywhere (vanishing only at $x = 0$). The function $y = x^2$ decreases

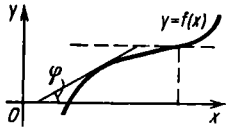


Fig. 202

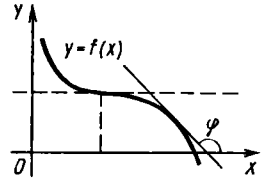


Fig. 203

in $(-\infty, 0]$ and increases in $[0, +\infty)$ (see Fig. 32). Accordingly, its derivative $y' = 2x$ is nonpositive for $x \leqslant 0$ and nonnegative for $x \geqslant 0$.

We have learnt to determine the sign of a derivative from the character of monotonicity of a function. But it is more important to solve the converse, that is, to find out the character of monotonicity of a function by the sign of its derivative. Consider once again the motion of a point along an axis. If at any instant of the time interval X its velocity is positive, then during this interval the point moves in the positive direction, and therefore its coordinate increases. And if at any instant of the time interval X its velocity is negative, then during this interval the point moves in the negative direction, and therefore its coordinate decreases. This shows the validity of the following assertion (the proof is left to the reader).

Theorem 8.5.3. *If for all $x \in X$ the inequality $f'(x) > 0$ is satisfied, then in the interval X the function $y = f(x)$ increases. And if for all $x \in X$ the inequality $f'(x) < 0$ is satisfied, then the function $y = f(x)$ decreases in the interval X.*

A more thorough reasoning shows that the theorem is also valid in the case when $f'(x) \geqslant 0$ ($f'(x) \leqslant 0$), $f'(x)$ vanishing only at a finite number of points.

Example 1. Prove that the function $y = x^3 + 3x$ increases everywhere.

○ We find the derivative $y' = 3x^2 + 3$. Since $x^2 \geqslant 0$ for any value of x, the inequality $y' > 0$ is satisfied for all x's. Hence, the function $y = x^3 + 3x$ increases everywhere. ●

Example 2. Prove that the equation $1 - x^3 - x^5 = 0$ has only one real root, this root lying within the interval (0, 1).

○ Consider the function $y = 1 - x^3 - x^5$ and find its derivative $y' = -3x^2 - 5x^4$. It is obvious that $y' \leqslant 0$, and therefore the function $y = 1 - x^3 - x^5$ decreases everywhere. Since $f(0) = 1 > 0$, and $f(1) = -1 < 0$, the graph of the continuous function $f(x) = 1 - x^3 - x^5$ intersects the x-axis within the interval (0, 1). Since the function is monotonic, the point of intersection of its graph with the x-axis is unique (see p. 307). Thus, the equation has only one root which is found within the interval (0, 1). ●

Example 3. Test the function $y = x^3 - 3x$ for monotonicity and sketch its graph.

○ We find $y' = (x^3 - 3x)' = 3x^2 - 3 = 3(x - 1)(x + 1)$. In the intervals $(-\infty, -1)$ and $(1, +\infty)$ we have $y' > 0$, and in the

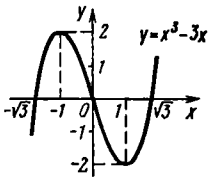


Fig. 204

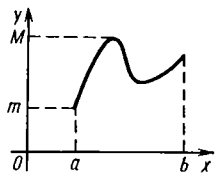


Fig. 205

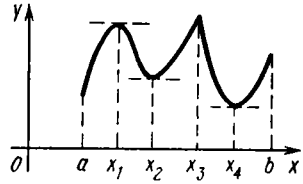


Fig. 206

interval $(-1, 1)$ $y' < 0$. Hence, the given function increases in the intervals $(-\infty, -1)$ and $(1, +\infty)$ and decreases in the interval $(-1, 1)$.

In order to construct the graph of the given function, we find several check points. Note first that if $x = -1$, then $y = 2$, and if $x = 1$ then $y = -2$. In addition, from the equation $x^3 - 3x = 0$ we find the points of intersection of the graph and the axis of abscissas: $x_1 = 0$, $x_2 = -\sqrt{3}$, $x_3 = \sqrt{3}$. The required graph is represented in Fig. 204. ●

2. Extremum of a Function. In practice, we often have to solve optimization problems (from the Latin "optimum" which means "the best"), that is, to choose from various variants the best one for a number of reasons. For instance, builders must know how to select the dimensions of a square beam to ensure its best tensile strength, artillerymen what inclination of a gun tube will result in the greatest range of fire, space-ship designers what orbit ensures a minimum fuel consumption, agronomists what seeding rate will guarantee the richest harvest, and so on.

Most optimization problems are reduced to finding the greatest and the least value of functions. Such values can be attained either at the end points of the interval of function change or inside this interval (Fig. 205). If the greatest value occurs inside the interval, then it is the greatest as compared with values of the function in the vicinity of this point. The converse is false since it may happen that

the value of the function at the point $x = c$ is the greatest of all of its values in some neighbourhood of this point, but far from $x = c$ there are points at which the function takes on still greater values.

Definition. A point $x = c$ is called the *point of maximum* (*minimum*) for a function $y = f(x)$ if there is a neighbourhood $(c - \delta, c + \delta)$ of this point in which the inequality $f(x) \leqslant f(c)$ $(f(x) \geqslant f(c))$ holds, the equality being the case only for $x = c$.

Figure 206 represents the graph of a continuous function $y = f(x)$, $a \leqslant x \leqslant b$. This function has maximum points at x_1 and x_3, and minimum points at x_2 and x_4.

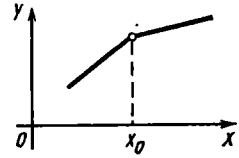


Fig. 207

The property of a point to be a point of maximum or minimum of a function depends only on the behaviour of the function in the vicinity of the point, that is, in the sufficiently small neighbourhood of the point. Points of maximum or minimum of a function are termed the *points of extremum* (from the Latin "extremum" meaning "the extreme").

Note that at the points of extremum x_1, x_2, x_4 (see Fig. 206) the tangent line to the graph of the function is parallel to the axis of abscissas, that is, at each of these points the derivative equals zero. We cannot draw a tangent line at the point x_3, hence the derivative does not exist at this point. We can prove that there are no points of extremum of other kind, that is, that the following theorem holds true.

Theorem 8.5.4 (a necessary condition for an extremum to exist). *If a function $y = f(x)$, $x \in X$ has an extremum at a point $x_0 \in X$, then the derivative of the given function either does not exist or is equal to zero at this point.*

Geometrically, this theorem implies that at a point of extremum the tangent line to the graph of the function is either horizontal or does not exist at all, that is, the graph has a cusp (a sharp bend or break).

Note that a necessary condition for an extremum to exist **is not sufficient**: neither vanishing of the derivative at $x = x_0$, nor absence of the derivative for this value of x implies that the function has an extremum at the point $x = x_0$. For example, the derivative of the function $y = x^3$ equal to $3x^2$ becomes zero at $x = 0$. But this function increases throughout the number line, and therefore its value at the point $x = 0$ is neither the greatest nor the least as compared with the values in the vicinity of this point (see Fig. 62). Figure 207 represents the graph of a function which has no derivative at the point x_0: the graph has a break at this point. It is obvious that x_0 is not a point of extremum of the graphed function.

In order to determine whether or not a function $y = f(x)$ has an extremum at a point c, where its derivative vanishes or does not exist at all, we need to know sufficient conditions for an extremum to exist. Let there exist a neighbourhood $(c - \delta, c + \delta)$ of the point c such that the function $y = f(x)$ increases in the interval $(c - \delta, c]$ and decreases in the interval $[c, c + \delta)$. Then for all x's, for which $c - \delta < x < c$, the inequality $f(x) < f(c)$ is satisfied, and for all x's for which $c < x < c + \delta$, the inequality $f(c) > f(x)$ holds. But this means that the value of the function $f(x)$ at the point c is greater than its values at other points of the neighbourhood $(c - \delta, c + \delta)$, that is, c is a point of maximum for $f(x)$.

In similar fashion, if a function $f(x)$ decreases in the interval $(c - \delta, c]$ and increases in the interval $[c, c + \delta)$, then c is a point of minimum for $f(x)$. And if the function $f(x)$ increases both in the interval $(c - \delta, c]$ and $[c, c + \delta)$, then c is neither a point of maximum nor a point of minimum since, to the left of it, the values of the function are smaller than at c, while, to the right of it, they are greater than at this point. If the function decreases both to the left and to the right of the point c, then the situation is the same.

We already know the conditions for an increase and a decrease of a function on an interval, they are related with the sign of the derivative on this interval. Applying them, we get the following assertion.

Theorem 8.5.5. *Let a function $y = f(x)$ be continuous at a point $x = c$, and there exist a neighbourhood $(c - \delta, c + \delta)$ of this point such that the inequality $f'(x) > 0$ holds in the interval $(c - \delta, c)$, and the inequality $f'(x) < 0$ in the interval $(c, c + \delta)$. Then $x = c$ is a point of maximum for $f(x)$.*

□ Since $f'(x) > 0$ in the interval $(c - \delta, c)$, by Theorem 8.5.3, the function $y = f(x)$ increases in this interval. Note that, by virtue of the continuity of the function at the point c, we may speak of its increase not only in the interval $(c - \delta, c)$, but also in the interval $(c - \delta, c)$, that is, in the interval containing the point c. Similarly, from the condition that $f'(x) < 0$ in the interval $(c, c + \delta)$ we obtain that the function $f(x)$ decreases in the interval $[c, c + \delta)$. Consequently, as it was shown above, c is a point of maximum for $f(x)$. ■

The following theorem is proved in a similar way.

Theorem 8.5.6. *If for some $\delta > 0$ we have $f'(x) < 0$ in the interval $(c - \delta, c)$ and $f'(x) > 0$ in the interval $(c, c + \delta)$, $f(x)$ being continuous at the point $x = c$, then $x = c$ is a point of minimum for $f(x)$.*

If $f'(x) > 0$ in both intervals $(c - \delta, c)$ and $(c, c + \delta)$, then c is not a point of extremum since the function $f(x)$ increases in the neighbourhood of this point. When $f'(x) < 0$ over both $(c - \delta, c)$ and $(c, c + \delta)$ then there is no extremum at c. In this case, the function $y = f(x)$ decreases in the neighbourhood of the point c.

As a result, we get the following **algorithm for investigating a function $y = f(x)$ for an extremum:**

1^0. Find the derivative $f'(x)$.

2^0. Find the critical points, that is, the points at which the function is continuous, and the derivative is either equal to zero or does not exist.

3^0. Consider the neighbourhood of each point found which does not contain other critical point and investigate the sign of the

	$(x_0-\delta; x_0)$	$(x_0; x_0+\delta)$	Graphs	Concl.
$f'(x)$	+	−		max
$f'(x)$	−	+		min
$f'(x)$	+	+		none
$f'(x)$	−	−		none

Fig. 208

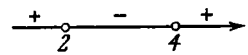


Fig. 209

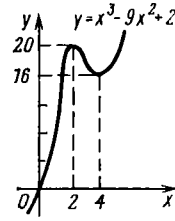


Fig. 210

derivative to the left and to the right of the point under consideration.

4^0. Using the sufficient conditions for an extremum, draw relevant conclusions (Fig. 208).

Example. Investigate the function $y = x^3 - 9x^2 + 24x$ for an extremum.

○ 1^0. We have $y' = 3x^2 - 18x + 24 = 3(x - 2)(x - 4)$.

2^0. Equating the derivative to zero, we find $x_1 = 2$, $x_2 = 4$. In this case, the derivative is defined everywhere; hence there are no other critical points.

3^0. The sign of the derivative $y' = 3(x - 2)(x - 4)$ changes depending on the interval as shown in Fig. 209. When passing through the point $x = 2$, the derivative changes sign from plus to minus, and when passing through the point $x = 4$, from minus to plus.

4^0. At the point $x = 2$, the function has a maximum $y_{\max} = 20$, while at the point $x = 4$ a minimum $y_{\min} = 16$. Figure 210 represents the outline of the graph of the function $y = x^3 - 9x^2 + 24x$. ●

3. Finding the Greatest and the Least Value of a Continuous Function on a Closed Interval. Let a function $y = f(x)$ be continuous on a closed interval $[a, b]$. Then it attains its greatest and least values on this interval (see Theorem 8.1.7).

A continuous function can attain its greatest value M and least value m both inside the interval and at its end points (Fig. 211). If the function reaches its greatest (least) value at an interior point of the interval, then this point is a point of extremum; however, for the practical purposes it suffices that this is a critical point.

Here is an **algorithm for finding the greatest and the least value of a continuous function $y = f(x)$ on a closed interval $[a, b]$:**

1°. Find $f'(x)$.

2°. Find the points at which $f'(x) = 0$ or does not exist at all and choose those lying inside the interval $[a, b]$.

3°. Evaluate the function $y = f(x)$ at the points obtained in 2° as well as at the end points of the interval and pick up the greatest and the least value from among them; they are the greatest y_{great}, and the least y_{least} value of the function $y = f(x)$ on the interval $[a, b]$.

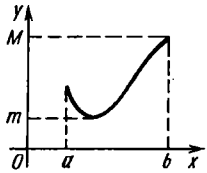


Fig. 211

Example 1. Find the greatest and the least value of the continuous function $y = x^3 - 3x^2 - 45x + 225$ on the interval $[0, 6]$.

○ 1°. We find $y' = 3x^2 - 6x - 45$.

2°. The derivative y' exists for all x's. We find the points at which $y' = 0$; we have $3x^2 - 6x - 45 = 0$, $x^2 - 2x - 15 = 0$, $x_1 = -3$, $x_2 = 5$. Of these two, only the point $x = 5$ belongs to the interval $[0, 6]$.

3°. We evaluate the function at the points $x = 0$, $x = 5$, and $x = 6$;

x	0	5	6
y	225	50	63

The greatest of the found values of the function is the number 225, the least the number 50. Thus, $y_{\text{great}} = 225$, $y_{\text{least}} = 50$. ●

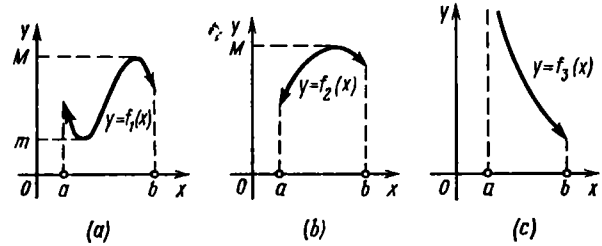


Fig. 212

Consider now the problem of finding the greatest or the least value of a continuous function on an open interval. This problem not always has a solution. Figure 212 represents the graphs of three functions continuous on the interval (a, b). The function $y = f_1(x)$

reaches both the greatest and the least value; the function $y = f_2(x)$ attains the greatest value, but has no least value on (a, b); while the function $y = f_3(x)$ has neither the greatest nor the least value on (a, b).

If it is required to find the greatest or least value of a function continuous on (a, b), we proceed in the same way as in finding these values on the closed interval $[a, b]$. However, instead of evaluating the function at the end points of the closed interval, we find the limits of the function as the latter approaches the end points of the interval.

Example 2. Find the least value of the function $y = \left(\frac{2+\cos x}{\sin x}\right)^2$ on the interval $(0, \pi)$.

○ 1°. Find the derivative

$$y' = 2\left(\frac{2+\cos x}{\sin x}\right)\cdot\left(\frac{2+\cos x}{\sin x}\right)' = 2\cdot\frac{2+\cos}{\sin x}\cdot\frac{(2+\cos x)'\sin x-(2+\cos x)(\sin x)'}{\sin^2 x}$$

$$= 2\cdot\frac{2+\cos x}{\sin x}\cdot\frac{(-\sin x)\sin x-(2+\cos x)\cos x}{\sin^2 x}$$

$$= \frac{2(2+\cos x)(-\sin^2 x-\cos^2 x-2\cos x)}{\sin^2 x} = \frac{-2(2+\cos x)(1+2\cos x)}{\sin^2 x}.$$

2°. The derivative $y' = 0$ if $1 + 2\cos x = 0$ or $2 + \cos x = 0$. The second equation has no solutions since $|\cos x| \leqslant 1$, while from the first equation we find $\cos x = -1/2$, that is, $x = \pm 2\pi/3 + 2\pi k$. Of these values only $x = 2\pi/3$ belongs to the interval $(0, \pi)$.

The derivative y' does not exist if $\sin^3 x = 0$. But this equation has no solution on $(0, \pi)$.

Thus, the function has only one critical point $x = 2\pi/3$ inside the interval $(0, \pi)$.

3°. If $x = 2\pi/3$, then $y = \left(\frac{2+\cos(2\pi/3)}{\sin(2\pi/3)}\right)^2 = \left(\frac{2-0.5}{0.5\sqrt{3}}\right)^2 = 3$. As x approaches the end points of the interval, that is, as $x \to 0$ or $x \to \pi$, the denominator of the fraction $\frac{(2+\cos x)^2}{\sin^2 x}$ tends to zero, while the numerator to 9 or to 1, respectively. Hence, in both cases $y \to +\infty$ (see page 313).

Since when approaching the end points of the interval $(0, \pi)$ the function increases without bound, it reaches the least value at a single critical point, that is, at the point $x = 2\pi/3$. Thus, $y_{\text{least}} = 3$. ●

Sometimes, the following assertions turn out to be useful for finding the greatest or the least value on an open interval (a, b):

1. *If a function $y = f(x)$ has only one point of extremum $x = a$ in an interval X, this point being a point of maximum, then $f(a)$ is the greatest value of the function on the interval X.*

2. *If a function $y = f(x)$ has only one point of extremum $x = a$ in the interval X, this point being a point of minimum, then $f(a)$ is the least value of the function on the interval X.*

Thus, in the example considered above the function has only one critical point $x = 2\pi/3$ in the interval $(0, \pi)$. When passing through this point, the derivative changes sign from minus to plus. Hence, $2\pi/3$ is a point of minimum, and therefore $f(2\pi/3) = 3$ is the least value of the function in the interval $(0, \pi)$.

4. Problems on Finding the Greatest or the Least Value of Quantities. Such problems are easily solved according to the following schedule:

1^0. The quantity to be optimized (that is, the quantity whose greatest or least value is required to be found) is found and is denoted by y (or S, p, r, R, etc.) depending on the problem to be solved.

2^0. One of the unknown quantities (side, angle, etc.) is assumed to be an independent variable and is denoted by x. The limits over which x may vary are set by the problem.

3^0. Proceeding from the concrete conditions of the problem, y is expressed in terms of x and of known quantities.

4^0. The greatest or the least value of the function $y = f(x)$ obtained at the preceding step (depending on the requirements of the problem) are found by making use of the interval of real changes in x found in 2^0.

5^0. The result of the preceding item is interpreted for the given concrete problem.

Going through steps 1^0-3^0, we set up, as we usually say, a mathematical model of the problem. A success in solution then often depends on a clever choice of the independent variable. It is important that y could be easily expressed in terms of x. At step 4^0, the mathematical model formed is investigated, as a rule, with the help of the derivative, sometimes by an elementary method. In carrying out such an investigation, the problem itself is of no importance. Only when the problem is completely solved within the mathematical model, can the result be interpreted for the initial problem (step 5^0).

Example 1. A party of geologists is camped on the steppe 9 km to the north of a highway running from west to east. There is a town 15 km to the east of the nearest point on the highway to the camp. The geologists send a messenger on by bicycle to the town. What route should the messenger follow to reach the town in the shortest time if he can ride at 8 km/h across the steppe and at 10 km/h along the highway?

○ Let us make a drawing (Fig. 213). The point P denotes the location of the party, the straight line l the highway, B the town, $PA = 9$ km, $AB = 15$ km, PMB the route of the messenger, the position of the point M between A and B is unknown yet.

1°. The quantity to be optimized is the time t during which the messenger will move from P to B; we have to find t_{least}.

2°. We set $AM = x$. According to the problem, the point M may be anywhere between A and B including points A and B. Hence, the real bounds within which x varies are: $0 \leqslant x \leqslant 15$.

3°. We express t in terms of x. We have $PM = \sqrt{PA^2 + AM^2} = \sqrt{81 + x^2}$. The cyclist covers this distance at a speed of 8 km/h, that is, the time t_1 spent to cover this distance is expressed by the formula $t_1 = \frac{\sqrt{81+x^2}}{8}$. Further, $MB = 15 - x$. This distance is covered by the cyclist at a speed of 10 km/h, that is, the time t_2

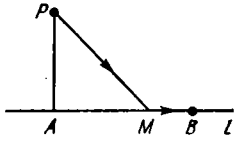


Fig. 213

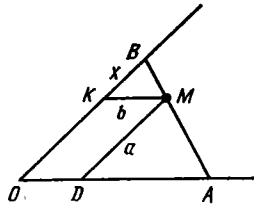


Fig. 214

spent to cover this distance is expressed by the formula $t_2 = \frac{15-x}{10}$. Thus, the total time t the messenger spent to cover the entire distance is equal to $t_1 + t_2$, that is,

$$t = \frac{\sqrt{81+x^2}}{8} + \frac{15-x}{10}.$$

4°. We have to find the least value of the function $t = \frac{\sqrt{81+x^2}}{8} + \frac{15-x}{10}$ in the interval [0, 15]. To this end, we make use of the above algorithm (see p. 346).

1. We find t':

$$t' = \frac{1}{8} \cdot \frac{1}{2} (81 + x^2)^{-1/2} \cdot 2x + \frac{1}{10}(-1) = \frac{x}{8\sqrt{81+x^2}} - \frac{1}{10}.$$

2. The derivative t' exists for all x. We find the points at which $t' = 0$. We have

$$\frac{x}{8\sqrt{81+x^2}} - \frac{1}{10} = 0, \quad 5x = 4\sqrt{81+x^2}, \quad 25x^2 = 16(81 + x^2),$$

$$9x^2 = 16 \cdot 81, \quad x^2 = 16 \cdot 9, \quad x = 4 \cdot 3 = 12.$$

The value $x = 12$ belongs to the closed interval [0, 15].

3. We compile a table of values attained by the function at the end points of the interval and at the critical point found:

x	0	12	15
t	105/40	87/40	$5\sqrt{306}/40$

Consequently, $t_{\text{least}} = 87/40$.

The fourth step of the solution has been completed; it remains to interpret the result obtained in conformity with the original problem.

5°. Since t_{least} is reached for $x = 12$, the cyclist has to ride along a route PMB such that the distance between the points A and M on the highway be equal to 12 km. ●

Example 2. Draw a straight line through a fixed point M inside an angle to cut off a triangle having the least area (Fig. 214).

○ 1°. The quantity to be optimized is the area S of the triangle AOB.

2°. We draw $DM \parallel OB$, $MK \parallel OA$. We set $KB = x$; the real boundaries of change for x are: $0 < x < +\infty$.

3°. Since M is a fixed point, the line segments DM and KM are also fixed; we set $DM = a$, $KM = b$ and express S in terms of x, a, and b.

Consider the triangles MKB and AOB; they are similar, hence $\frac{MK}{AO} = \frac{KB}{OB}$, that is, $\frac{b}{AO} = \frac{x}{a+x}$. Hence we find $AO = \frac{b(a+x)}{x}$.

Further, we have $S = \frac{1}{2} AO \cdot OB \cdot \sin\alpha$, where $\alpha = \angle AOB$. Hence,

$$S = \frac{1}{2} \frac{b(a+x)}{x} (a+x) \sin\alpha = \frac{b \sin\alpha}{2} \cdot \frac{(a+x)^2}{x}.$$

A mathematical model of the problem has been formed.

4°. Consider the function $S = k \frac{(a+x)^2}{x}$, $0 < x + \infty$, where $k = \frac{b \sin\alpha}{2}$. We find its least value.

1. We have

$$S' = k \frac{2(a+x)x - (a+x)^2}{x^2} = k \frac{(a+x)(x-a)}{x^2}.$$

2. The derivative does not exist at the point $x = 0$ and vanishes at the points $x = -a$ and $x = a$. Only one of these three points belongs to the open interval $(0, +\infty)$.

3. We have $S \to \infty$ both as $x \to 0$ and as $x \to +\infty$. Hence, the least value of the function is reached at the point $x = a$.

5^0. We return to the original geometrical problem. Since $x = KB = a$, and $OK = a$, MK is a midline of the triangle AOB; hence M is the midpoint of AB. Thus, to cut off the triangle of least area, we have to draw a straight line through the point M such that its segment enclosed between the sides of the given angle is bisected at the point M. ●

5. Applying the Derivative for Proving Identities and Inequalities. The proof of identities with the help of the derivative is based on the following theorem (its proof is left to the reader).

Theorem 8.5.7 (condition for a function to be constant). *For a continuous function $y = f(x)$ to be constant in an interval X, it is necessary and sufficient that its derivative be zero at all interior points of the interval.*

Example 1. Prove the identity

$$\cos^4 x - \frac{1}{8}\cos 4x = 2\cos^2 x - \frac{1}{2}\cos 2x - \frac{5}{8}.$$

○ Consider the function

$$f(x) = \cos^4 x - \frac{1}{8}\cos 4x - 2\cos^2 x + \frac{1}{2}\cos 2x + \frac{5}{8}$$

and find its derivative. We have

$$\begin{aligned} f'(x) &= 4\cos^3 x(-\sin x) - \frac{1}{8}(-\sin 4x)\cdot 4 \\ &\quad - 2\cdot 2\cos x(-\sin x) + \frac{1}{2}(-\sin 2x)\cdot 2 \\ &= -4\sin x\cos^3 x + \frac{1}{2}\sin 4x + 4\sin x\cos x - \sin 2x. \end{aligned}$$

Making use of the formula $2\sin x\cos x = \sin 2x$, we get

$$\begin{aligned} f'(x) &= -2\sin 2x\cos^2 x + \sin 2x\cos 2x + 2\sin 2x - \sin 2x \\ &= \sin 2x(-2\cos^2 x + \cos 2x + 1). \end{aligned}$$

But $1 + \cos 2x = 2\cos^2 x$. Hence, $f'(x) = 0$ for all x's, and therefore $f(x)$ is a constant function, that is, $f(x) = C$. It remains to find the value of the constant C. For this purpose, it suffices to evaluate $f(x)$ for any value of x, say, for $x = 0$. We have $f(0) = 1 - \frac{1}{8} - 2 + \frac{1}{2} + \frac{5}{8} = 0$. Thus, $f(0) = 0$, and therefore $C = 0$. Consequently, the following identity holds true

$$\cos^4 x - \frac{1}{8}\cos 4x - 2\cos^2 x + \frac{1}{2}\cos 2x + \frac{5}{8} = 0,$$

whence

$$\cos^4 x - \frac{1}{8}\cos 4x = 2\cos^2 x - \frac{1}{2}\cos 2x - \frac{5}{8}. \quad ●$$

In most cases, application of derivatives for proving inequalities involves the investigation of functions for monotonicity and extremum.

Example 2. Prove that the inequality

$$2x+\frac{1}{x^2}>5$$

is valid for $0<x<0.5$.

○ Consider the function $f(x)=2x+\frac{1}{x^2}$ and find its derivative: $f'(x)=2-\frac{2}{x^3}=\frac{2(x^3-1)}{x^3}$. We note that $f'(x)<0$ on the open interval $(0, 1)$; hence the function $f(x)$ is decreasing on this interval. Therefore, in particular for $0<x<0.5$ the inequality $f(x)>f(0.5)$ is true. But $f(0.5)=2\cdot 0.5+\frac{1}{0.5^2}=5$.

Thus, $f(x)>5$, that is, $2x+\frac{1}{x^3}>5$. ●

Example 3. Prove that if $\alpha<\beta$, then $\alpha+\cos\alpha<\beta+\cos\beta$.

○ Consider the function $f(x)=x+\cos x$ and find its derivative: $f'(x)=1-\sin x$. Since $f'(x)\geqslant 0$, the function increases throughout the number line. Hence $\alpha<\beta$ implies that $f(\alpha)<f(\beta)$, that is, $\alpha+\cos\alpha<\beta+\cos\beta$. ●

Example 4. Prove that for all x's the following inequality is valid:

$$x^5+(1-x)^5\geqslant\frac{1}{16}.$$

○ Consider the function $f(x)=x^5+(1-x)^5$ and investigate it for extremum. We have

$$f'(x)=5x^4-5(1-x)^4=5(x^2-(1-x)^2)(x^2+(1-x)^2)$$
$$=5(2x-1)(2x^2-2x+1),\ f'(x)=0 \text{ for } x=1/2.$$

The function has no other critical points (the equation $2x^2-2x+1=0$ has no roots). If $x<1/2$, then $f'(x)<0$, and if $x>1/2$, then $f'(x)>0$, hence $x=1/2$ is a point of minimum. Since the given continuous function has no other points of extremum $f(1/2)$ is the least value of the function (see assertion 2 from Item 4). But $f(1/2)=(1/2)^5+(1/2)^5=1/16$. Thus, $f(x)\geqslant 1/16$, that is, $x^5+(1-x)^5\geqslant 1/16$. ●

6. Constructing the Graph of a Function. In practice, to graph a function $y=f(x)$, we sometimes proceed as follows: from the equation $y=f(x)$ we find several points on the graph of the function $y=f(x)$ and draw a smooth curve through these points. However, making use of this method, we run the risk of leaving out some important features of the graph and thus make a mistake in its construction.

To construct the graph of a function, we have to investigate its properties. First we have to find the domain of the function, and then to investigate the function for parity and periodicity. Since the graph of an even function is symmetric with respect to the y-axis, and the graph of an odd function with respect to the origin, we may confine ourselves to investigating the properties of even and odd functions only for $x \geqslant 0$. If a function is periodic, and T is its fundamental period, then we may confine ourselves to investigating function's properties on the interval of length T.

It is good practice to find the points of intersection of the graph and the coordinate axes and to determine the intervals of constant sign of the function. The thing is that if, say, a function $y = f(x)$ takes on only positive values on an interval (a, b), then its graph on this interval lies above the x-axis. Hence, the part of the plan under the indicated interval may be hatched (as in Fig. 215), since there is no graph there. This step of investigation enables us to indicate the regions of the expected location of the graph of the function. This done, we have to study the behaviour of the function on the boundaries of the domain of definition of the function, determine the character of discontinuities (if any), and find the asymptotes. Finally, we have to find the intervals of the increase and decrease of the function, and to investigate it for extremum.

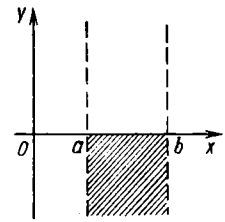


Fig. 215

Summing up the aforesaid, we get the following scheme for investigating the properties of the function and constructing its graph.

1°. Find the domain of the function.

2°. Investigate the function for evenness.

3°. Test the function for periodicity.

4°. Find the points at which the graph intersects the coordinate axes.

5°. Determine the intervals of constant signs.

6°. Study the behaviour of the function at the boundaries of the domain. Find the asymptotes.

7°. Investigate the function for extremum.

8°. Compile a table of function values for some values of the argument.

9°. Making use of the results obtained, construct the graph of the function.

Example 1. Graph the function $y = x^4 - 2x^2 - 8$.

○ 1°. The function is defined for any value of x, that is $D(f) = \mathbf{R}$.

2°. Since the domain of the function is a symmetric set and $f(-x) = f(x)$ the function is even. Consequently, the graph of the function is symmetric about the y-axis, and we may confine our further investigation to the interval $[0, +\infty)$. But we are not going to do this here.

3°. The function is nonperiodic.

4°. We find the points at which the graph intersects the x-axis. To this end, we solve the equation $x^4 - 2x^2 - 8 = 0$. Setting $u = x^2$, we get the quadratic equation $u^2 - 2u - 8 = 0$ having the roots 4 and -2. From the equation $x^2 = 4$ we find $x_1 = 2$ and $x_2 = -2$; the equation $x^2 = -2$ has no solution. Thus, we have found two points of intersection of the graph and the x-axis: $(2, 0)$ and $(-2, 0)$.

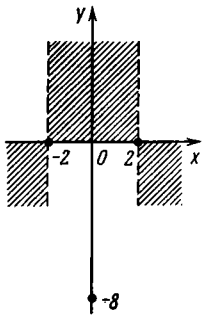


Fig. 216

The graph of the function $y = x^4 - 2x^2 - 8$ intersects the y-axis at the point $(0, -8)$.

5°. We find the intervals of constant signs. The given function is continuous throughout the number line and vanishes at the points 2 and -2. Hence, it retains a constant sign within the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, +\infty)$. In order to determine the sign of the function on each of the indicated intervals it suffices to take one "sampling" point from each interval.

We have $-100 \in (-\infty, -2)$, $f(-100) = (-100)^4 - 2(-100)^2 - 8 > 0$. Hence, $f(x) > 0$ in the interval $(-\infty, -2)$. Further, $0 \in (-2, 2)$, $f(0) = -8 < 0$. Therefore, $f(x) < 0$ in the interval $(-2, 2)$. Finally, $100 \in (2, +\infty)$, $f(100) = f(-100)$, and we saw above that $f(-100) > 0$. Consequently, $f(100) > 0$, and therefore $f(x) > 0$ in the interval $(2, +\infty)$.

Figure 216 gives a geometrical illustration of the data we have obtained. The regions of the coordinate plane containing no graph are hatched, the known points $(0, -8)$, $(2, 0)$, and $(-2, 0)$ are marked. This is the answer to the question where the graph is located. Further investigation enables us to answer the question how to construct the graph.

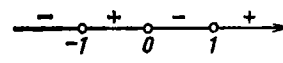


Fig. 217

6°. We study the behaviour of the function in the vicinity of the boundaries of the domain of its definition. Since $D(f) = (-\infty, +\infty)$, $-\infty$ and $+\infty$ may be regarded as such "boundaries". Transforming the expression $x^4 - 2x^2 - 8$ to the form $x^2\left(x^2 - 2 - \frac{8}{x^2}\right)$, we note that $y \to +\infty$ as $x \to -\infty$ or $x \to +\infty$.

The graph has no asymptotes.

7°. We investigate the function for extremum; we have

$$y' = 4x^3 - 4x = 4x(x - 1)(x + 1).$$

Equating the derivative to zero, we find three roots: 0, 1, and -1. These points divide the number line into four intervals: $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, +\infty)$. If $x > 1$, then $y' > 0$, and in all the other intervals the signs alternate from right to left (Fig. 217).

We now compile a table:

x	$-\infty < x < -1$	-1	$-1 < x < 0$	0	$0 < x < 1$	1	$1 < x < +\infty$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\searrow$ decr.	-9 min	$\nearrow$ incr.	-8 max	$\searrow$ decr.	-9 min	$\nearrow$ incr.

Thus, the function has a minimum at the points $(-1, -9)$ and $(1, -9)$ and a maximum at the point $(0, -8)$.

8°. We compile a table of the function values for some values of the argument including those already mentioned in the course of investigation:

x	-2	-1	0	1	2	-2.5	2.5
y	0	-9	-8	-9	0	≈ 6	≈ 6

9°. We construct the graph of the function $y = x^4 - 2x^2 - 8$ (Fig. 218). ●

Example 2. Graph the function $y = \frac{x^2 - 1}{x}$.

○ 1°. The function is defined only at the point $x = 0$.

2°. We have $f(-x) = \frac{(-x)^2 - 1}{-x} = -\frac{x^2 - 1}{x} = -f(x)$. Hence, $y = f(x)$ is an odd function. Therefore its graph is symmetric with respect to the origin, and we may confine ourselves to the interval $(0, +\infty)$.

3°. The function is nonperiodic.

4°. We find the points of intersection of the graph with the positive ray of the x-axis. From the equation $\frac{x^2 - 1}{x} = 0$ we find $x = 1$ (the root $x = -1$ being neglected so far). Thus, we have found one point $(1, 0)$ of intersection with the x-axis.

The graph does not intersect the y-axis since the point $x = 0$ does not belong to the domain of the function.

5°. We find the intervals of constant signs: $(0, 1)$ and $(1, +\infty)$. In the first of them $f(x) < 0$, in the second $f(x) > 0$.

Figure 219 gives a geometrical illustration of the data we have obtained.

6°. We now study the behaviour of the function in the vicinity of the boundaries of the domain of its definition, that is, in the vicinity of the point 0, and as $x \to +\infty$. If $x \to 0$ (recall that we

consider only the case when $x > 0$), then $\frac{x^2-1}{x} \to -\infty$. And if $x \to +\infty$, then $\frac{x^2-1}{x} = x - x \to +\infty$.

The straight line $x = 0$ is a vertical asymptote. Further, since the dergee of the numerator of the expression $\frac{x^2-1}{x}$ is by unity

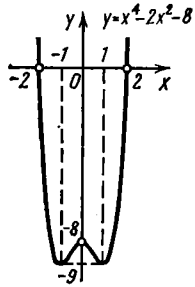


Fig. 218

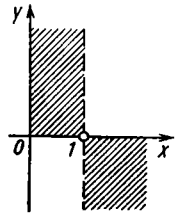


Fig. 219

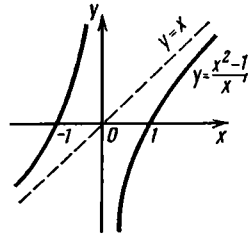


Fig. 220

greater than that of the denominator, the graph must have an oblique asymptote. Indeed, since $\frac{x^2-1}{x} = x - \frac{1}{x}$ and $\frac{1}{x}$ tends to zero as $x \to +\infty$, the straight line $y = x$ serves as an oblique asymptote.

7°. We investigate the function for an extremum; we have

$$y' = \left(\frac{x^2-1}{x}\right)' = \left(x - \frac{1}{x}\right)' = 1 + \frac{1}{x^2}.$$

We note that $y' > 0$ for any x. Hence, the function is increasing on the ray $(0, +\infty)$ and has no extrema.

8°. We compile a table of the function's values:

x	1	0.5	0.25	2	3	4
y	0	−1.5	−3.75	1.5	2.67	3.75

9°. Plotting the found points on the coordinate plane and taking account of the results of investigation, we sketch a branch of the graph for $x > 0$ (Fig. 220).

Since the graph of the function $y = \frac{x^2-1}{x}$ is symmetric with respect to the origin, adding a symmetric branch to the branch constructed, we obtain the required graph (Fig. 220). ●

Example 3. Investigate the function $y = \sin x - \frac{1}{2}\sin 2x$ and construct its graph.

○ 1°. We find $D(f) = \mathbf{R}$.

2°. We have $f(-x) = \sin(-x) - \frac{1}{2}\sin(-2x) = -\sin x + \frac{1}{2}\sin 2x = -f(x)$. Hence, the function is odd.

3°. The function is periodic with the fundamental period $T = 2\pi$. Since the period of the function is 2π, it suffices to carry out the investigation only from $-\pi$ to π, construct the graph of the function on the interval $[-\pi, \pi]$, and continue it by making use of periodicity. But since the function is odd, it suffices to investigate it and construct its graph on the interval $[0, \pi]$, then, by making use of the symmetry with respect to the origin, reflect it in the closed interval $[-\pi, 0]$, and then take advantage of the periodicity. Thus, we shall carry out our investigation for the closed interval $[0, \pi]$.

4°. We find the points at which the graph intersects the x-axis. For this purpose, we solve the equation $\sin x - \frac{1}{2}\sin 2x = 0$; we have

$$\sin x - \sin x \cos x = 0, \quad \sin x (1 - \cos x) = 0.$$

On the interval $[0, \pi]$ the last equation has two roots: $x_1 = 0$ and $x_2 = \pi$. Hence, the graph of the function intersects the axis of abscissas at no interior point of the interval $[0, \pi]$.

5°. In the interval $(0, \pi)$ the function takes on only positive values.

6°. The function is continuous and periodic, consequently, its graph has no asymptotes. We find the values of the function at the end points of the interval $[0, \pi]$; we have $f(0) = 0$, $f(\pi) = 0$.

7°. We find the points of extremum. Since $y' = \cos x - \cos 2x$, equating the derivative to zero, we get $\cos x - \cos 2x = 0$. Further,

$$\cos x - (1 + \cos 2x) + 1 = 0,$$

$$2\cos^2 x - \cos x - 1 = 0, \quad \cos x = 1, \quad \cos x = -1/2.$$

From the first equation we find $x_1 = 0$, from the second $x_2 = 2\pi/3$ (recall that we confine ourselves to the closed interval $[0, \pi]$).

Thus, there is only one "suspicious" point $x = 2x/3$ inside the interval $[0, \pi]$. It is clear that this is a point of maximum since, as we noted above, the function vanishes at the end points of the interval $[0, \pi]$, while inside this interval it is positive. We evaluate the function at the point of maximum:

$$y_{\max} = f\left(\frac{2\pi}{3}\right) = \sin\frac{2\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3} = \frac{3\sqrt{3}}{4}.$$

8°. We compile a table of the function's values for some values of the argument:

x	0	$2\pi/3$	$\pi/2$	π
y	0	$3\sqrt{3}/4$	1	0

9°. Using the results obtained, we first construct the graph of the function on the interval $[0, \pi]$ and then on the entire number line (Fig. 221). ●

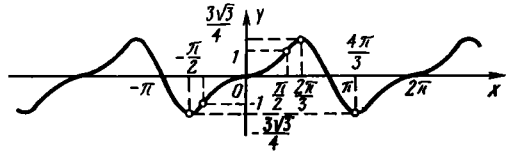


Fig. 221

Exercises

1. Determine the monotonicity intervals of the function $y = 3x^2 - 2x + 2$.
2. Investigate the function $y = \frac{x}{7} + \frac{7}{x}$ for extremum.
3. Investigate the following functions for monotonicity and for extremum: (a) $f(x) = 3 + x - 3x^3$, (b) $f(x) = x^3 - 3x^2 + 2$. Sketch the graphs of the functions.
4. Find the greatest and the least value of the function $\varphi(x) = x^3/3 - x^2 + 1$ on the closed interval $[-1, 1]$.
5. Find the triangle having the greatest area among the isosceles triangles with a given lateral side a.
6. A rectangle whose base lies on the hypotenuse is inscribed into a right triangle with hypotenuse 16 cm long and an acute angle of 60°. What dimensions must the rectangle have to have the greatest area?
7. It is required to make a conic funnel with generatrix $l = 15$ cm. What must the height of the funnel be to ensure the greatest volume?
8. It is required to make a closed cylindrical tank of capacity V. For what radius of the base will the material consumption be the least?
9. A piece of wire of length l is bent to form a sector of a circle of maximum area. Find the central angle of the sector.
10. Construct the graph of the velocity of a body if the law of motion is specified as follows:

$$s(t) = \begin{cases} -(1/3)t^3 + 7t & \text{for } 0 \leqslant t \leqslant 2, \\ -0.5t^2 + 5t & \text{for } 2 < t \leqslant 5. \end{cases}$$

Determine from the graph the velocity of the body at the instants $t = 0, 2, 5$.

11. A body moves according to the law $s(t) = 62.6 + 54t^2 - 0.2t^5$ (s in metres, t in seconds). Determine: (a) at what instant of time the body has the greatest velocity, (b) the velocity and the acceleration of the body at this instant, (c) what distance will the body cover by the instant when its velocity is the greatest.
12. Prove that $2x^3 + 6x > 3x^2 + 5$ for $x > 1$.

In Exercises 13 to 20, investigate the given functions and construct their graphs:

13. $y=x^2-5x-6$. 14. $y=x^3-3x^2-9x+15$.

15. $y=x^4+3x^2-10$. 16. $y=\frac{x^2}{x^2+1}$.

17. $y=\frac{x}{x^2-1}$. 18. $y=\frac{2x^2-5x-1}{x-3}$.

19. $y=\sin^2 x$. 20. $y=\cos x-\frac{1}{3}\cos 3x$.

§ 8.6. Indefinite Integral

1. Primitive Functions (or Antiderivatives) and Indefinite Integral. We noted above that a number of physical and geometrical problems are reduced to finding derivatives of functions. In this section we shall discuss a process inverse to differentiation, that is, the process of finding a function by its derivative. This operation is called *integration*, and the section of mathematical analysis which studies the methods of finding a function by its derivative together with applications of this operation is called *integral calculus*.

One of the basic concepts of integral calculus is the notion of primitive function (or antiderivative).

Definition 1. A function $y = F(x)$ defined in some interval X is called a *primitive function* or *antiderivative* of a function $y = f(x)$ specified in the same interval if for all $x \in X$ the equality $F'(x) = f(x)$ holds.

Example 1. Let $f(x)=x^3$. Then a primitive function $F(x)$ has the form $F(x)=\frac{x^4}{4}$, since $F'(x)=\left(\frac{x^4}{4}\right)'=x^3=f(x)$.

Example 2. Let $f(x)=\sin 3x$. Then a primitive function has the form $F(x)=-\frac{1}{3}\cos 3x$, since $F'(x)=\left(-\frac{1}{3}\cos 3x\right)'=-\frac{1}{3}\cdot(-\sin 3x)\cdot 3=\sin 3x=f(x)$.

For the function $f(x)=x^3$ from Example 1 we have found a primitive function $F(x)=\frac{x^4}{4}$. But this is not a unique solution to the problem, therefore we say "a primitive function" and not "the primitive function". For example, as a primitive function for the function $f(x)=x^3$, we could have taken the function $F_1(x)=\frac{x^4}{4}+3$ $\left(\text{since } \left(\frac{x^4}{4}+3\right)'=x^3\right)$, and the function $F_2(x)=\frac{x^4}{4}-5$ $\left(\text{since } \left(\frac{x^4}{4}-5\right)'=x^3\right)$, and, in general, any function of the form $\frac{x^4}{4}+C$ as in Example 2, where we could take any function of the form $-\frac{1}{3}\cos 3x+C$ as a primitive function.

Theorem 8.6.1 (on the fundamental property of a primitive function). *If $F(x)$ is a primitive function of $f(x)$ in some interval X, then the function $f(x)$ has an infinite set of primitive functions, all of them having the form $F(x) + C$.*

□ Let $F(x)$ be a primitive function of $f(x)$. Then $(F(x) + C)' = F'(x) + C' = f(x) + 0 = f(x)$, and this means that $F(x) + C$ is also a primitive function of $f(x)$.

Let now $F(x)$ and $\Phi(x)$ be two primitive functions of the function $y = f(x)$. We set $H(x) = \Phi(x) - F(x)$. Then $H'(x) = \Phi'(x) - F'(x)$. Since $\Phi(x)$ and $F(x)$ are two primitive functions of $y = f(x)$, we have $\Phi'(x) = f(x)$ and $F'(x) = f(x)$ and therefore $H'(x) = f(x) - f(x) = 0$. But if the equality $H'(x) = 0$ is valid in the interval X, then $H(x) = C$ (see Theorem 8.5.7).

Thus, $H(x) = C$, that is, $\Phi(x) - F(x) = C$, whence we get $\Phi(x) = F(x) + C$. ■

If a function $y = f(x)$ has at least one primitive function in some interval X, then it is called *integrable* in this interval. We can prove that *any function continuous in a closed interval $[a, b]$ is integrable in this interval.*

The operation of integration associates every integrable function $y = f(x)$ with a whole family of functions differing from one another by a constant term. This family of functions, that is, the set of all primitive functions of $y = f(x)$ in the interval X, is called the indefinite integral of $f(x)$ and is denoted $\int f(x)\,dx$ (read: "integral $f(x)\,dx$").

Definition 2. If a function $y = f(x)$ has a primitive function $F(x)$ in an interval X, then the set of all primitive functions is called the *indefinite integral* of the function $f(x)$:

$$\int f(x)\,dx = F(x) + C. \tag{1}$$

In equality (1), the function $y = f(x)$ is called the *integrand function*, $f(x)\,dx$ the *integrand* (or the integration element), the variable x the *integration variable*, C the *constant of integration.*

2. Table of Basic Integrals. Each integration formula follows from the corresponding differentiation formula. For example, $\left(\frac{x^{n+1}}{n+1}\right)' = x^n$ implies the equality $\int x^n dx = \frac{x^{n+1}}{n+1} + C$. Given below is the table of basic integrals, which is obtained from the corresponding formulas for differentiating elementary functions:

I. $\int dx = x + C.$

II. $\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C$, where $\alpha \neq -1$, $\alpha \in \mathbf{Q}$.

III. $\int \sin x\,dx = -\cos x + C.$

IV. $\int \cos x\,dx = \sin x + C.$

V. $\int \frac{dx}{\sin^2 x} = -\cot x + C.$

VI. $\int \frac{dx}{\cos^2 x} = \tan x + C.$

Integrals I-VI are usually called the **tabular** integrals.

Note that by saying "take an indefinite integral" or "compute an indefinite integral" we understand finding the set of primitive functions of the integrand function.

Example. Compute $\int \frac{dx}{\sqrt[3]{x}}$.

○ We have $\int \frac{dx}{\sqrt[3]{x}} = \int x^{-1/3}\,dx$. Making use of formula II for $\alpha = -1/3$, we get

$$\int x^{-1/3}\,dx = \frac{x^{-1/3+1}}{-1/3+1} + C = \frac{3}{2}\,x^{2/3} + C. \bullet$$

3. Properties of the Indefinite Integral. Consider the basic properties of the indefinite integral.

1°. *The integral of an algebraic sum of functions is equal to the sum of the integrals of the summands.*

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

□Let $F(x)$ be a primitive function of $f(x)$, and $G(x)$ a primitive function of $g(x)$. Since

$$(F(x) + G(x))' = F'(x) + G'(x) = f(x) + g(x),$$

$F(x) + G(x)$ is a primitive function of $f(x) + g(x)$. Hence,

$$\int (f(x) + g(x))\,dx = F(x) + G(x) + C.$$

On the other hand, we have

$$\int f(x)\,dx + \int g(x)\,dx = (F(x) + C_1) + (G(x) + C_2) = F(x) + G(x) + C$$

(we have set $C_1 + C_2 = C$).

Thus,

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx. \blacksquare$$

The next property is proved in a similar way.

2^0. *A constant factor may be taken outside the integral sign*:

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

Example 1. Compute $\int \frac{\sqrt{x}+x^2-5}{x^3\sqrt{x}}\,dx$.

○ Dividing the numerator by the denominator termwise and making use of properties 1^0 and 2^0, we get

$$\int \frac{\sqrt{x}+x^2-5}{x^2\sqrt{x}}\,dx = \int\left(\frac{1}{x^2}+\frac{1}{\sqrt{x}}-\frac{5}{x^2\sqrt{x}}\right)dx$$
$$= \int x^{-2}\,dx+\int x^{-1/2}\,dx-5\int x^{-5/2}\,dx.$$

We have obtained a sum of tabular integrals. Making use of formula II for each term (setting $\alpha=-2$, $\alpha=-1/2$, and $\alpha=-5/2$, respectively) we find

$$\int x^{-2}\,dx+\int x^{-1/2}\,dx-5\int x^{-5/2}\,dx$$
$$=\frac{x^{-2+1}}{-2+1}+\frac{x^{-1/2+1}}{-1/2+1}-5\cdot\frac{x^{-5/2+1}}{-5/2+1}+C$$
$$=-\frac{1}{x}+2\sqrt{x}+\frac{10}{3x\sqrt{x}}+C.$$

Note that each of the three indefinite integrals contains its own constant. However, the total constant C is written only in the final answer. In our further considerations we shall stick to such a procedure. ●

Example 2. Compute $\int \frac{dx}{\sin^2 x\cos^2 x}$.

○ We have

$$\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}\,dx$$
$$= \int\left(\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}\right)dx = \int \frac{dx}{\cos^2 x}+\int \frac{dx}{\sin^2 x}.$$

We have obtained a sum of two tabular integrals. Making use of formula VI for the first term and formula V for the second, we get

$$\int \frac{dx}{\cos^2 x}+\int \frac{dx}{\sin^2 x} = \tan x-\cot x+C. \;●$$

Note one more property of the indefinite integrals which proves to be of use in practice.

3^0. *If* $\int f(x)\,dx = F(x)+C$, *then*

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+C, \quad a\neq 0.$$

□Let us show that $H(x) = \frac{1}{a} F(ax + b)$ is a primitive function of $f(ax + b)$. Indeed,

$$H'(x) = \left(\frac{1}{a} F(ax+b)\right)' = \frac{1}{a} F'(ax+b).$$

Since, by the hypothesis, $F'(x) = f(x)$, applying the rule for differentiating a composite function, we get

$$F'(ax + b) = f(ax + b)(ax + b)' = f(ax + b) \cdot a.$$

Therefore

$$H'(x) = \frac{1}{a} F'(ax+b) = \frac{1}{a} \cdot f(ax+b) \cdot a = f(ax+b).$$

Thus, $H'(x) = f(ax+b)$, and therefore

$$\int f(ax+b)\,dx = H(x) + C, \text{ that is,}$$

$$\int f(ax+b)\,dx = \frac{1}{a} F(ax+b) + C. \blacksquare$$

Example 3. Compute $\int \frac{dx}{\sqrt{4x+1} + \sqrt{4x-2}}$.

○Multiplying the numerator and the denominator of the integrand by the expression conjugate to the denominator, we get

$$\int \frac{dx}{\sqrt{4x+1} + \sqrt{4x-2}} = \int \frac{\sqrt{4x+1} - \sqrt{4x-2}}{(\sqrt{4x+1})^2 - (\sqrt{4x-2})^2}\,dx$$

$$= \int \frac{\sqrt{4x+1} - \sqrt{4x-2}}{3}\,dx$$

$$= \frac{1}{3}\int (4x+1)^{1/2}\,dx - \frac{1}{3}\int (4x-2)^{1/2}\,dx.$$

Applying formula II (for $\alpha = 1/2$) and property 3^0 to either of the two integrals obtained, we get

$$\frac{1}{3}\int (4x+1)^{1/2}\,dx - \frac{1}{3}\int (4x-2)^{1/2}\,dx$$

$$= \frac{1}{3} \cdot \frac{(4x+1)^{1/2+1}}{(1/2+1)\cdot 4} - \frac{1}{3} \cdot \frac{(4x-2)^{1/2+1}}{(1/2+1)\cdot 4} + C$$

$$= \frac{1}{18}\left(\sqrt{(4x+1)^3} - \sqrt{(4x-2)^3}\right) + C. \bullet$$

Example 4. Compute $\int \sin x \sin 4x\,dx$.

○Using the formula $\sin x \sin 4x = (\cos 3x - \cos 5x)/2$, we have

$$\int \sin x \sin 4x\,dx = \frac{1}{2}\int \cos 3x\,dx - \frac{1}{2}\int \cos 5x\,dx.$$

Applying formula IV and property 3^0 to each term, we get

$$\frac{1}{2}\int \cos 3x\,dx-\frac{1}{2}\int \cos 5x\,dx=\frac{1}{2}\cdot\frac{\sin 3x}{3}-\frac{1}{2}\cdot\frac{\sin 5x}{5}+C$$
$$=\frac{\sin 3x}{6}-\frac{\sin 5x}{10}+C. \bullet$$

Exercises

1. Prove that $F(x)$ is a primitive function for $f(x)$ if: (a) $F(x)=x^3-2x+1$, $f(x)=3x^2-2$, (b) $F(x)=2\sin 2x+2$, $f(x)=4\cos 2x$, (c) $F(x)=\sin^2 x+1$, $f(x)=\sin 2x$.

2. Prove that $F(x)=\sqrt{-x}$ is a primitive function of $f(x)=-1/(2\sqrt{-x})$ on the interval $(-\infty, 0)$.

3. Find a primitive function of the function $f(x)=\sin x$ whose graph passes through the point $(\pi/3, 2)$.

4. Find primitive functions of the functions: (a) $x^2+5\sqrt[4]{x}$, (b) $\sin(2x-3)-1/\cos^2 x$.

In Exercises 5 to 13, compute the given indefinite integrals:

5. $\int \frac{2+3\sqrt[3]{x^2}+5\sqrt{x^3}}{\sqrt{x^3}}\,dx$. 6. $\int (2\sin x-3\cos x)\,dx$.

7. $\int \frac{2x\sin^2 x+1}{\sin^2 x}\,dx$. 8. $\int \frac{\cos^3 x-\sin x\cos^2 x+1}{\cos^2 x}\,dx$.

9. $\int \sqrt{4-3x}\,dx$. 10. $\int (2x-5)^{100}\,dx$. 11. $\int \sin^2 x\,dx$.

12. $\int \cos x\cos 3x\,dx$. 13. $\int \sin^2 x\cos x\,dx$.

§ 8.7. Definite Integral

1. Problem on Computing the Area of a Curvilinear Trapezoid. Let in the xy-plane there be given a figure bounded by straight lines $x=a$, $x=b$ $(a<b)$, $y=0$ and by the graph of a function $y=f(x)$, continuous and nonnegative in a closed interval $[a, b]$.

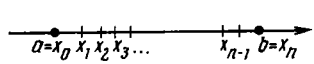


Fig. 222

How to find the area S of such a curvilinear trapezoid?

Consider one of the possible methods of solving this problem. We subdivide the interval $[a, b]$ into n parts by points $x_1, x_2, \ldots, x_{n-1}$; and for uniformity of designation, set $a=x_0$, $b=x_n$ (Fig. 222). We denote: $x_1-x_0=\Delta x_0$, $x_2-x_1=\Delta x_1$, $x_3-x_2=\Delta x_2$, $\ldots$, $x_n-x_{n-1}=\Delta x_{n-1}$ and consider the sum

$$f(x_0)\,\Delta x_0+f(x_1)\Delta x_1+f(x_2)\Delta x_2+\ldots+f(x_{n-1})\Delta x_{n-1}. \quad (1)$$

From the geometrical point of view, this sum represents the area P of the figure shown in Fig. 223, S being approximately equal to P.

Note that, instead of sum (1), we may consider the sum

$$f(x_1)\,\Delta x_0 + f(x_2)\Delta x_1 + f(x_3)\Delta x_2 + \ldots + f(x_n)\,\Delta x_{n-1}. \qquad (2)$$

The difference between sum (2) and sum (1) consists in that in the first case the value of the function is chosen at the left-hand end of each of the subintervals $[x_0, x_1]$, $[x_1, x_2]$, . . ., $[x_{n-1}, x_n]$ whereas in the second case at the right-hand end. From the geometrical point of

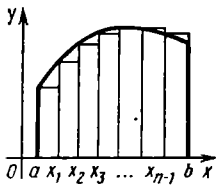


Fig. 223

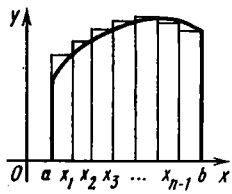


Fig. 224

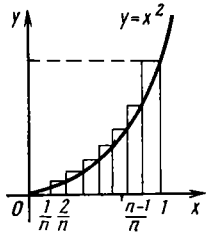


Fig. 225

view, sum (2) represents the area P_1 of the figure shown in Fig. 224 which differs from the figure depicted in Fig. 223. However, the approximate equality $S \approx P_1$ is satisfied in this case as well.

In practice, it is more convenient to divide the closed interval $[a, b]$ into n equal parts. Then $\Delta x_0 = \Delta x_1 = \Delta x_2 = \ldots = \Delta x_{n-1} = \frac{b-a}{n}$ and sum (1) takes the form $\frac{b-a}{n}(f(x_0) + f(x_1) + \ldots + f(x_{n-1}))$. The value of the sum depends on the number n, therefore it can be denoted by S_n:

$$S_n = \frac{b-a}{n}(f(x_0) = f(x_1) + \ldots + f(x_{n-1})). \qquad (3)$$

Then for the area of a curvilinear trapezoid we get an approximate equality $S \approx S_n$; the greater n, the more exact this equality is. Therefore *by the area S of the curvilinear trapezoid* we quite naturally understand $\lim\limits_{n\to\infty} S_n$ or $\lim\limits_{n\to\infty} S'_n$, where

$$S'_n = \frac{b-a}{n}(f(x_1) + f(x_2) + \ldots + f(x_n)). \qquad (4)$$

This is a sum of form (2) obtained by dividing the closed interval $[a, b]$ into n equal parts. It is proved in the course of mathematical analysis that the two indicated limits are equal.

Example. Find the area of the curvilinear trapezoid bounded by a parabola $y = x^2$ and straight lines $x = 0$, $x = 1$, and $y = 0$ (Fig. 225).

○ Divide the interval $[0, 1]$ into n equal parts by points $x=0$, $x_1=\frac{1}{n}$, $x_2=\frac{2}{n}$, $\ldots$, $x_{n-1}=\frac{n-1}{n}$, $x_n=\frac{n}{n}=1$. Then $f(x_0)=0$, $f(x_1)=\frac{1}{n^2}$, $f(x_2)=\frac{2^2}{n^2}$, $\ldots$, $f(x_{n-1})=\frac{(n-1)^2}{n^2}$, $f(x_n)=\frac{n^2}{n^2}=1$. Form the sum S_n' (the area of the figure in Fig. 225):

$$S_n'=\frac{1}{n}\left(\frac{1^2}{n^2}+\frac{2^2}{n^2}+\frac{3^2}{n^2}+\ldots+\frac{n^2}{n^2}\right)=\frac{1}{n^3}(1^2+2^2+3^2+\ldots+n^2).$$

Applying the method of mathematical induction we can prove that

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

(see § 1.7, Exercise 5). Hence

$$S_n'=\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}=\frac{2n^2+3n+1}{6n^2}.$$

It remains to compute $\lim\limits_{n\to\infty} S_n'$. We have

$$\lim_{n\to\infty} S_n'=\lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2}=\lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)=\frac{1}{3}.$$

Thus, the required area is equal to 1/3. ●

2. The Notion of the Definite Integral. Let a function $y=f(x)$ be continuous in the interval $[a, b]$; we form for it the sum of form (3):

$$S_n=\frac{b-a}{n}(f(x_0)+f(x_1)+\ldots+f(x_{n-1})),$$

which is called the *integral sum*. Consider the sequence of integral sums $S_1, S_2, S_3, \ldots, S_n, \ldots$, where

$$S_1=(b-a)f(x_0),$$

$$S_2=\frac{b-a}{2}(f(x_0)+f(x_1)),$$

$$S_3=\frac{b-a}{3}(f(x_0)+f(x_1)+f(x_2)),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$S_n=\frac{b-a}{n}(f(x_0)+f(x_1)+\ldots+f(x_{n-1})),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

It is possible to prove that for a function $y=f(x)$, continuous on a closed interval $[a, b]$, this sequence converges, that is, the limit $\lim\limits_{n\to\infty} S_n$ exists.

Definition. Let $y = f(x)$ be continuous on a closed interval $[a, b]$ and (S_n) a sequence of integral sums. Then $\lim\limits_{n\to\infty} S_n$ is called the *definite integral* (or simply the *integral*) of the function $f(x)$ with respect to x over the interval $[a, b]$ and is denoted $\int\limits_a^b f(x)\,dx$ (read: "the integral of $f(x)\,dx$ from a to b"). The numbers a and b are called, respectively, the *lower* and the *upper limit* * *of integration*, the symbol $\int\limits_a^b$ the *integral sign*, and the function $f(x)$ the *integrand*.

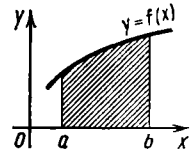


Fig. 226

Comparing this definition with the notion of the area of a curvilinear trapezoid introduced in subsection 1, we get the following result: *if* $y = f(x)$ *is a function continuous and nonnegative on a closed interval* $[a, b]$, *then* $\int\limits_a^b f(x)\,dx = S$, *where* S *is the area of a curvilinear trapezoid bounded by straight lines* $y = 0$, $x = a$, $x = b$ *and by the graph of the function* $y = f(x)$. This is just the **geometrical meaning of the definite integral** (Fig. 226).

Example 1. Compute $\int\limits_0^1 x\,dx$.

○ **Method I.** We divide the closed interval $[0, 1]$ into n equal parts by points $x_0 = 0$, $x_1 = \frac{1}{n}$, $x_2 = \frac{2}{n}$, ..., $x_{n-1} = \frac{n-1}{n}$. Then $f(x_0) = 0$, $f(x_1) = \frac{1}{n}$, $f(x_2) = \frac{2}{n}$, ..., $f(x_{n-1}) = \frac{n-1}{n}$. We form the integral sum:

$$S_n = \frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \ldots + \frac{n-1}{n}\right) = \frac{1+2+3+\ldots+(n-1)}{n^2}.$$

The numerator represents the sum of the first n terms of the arithmetic progression whose first term is 1 and $(n-1)$st term is equal to $n-1$ (that is, $a_1 = 1$, $a_{n-1} = n-1$). This sum can be computed by the formula $\frac{a_1+a_{n-1}}{2}(n-1)$ (see page 287). Hence

$$S_n = \frac{1}{n^2} \cdot \frac{1+(n-1)}{2} \cdot (n-1) = \frac{n(n-1)}{2n^2} = \frac{n-1}{2n}.$$

* It is unfortunate that the word "limit" is used in this connection. The limits of integration have nothing to do with the notion of the limit of a function.

Now we can compute $\int_0^1 x\,dx$. We have

$$\int_0^1 x\,dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n-1}{2n} = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{2n}\right) = \frac{1}{2}.$$

Thus, $\int_0^1 x\,dx = \frac{1}{2}$.

Method II. We take advantage of the geometrical interpretation of the integral: $\int_0^1 x\,dx$ is the area of the figure bounded by the straight lines $x = 0, x = 1, y = 0$ and by the graph of the function

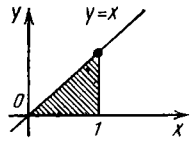


Fig. 227

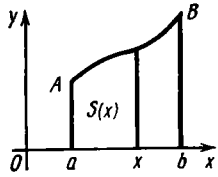


Fig. 228

$y = x$ (Fig. 227), that is, the area of a right isosceles triangle whose legs a and b are egual to 1. We find $S = \frac{1}{2}ab = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$. Thus, $\int_0^1 x\,dx = \frac{1}{2}$. ●

Example 2. Compute $\int_0^1 x^2\,dx$.

○ The integral $\int_0^1 x^2\,dx$ is the area of a curvilinear trapezoid bounded by three straight lines $x = 0$, $x = 1$, $y = 0$ and a parabola $y = x^2$. In the example considered in subsection 1 we determined that this area is equal to 1/3. Hence $\int_0^1 x^2 dx = \frac{1}{3}$. ●

3. Relation Between an Integral and a Primitive Function (the Newton-Leibniz formula). Let us consider again the problem of computing the area of a curvilinear trapezoid bounded by straight lines $x = a$, $x = b$ $(a < b)$, $y = 0$ and by the graph of a function $y = f(x)$, continuous and nonnegative in a closed interval $[a, b]$. Consider another method of solving the problem on finding the area of such a trapezoid $aABb$ (Fig. 228).

Let $S(x)$ denote the area of the curvilinear trapezoid cut off from the trapezoid $aABb$ by a straight line parallel to the axis of ordinates and intersecting the axis of abscissas at the point x (Fig. 228). Then $S(a) = 0$, and $S(b) = S$, that is, we may write $S = S(b) - S(a)$.

We estabilished above (see p. 327) that $S'(x) = f(x)$, that is, $S(x)$ is one of the primitive functions of the function $f(x)$. Let $F(x)$ be another primitive function of $f(x)$. Then $F(x) = S(x) + C$, and we get

$$F(b) - F(a) = (S(b) + C) - (S(a) + C) = S(b) - S(a) = S.$$

On the other hand, $S = \int_a^b f(x)\,dx$ (see subsection 2). Thus,

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{5}$$

Note that there is a rigorous proof of this formula for any (not necessarily nonnegative) function $y = f(x)$ continuous in a closed interval $[a, b]$ which is not based on geometrical illustration, but we shall not consider it here.

Formula (5) is called the **Newton-Leibniz formula.** It establishes the relation between the integral and a primitive function of the integrand and provides the main method of computing the integral, using the designation $F(x)\,|_a^b = F(b) - F(a)$.

Example 1. Compute $\int_0^1 x\,dx$.

○ In subsection 2, we computed this integral by two methods. We now take advantage of the third method, that is, apply the Newton-Leibniz formula. We find a primitive function of the integrand $f(x) = x$: $F(x) = \frac{x^2}{2}$. Then $\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$. ●

Example 2. Compute $\int_0^1 x^2\,dx$.

○ We have $\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$ (see Example 2 from subsection 2). ●

Example 3. Find the area of the figure bounded by the axis of abscissas and by an arc of the sine curve $y = \sin x$ on the interval $[0, \pi]$ (Fig. 229).

○ We have $S = \int_0^{\pi} \sin x\, dx = -\cos x \Big|_0^{\pi} = -(\cos \pi - \cos 0) = -(-1-1) = 2$. ●

4. Some Physical Applications of the Integral. Consider a couple of problems reduced to computing definite integrals.

Problem 1. Let a point M move in a straight line, and let there be known the velocity of this point $v = v(t)$ at any instant of time t on the interval $[a, b]$. Find the displacement s of the point M during this interval of time.

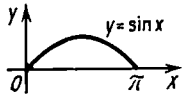


Fig. 229

○ We know that if $x = x(t)$ is the law of motion of the point (the value of its coordinate at time t), then $v(t) = x'(t)$. Therefore $x(t)$ is one of the primitive functions of the function $v = v(t)$. But the displacement of the point M during the time interval $[a, b]$ is equal to the difference between its coordinates at instants b and a, that is, $x(b) - x(a)$. In other words, this displacement is equal to the difference of values of the primitive function of the function $v = v(t)$ at the instants of time b and a, that is, $s = \int_a^b v(t)\, dt$. ●

For example, the velocity of a freely falling body is expressed by the formula $v = gt$. Therefore the distance covered by the falling body during b seconds from the beginning of fall is expressed as follows:

$$s = \int_0^b gt\, dt = \frac{gt^2}{2}\Big|_0^b = \frac{gb^2}{2}.$$

Problem 2. Let $I = I(t)$ be the current strength at instant t. Find the quantity of electricity q flowing through the cross section of the conductor during the time interval $[a, b]$.

○ As is known, $q'(t) = I(t)$ (see page 327), that is, $q(t)$ is a primitive function of $I(t)$. Thus, reasoning as in Problem 1, we get $q = \int_a^b I(t)\, dt$. ●

Similarly, taking advantage of the fact that the mass of a rod is a primitive function of line density $\rho(x)$ (see page 327), we find that the mass m of the section $[a, b]$ of the rod is expressed by the formula $m = \int_a^b \rho(x)\, dx$.

5. Properties of the Definite Integral. The basic properties of the definite integral are easily derived from the Newton-Leibniz

formula. When deriving these properties, we assume that the functions under consideration have primitive functions in the given intervals.

1°. *The integral of the sum of two functions* $f_1(x)$ *and* $f_2(x)$ *over a closed interval* $[a, b]$ *is equal to the sum of values of the integrals of these functions over the same interval*:

$$\int_a^b (f_1(x)+f_2(x))\,dx=\int_a^b f_1(x)\,dx+\int_a^b f_2(x)\,dx.$$

□Indeed, it follows from the properties of the indefinite integral that if $F_1(x)$ is a primitive function of $f_1(x)$ and $F_2(x)$ is a primitive function of $f_2(x)$, then $F_1(x)+F_2(x)$ is a primitive function of $f_1(x)+f_2(x)$. Therefore

$$\begin{aligned}\int_a^b (f_1(x)+f_2(x))\,dx&=(F_1(b)+F_2(b))-(F_1(a)+F_2(a))\\&=(F_1(b)-F_1(a))+(F_2(b)-F_2(a))\\&=\int_a^b f_1(x)\,dx+\int_a^b f_2(x)\,dx.\ \blacksquare\end{aligned}$$

2°. *A constant factor can be taken outside the integral sign*:

$$\int_a^b \lambda f(x)\,dx=\lambda\int_a^b f(x)\,dx.$$

□Indeed, if $F(x)$ is a primitive function of $f(x)$, then $\lambda F(x)$ is a primitive function of $\lambda f(x)$. Therefore

$$\int_a^b \lambda f(x)\,dx=(\lambda F(b)-\lambda F(a))=\lambda(F(b)-F(a))=\lambda\int_a^b f(x)\,dx.\ \blacksquare$$

Example 1. Compute $\int_{-2}^{1}(2x^3+3x-4)\,dx$.

$$\begin{aligned}\circ\ \int_{-2}^{1}(2x^3+3x-4)\,dx&=\int_{-2}^{1}2x^3\,dx+\int_{-2}^{1}3x\,dx+\int_{-2}^{1}-4\,dx\\&=2\int_{-2}^{1}x^3\,dx+3\int_{-2}^{1}x\,dx-4\int_{-2}^{1}1\,dx\\&=2\cdot\frac{x^4}{4}\Big|_{-2}^{1}+3\cdot\frac{x^2}{2}\Big|_{-2}^{1}-4\cdot x\Big|_{-2}^{1}\\&=\frac{1}{2}(1^4-(-2)^4)+\frac{3}{2}(1^2-(-2)^2)\\&\quad-4(1-(-2))=-24.\ \bullet\end{aligned}$$

Example 2. Compute $\int_{\pi/6}^{\pi/4} \tan^2 x\,dx$.

$$\circ \quad \int_{\pi/6}^{\pi/4} \tan^2 x\,dx = \int_{\pi/6}^{\pi/4} \frac{\sin^2 x}{\cos^2 x}\,dx = \int_{\pi/6}^{\pi/4} \frac{1-\cos^2 x}{\cos^2 x}\,dx$$

$$= \int_{\pi/6}^{\pi/4} \left(\frac{1}{\cos^2 x} - 1\right) dx = \int_{\pi/6}^{\pi/4} \frac{1}{\cos^2 x}\,dx - \int_{\pi/6}^{\pi/4} 1 \cdot dx$$

$$= \tan x \Big|_{\pi/6}^{\pi/4} - x \Big|_{\pi/6}^{\pi/4}$$

$$= \left(\tan\frac{\pi}{4} - \tan\frac{\pi}{6}\right) - \left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= 1 - \frac{\sqrt{3}}{3} - \frac{\pi}{12}. \quad \bullet$$

3°. *If a point c lies in a closed interval $[a, b]$, then*

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx. \tag{6}$$

□ Indeed, let $F(x)$ be a primitive function of $f(x)$. Then we have

$$\int_a^c f(x)\,dx = F(c) - F(a), \quad \int_c^b f(x)\,dx = F(b) - F(c),$$

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

But $F(c) - F(a) + F(b) - F(c) = F(b) - F(a)$, and therefore equality (6) is valid. ■

The equality proved above has a vivid geometrical meaning: it expresses the additivity property of the area of a plane figure. Thus, the area S of the curvilinear trapezoid $aABb$ represented in Fig. 230 is equal to $S_1 + S_2$, where S_1 is the area of the trapezoid $aACc$, and S_2 the area of the trapezoid $cCBb$. But $S_1 = \int_a^c f(x)\,dx$, $S_2 = \int_c^b f(x)\,dx$, $S = \int_a^b f(x)\,dx$, whence we get equality (6).

From the physical point of view, equality (6) expresses the additivity property of the mass of a rod or, similarly, the additivity property of displacement when a point moves in a straight line.

4^0. *If* $f(x) \geqslant 0$ *in a closed interval* $[a, b]$, *then*

$$\int_a^b f(x)\,dx \geqslant 0. \tag{7}$$

□ Indeed, let $F(x)$ be a primitive function of $f(x)$, that is, $F'(x) = f(x)$. Since, by the hypothesis, $f(x) \geqslant 0$ on $[a, b]$, that is, $F'(x) \geqslant 0$ in $[a, b]$, the function $y = F(x)$ increases in the interval $[a, b]$. An increase in this function implies that $F(a) \leqslant F(b)$. Hence

$$\int_a^b f(x)\,dx = F(b) - F(a) \geqslant 0. \ \blacksquare$$

Inequality (7) allows a simple geometrical interpretation: the area of a curvilinear trapezoid bounded from above by the graph of the function taking on only nonnegative values, is a nonnegative number.

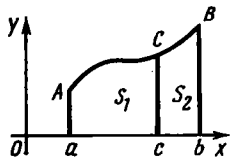


Fig. 230

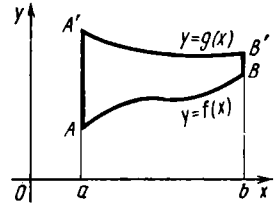


Fig. 231

The physical meaning of inequality (7) is also obvious: if the velocity is nonnegative during the time interval $[a, b]$, then the displacement during this time interval is also nonnegative.

5^0. *If an inequality* $f(x) \leqslant g(x)$ *is satisfied in an interval* $[a, b]$, *then*

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx. \tag{8}$$

□ Indeed, $g(x) - f(x) \geqslant 0$, and then, according to property 4^0, we have $\int_a^b (g(x) - f(x))\,dx \geqslant 0$, that is, $\int_a^b g(x)\,dx - \int_a^b f(x)\,dx \geqslant 0$, whence inequality (8) follows. ■

The geometrical meaning of inequality (8) is clear from Fig. 231: $S_{aABb} = \int_a^b f(x)\,dx$, $S_{aA'B'b} = \int_a^b g(x)\,dx$, it is obvious that $S_{aABb} < S_{aA'B'b}$.

The physical interpretation of inequality (8) is also sufficiently obvious:

(1) if one point moves in a straight line at a lower velocity than the other, that is, $v_1(t) < v_2(t)$, then the displacement of the first point during the time interval $[a, b]$ is less than that of the second point during the same interval $\int_a^b v_1(t)\,dt < \int_a^b v_2(t)\,dt$,

(2) if we are given two rods of equal length with end points at a and b, and the line density of the first rod at every point is less than that of the second rod at the corresponding point, that is, $\rho_1(x) < \rho_2(x)$, then the mass of the first rod is also less than that of the second rod $\int_a^b \rho_1(x)\,dx < \int_a^b \rho_2(x)\,dx$.

6^0. *If the inequality* $m \leqslant f(x) \leqslant M$ *is satisfied in a closed interval* $[a, b]$, *then*

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a). \tag{9}$$

□ Indeed, property 5^0 implies that

$$\int_a^b m\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b M\,dx.$$

But $\int_a^b m\,dx = m\int_a^b dx = m(b-a)$, similarly $\int_a^b M\,dx = M(b-a)$, and therefore inequality (9) is valid. ■

The geometrical meaning of inequality (9) is clear from Fig. 232: $m(b-a)$ is the area of the rectangle entirely contained inside the curvilinear trapezoid $aABb$, and $M(b-a)$ the area of the rectangle containing the trapezoid $aABb$. Hence, inequality (9) expresses the relation between the areas of three figures: an interior rectangle, a curvilinear trapezoid, and an exterior rectangle.

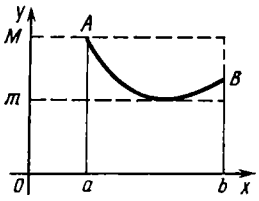


Fig. 232

6. Computing Areas of Plane Figures. With the help of an integral we can find the area not only of a curvilinear trapezoid bounded by straight lines $x = a$, $x = b$ $(a < b)$, $y = 0$, and by the graph of a function $y = f(x)$, continuous and nonnegative in $[a, b]$, but also of a plane figure more complicated in shape.

Consider the figure bounded by straight lines $x = a$, $x = b$ $(a < b)$, and by the graphs of the functions $y = f_1(x)$ and $y = f_2(x)$, contin-

uous in the interval $[a, b]$ and such that for all x from this interval the inequality $0 \leqslant f_2(x) \leqslant f_1(x)$ is satisfied (this figure is hatched in Fig. 233). The area S of such a figure can be computed in the following way:

$$S = S_{aA'B'b} - S_{aABb} = \int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx = \int_a^b (f_1(x) - f_2(x))\,dx.$$

Consider now a figure of the shape shown in Fig. 233 without the requirement that the functions $f_1(x)$ and $f_2(x)$ be nonnegative (Fig. 234). Let m be the least value of the function $y = f_2(x)$ in the

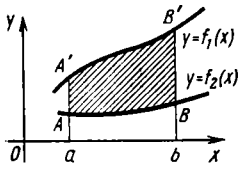


Fig. 233

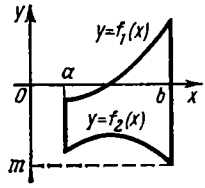


Fig. 234

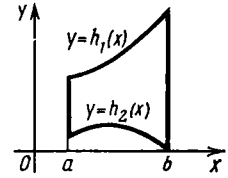


Fig. 235

interval $[a, b]$. Then the functions $h_1(x) = f_1(x) + |m|$ and $h_2(x) = f_2(x) + |m|$ are continuous and nonnegative in the interval $[a, b]$, and the area of the curvilinear trapezoid bounded by straight lines $x = a$ and $x = b$, and by the graphs of the functions $y = h_1(x)$ and $y = h_2(x)$ (Fig. 235) is computed, as is shown above, by the formula $S = \int_a^b (h_1(x) - h_2(x))\,dx$, that is,

$$S = \int_a^b ((f_1(x) + |m|) - (f_2(x) + |m|))\,dx = \int_a^b (f_1(x) - f_2(x))\,dx.$$

Since the given figure (Fig. 234) is equivalent to the obtained one (Fig. 235), the area S is computed by the formula

$$S = \int_a^b (f_1(x) - f_2(x))\,dx. \tag{10}$$

In particular, for the curvilinear trapezoid depicted in Fig. 226, we have

$$S = \int_a^b (f(x) - 0)\,dx = \int_a^b f(x)\,dx,$$

and for the curvilinear trapezoid represented in Fig. 236, we get

$$S=\int_a^b (0-f(x))\,dx=-\int_a^b f(x)\,dx.$$

Example 1. Compute the area of the figure bounded by the straight lines $y=x$, $y=5-x$, $x=1$, and $x=2$.

○The figure whose area is to be found is represented in Fig. 237. Making use of formula (10), we get

$$S=\int_1^2 ((5-x)-x)\,dx=\int_1^2 (5-2x)\,dx=\int_1^2 5\,dx-2\int_1^2 x\,dx$$

$$=5x\Big|_1^2-2\cdot\frac{x^2}{2}\Big|_1^2=5(2-1)-(4-1)=2.\ \bullet$$

Example 2. Compute the area of the figure bounded by the lines $y=x-2$ and $y=x^2-4x+2$.

○Constructing the straight line $y=x-2$ and the parabola $y=x^2-4x+2$, we get the figure whose area is required to be found (Fig. 238). We have $S=\int_a^b (f_1(x)-f_2(x))\,dx$, where $f_1(x)=x-2$, $f_2(x)=x^2-4x+2$, the limits of integration a and b being equal to the abscissas of the points of intersection of the parabola

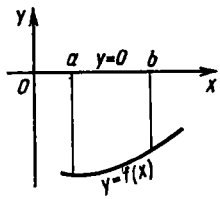


Fig. 236

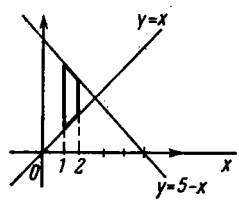


Fig. 237

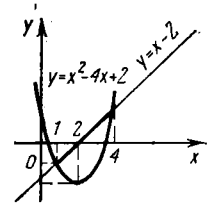


Fig. 238

and the straight line. To find the abscissas of these points, we solve the equation $f_1(x)=f_2(x)$, that is, $x-2=x^2-4x+2$. We have $x^2-5x+4=0$, whence $x_1=1$ and $x_2=4$. Thus, $a=1$ and $b=4$. Hence,

$$S=\int_1^4 ((x-2)-(x^2-4x+2))\,dx=\int_1^4 (5x-x^2-4)\,dx$$

$$=5\cdot\frac{x^2}{2}\Big|_1^4-\frac{x^3}{3}\Big|_1^4-4x\Big|_1^4=\frac{5}{2}(16-1)$$

$$-\frac{1}{3}(64-1)-4(4-1)=4.5\ \bullet$$

Example 3. Find the area of the figure bounded by the graphs of the functions $y = \sqrt{x}$, $y = |x - 2|$.

○ If $x \geqslant 2$, then $|x - 2| = x - 2$, if $x < 2$, then $|x - 2| = 2 - x$. Hence the function $y = |x - 2|$ may be rewritten as follows:

$$y = \begin{cases} x - 2 & \text{for } x \geqslant 2, \\ 2 - x & \text{for } x < 2, \end{cases}$$

Its graph is shown in Fig. 239*a*. The graph of the function $y = \sqrt{x}$ is represented in Fig. 239*b*, and the figure whose area we have to find in Fig. 239*c*. This figure is not a curvilinear trapezoid, but it can be

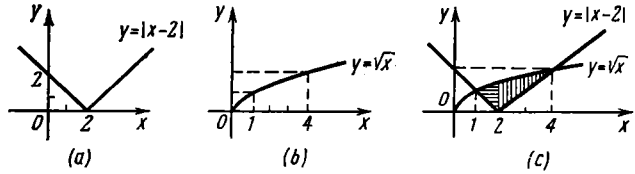


Fig. 239

represented as a combination of curvilinear trapezoids by drawing a straight line $x = 2$. Then the area S of the figure of interest is equal to $S_1 + S_2$, where S_1 and S_2 are the areas of the curvilinear trapezoids hatched in Fig. 239*c* horizontally and vertically, respectively. We have

$$S_1 = \int_1^2 (\sqrt{x} - (2 - x))\, dx \int_1^2 (x^{1/2} + x - 2)\, dx$$

$$= \frac{x^{3/2}}{3/2}\Big|_1^2 + \frac{x^2}{2}\Big|_1^2 - 2x\Big|_1^2$$

$$= \frac{2}{3}(2\sqrt{2} - 1) + \frac{1}{2}(4 - 1) - 2(2 - 1) = \frac{8\sqrt{2} - 7}{6},$$

$$S_2 = \int_2^4 (\sqrt{x} - (x - 2))\, dx = \frac{2x^{3/2}}{3}\Big|_2^4 - \frac{x^2}{2}\Big|_2^4 + 2x\Big|_2^4$$

$$= \frac{2}{3}(8 - 2\sqrt{2}) - \frac{1}{2}(16 - 4) + 2(4 - 2) = \frac{10 - 4\sqrt{2}}{3},$$

$$S = S_1 + S_2 = \frac{8\sqrt{2} - 7}{6} + \frac{10 - 4\sqrt{2}}{3} = \frac{13}{6}. \quad ●$$

7. Upper and Lower Integral Sums. Inequality (9) obtained in sub-

sectiod 5 yields only quite a rough estimate of the value of a definite integral. For example, in the interval [1, 5] the values of the function $y = x^2$ lie between 1 and 25, and therefore we have the inequality

$$4 = 1 \cdot (5-1) \leqslant \int_1^5 x^2\,dx \leqslant 25 \cdot (5-1) = 100.$$

In order to get a more exact estimate, we divide the interval $[a, b]$ into several parts by points $a = x_0 < x_1 < x_2 < \ldots x_{n-1} < x_n = b$ and apply inequality (9) to each part $[x_k, x_{k+1}]$. If the inequality $m_k \leqslant f(x) \leqslant M_k$ is satisfied in the interval $[x_k, x_{k+1}]$, then

$$m_k\,\Delta x_k \leqslant \int_{x_k}^{x_{k+1}} f(x)\,dx \leqslant M_k\,\Delta x_k.$$

where Δx_k denotes the difference $x_{k+1} - x_k$, that is, the length of the interval $[x_k, x_{k+1}]$. We write these inequalities for all values of k from 0 to $n-1$:

$$m_0\,\Delta x_0 \leqslant \int_{x_0}^{x_1} f(x)\,dx \leqslant M_0\,\Delta x_0,$$

$$m_1\,\Delta x_1 \leqslant \int_{x_1}^{x_2} f(x)\,dx \leqslant M_1\,\Delta x_1,$$

$$m_2\,\Delta x_2 \leqslant \int_{x_2}^{x_3} f(x)\,dx \leqslant M_2\,\Delta x_2,$$

$$\ldots \ldots \ldots \ldots \ldots \ldots$$

$$m_{n-1}\,\Delta x_{n-1} \leqslant \int_{x_{n-1}}^{x_n} f(x)\,dx \leqslant M_{n-1}\,\Delta x_{n-1}.$$

Adding these n inequalities together, we get

$$\sum_{k=0}^{n-1} m_k\,\Delta x_k \leqslant \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} f(x)\,dx \leqslant \sum_{k=0}^{n-1} M_k\,\Delta x_k.$$

According to Property 3° from subsection 5, the sum of the integrals over all the parts of the interval $[a, b]$ is equal to the

integral over this interval. Hence,

$$\sum_{k=1}^{n-1} \int_{x_k}^{x_{k+1}} f(x)\,dx = \int_a^b f(x)\,dx$$

and therefore

$$\sum_{k=0}^{n-1} m_k \Delta x_k \leqslant \int_a^b f(x)\,dx \leqslant \sum_{k=0}^{n-1} M_k \Delta x_k.$$

For example, if we divide the interval [1, 5] into ten equal parts of length 0.4 each, then in the interval $[1 + 0.4\,k,\ 1 + 0.4\,(k+1)]$ the inequality

$$(1 + 0.4\,k)^2 \leqslant x^2 \leqslant (1 + 0.4\,(k+1))^2$$

is satisfied.

Therefore we have

$$0.4 \cdot \sum_{k=0}^{9} (1+0.4k)^2 \leqslant \int_1^5 x^2\,dx \leqslant 0.4 \cdot \sum_{k=0}^{9} (1+0.4\,(k+1))^2.$$

Calculating, we find that $36.44 \leqslant \int_1^5 x^2\,dx \leqslant 46.04$. This estimate is much more exact than the one obtained before. In order to get a still more exact estimate of the integral, we have to divide the interval [1, 5] not into 10, but, say, into 100 or 1000 parts and compute the corresponding sums. Needless to say that the integral $\int_1^5 x^2\,dx$ can be easier calculated with the help of a primitive function. But if a primitive of $y = f(x)$ is unknown, we have to resort to some of the approximate methods of computing an integral.

The sums $\sum_{k=0}^{n-1} m_k \Delta x_k$ and $\sum_{k=0}^{n-1} M_k \Delta x_k$, which serve for estimating a definite integral from below and from above exist only provided that a function $y = f(x)$ is bounded in the interval $[a, b]$, that is, if there are numbers m and M such that $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$.

The numbers m_k and M_k for the same interval $[x_k, x_{k+1}]$ may be chosen in different ways. For example, if we have $2 \leqslant f(x) \leqslant 6$ in this interval, then the more so — $3 \leqslant f(x) \leqslant 100$. It is clear that it is most expedient to choose these numbers so that they provide the closest bounds for $f(x)$, that is, to choose m_k the greatest possible and M_k the least possible (the greatest lower and the least upper bounds for the values of the function in the given interval). If among the

values of the function $y = f(x)$ in the interval $[x_k, x_{k+1}]$ there are the least and the greatest values (this will be the case, for instance, if the function $y = f(x)$ is continuous in this interval), then they are the numbers m_k and M_k.

The sums $\sum_{k=0}^{n-1} m_k \Delta x_k$ and $\sum_{k=0}^{n-1} M_k \Delta x_k$, formed with m_k and M_k chosen as shown above are called the *lower* and the *upper Darboux* integral sums* for the given partition of the interval $[a, b]$. We already know that for any partition the double inequality

$$\sum_{k=0}^{n-1} m_k \Delta x_k \leqslant \int_a^b f(x)\, dx \leqslant \sum_{k=0}^{n-1} M_k \Delta x_k$$

holds. In other words, if we take all possible partitions of the interval $[a, b]$ and denote by X the set of all lower Darboux sums thus obtained and by Y the set of all upper Darboux sums, then the number $\int_a^b f(x)\, dx$ lies in between these sets, it separates them.

We can prove for an extremely wide class of functions that the definite integral of them is the only number that separates the lower and the upper Darboux sums. For example, this is true for all continuous functions.

Exercises

1. Compute:

(a) $\int_1^4 \frac{dx}{\sqrt{x}}$, (b) $\int_0^{\pi/2} \sin 2x\, dx$, (c) $\int_{-\pi/3}^{\pi/4} \left(\frac{2}{\sin^2 x} + \frac{3}{\cos^2 x} \right) dx$.

2. Find the areas of the figures bounded by the given lines: (a) $y = 4x - x^2$ and $y = 0$, (b) $y = x^2 - 6x + 8$ and $y = 0$, (c) $y = x^3 - 1$, $x = 0$, and $y = 0$, (d) $y = \cos 5x$, $y = 0$, $x = 0$, and $x = \pi/10$.

3. A particle moves in a straight line so that its velocity at an instant t is equal to $v(t) = 10 - 0.2t$ (t in seconds, v in m/s). Find the distance covered by the particle during the time interval from 3 s to 10 s.

4. A particle moves in a straight line so that its velocity at an instant t is equal to $v(t) = t^2 - t + 1$. It is known that at the initial instant $t = 0$ the particle has the coordinate -1. Find the coordinate and the acceleration of the particle at the instant t.

5. A particle moves in a straight line, its acceleration at an instant t being equal to $1 + \sin t$. Find the coordinate as a function of time if at the instant $t = 0$ the coordinate is equal to 1 and the velocity to 1.

In Exercises 6 to 8, find the area of the figure bounded by the given lines:

6. $y = \cos 3x$, $y = 1$, $x = \pi/3$.

7. $y = x^2 - 9$, $y = -x^2 - 2x + 3$.

8. $y = 4 - |x - 2|$, $y = -\sqrt{x + 2}$, $x = 6$.

* Darboux, Jean Gaston (1842-1917), French mathematician.

Chapter 9

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

§ 9.1. Exponential Function

1. Function $y = a^x$ ($x \in Q$, $a > 1$). Let a be an arbitrary real number such that $a > 1$. Whatever rational number r is taken, it is possible to compute a^r. Thus, on the set Q of all rational numbers a function $y = a^x$ is defined. It is called the *exponential function*. We already know (see § 2.4, Item 5) the following properties of this function: for any x_1, $x_2 \in Q$ we have

1^0. $a^{x_1} \cdot a^{x_2} = a^{x_1+x_2}$.

2^0. $a^{x_1} \div a^{x_2} = a^{x_1-x_2}$.

3^0. $(a^{x_1})^{x_2} = a^{x_1 x_2}$.

Let us consider some more properties of the exponential function with the base $a > 1$.

4^0. *If* $x > 0$, *then* $a^x > 1$.

□ Let $x = p/q$, where p, $q \in N$, $q \neq 1$, then $a^x = a^{p/q} = \sqrt[q]{a^p}$. Since for $x > 0$ the functions $y = x^p$ and $y = \sqrt[q]{x}$ are increasing, $a > 1 \Rightarrow a^p > 1^p \Rightarrow \sqrt[q]{a^p} > \sqrt[q]{1^p}$. But $\sqrt[q]{1^p} = 1$, hence, $a^x > 1$. ■

5^0. *If* $x < 0$, *then* $0 < a^x < 1$.

□ Let $x = -r$, then $r > 0$. According to 4^0, we have $a^r > 1$. Then $0 < 1/a^r < 1$, that is, $0 < a^{-r} < 1$. This just means that $0 < a^x < 1$. ■

6^0. *The function takes on only positive values.*

This directly follows from Properties 4^0 and 5^0 and also from the fact that for $x = 0$ we get $a^0 = 1$.

7^0 *The function is increasing on the set* Q.

□ Let x_1, $x_2 \in Q$ and $x_1 < x_2$, then $f(x_1) = a^{x_1}$, $f(x_2) = a^{x_2}$. Let us show that $f(x_1) < f(x_2)$.

We have $f(x_2) - f(x_1) = a^{x_2} - a^{x_1} = a^{x_1}(a^{x_2-x_1} - 1)$. Since $x_2 - x_1 > 0$, we have $a^{x_2-x_1} > 1$ (by Property 4^0), and therefore $a^{x_2-x_1} - 1 > 0$. In addition, by Property 6^0 we have $a^{x_1} > 0$. Hence, $a^{x_1}(a^{x_2-x_1} - 1) > 0$, that is, $f(x_2) - f(x_1) > 0$, whence $f(x_1) < f(x_2)$. ■

Note that the function increases gradually, without jumps. In other words, if the points x_1 and x_2 are chosen sufficiently close to each other, then a^{x_1} and a^{x_2} differ from each other but slightly.

This result is formulated more exactly as the following property (given without proof).

8^0. *For any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$|x_1 - x_2| < \delta \Rightarrow |a^{x_1} - a^{x_2}| < \varepsilon.$$

2. Function $y = a^x$ ($x \in \mathbf{Q}$, $0 < a < 1$). If $0 < a < 1$, then, as above, on the set $\mathbf{Q}$ there is defined an exponential function satisfying Properties 1^0, 2^0, and 3^0. Let us note some more properties of this function.

Let us set $b = 1/a$, then $b > 1$. For the function b^x, on the basis of the results of the preceding item, we have: $x > 0 \Rightarrow b^x > 1$, $x < 0 \Rightarrow 0 < b^x < 1$, $x = 0 \Rightarrow b^x = 1$. But from $b^x > 1$ it follows that $(1/a)^x > 1$, that is, $0 < a^x < 1$, from $0 < b^x < 1$ it follows: $0 < (1/a)^x < 1$, that is $a^x > 1$, if $x = 0$, then $a^x = 1$.

Thus, we get the following three properties of the exponential function with base $0 < a < 1$:

4^0. *If* $x > 0$, *then* $0 < a^x < 1$.

5^0. *If* $x < 0$, *then* $a^x > 1$.

6^0. *The function takes on only positive values.*

7^0. *The function is decreasing on the set* $\mathbf{Q}$.

□ Indeed, setting, as above, $b = 1/a$ and using the fact that the function b^x is increasing, we get

$$x_1 < x_2 \Rightarrow b^{x_1} < b^{x_2} \Rightarrow \left(\frac{1}{a}\right)^{x_1} < \left(\frac{1}{a}\right)^{x_2} \Rightarrow a^{x_1} > a^{x_2}.$$

Thus, $x_1 < x_2 \Rightarrow a^{x_1} > a^{x_2}$, and this just means that the considered function decreases on the set $\mathbf{Q}$. ■

8^0. *For any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$|x_1 - x_2| < \delta \Rightarrow |a^{x_1} - a^{x_2}| < \varepsilon.$$

Remark. We have considered the exponential function $y = a^x$, $x \in \mathbf{Q}$ in two cases: $a > 1$ and $0 < a < 1$. The case $a = 1$ is usually not considered, since here we get the function 1^x taking on the only value 1, such a function is of no interest. The case $a \leqslant 0$ is not considered either, since the raising of nonpositive numbers to a rational power is not always possible.

3. Power with an Irrational Exponent. Let $a > 0$ and α be an irrational number. Let us define what will be understood by the notation a^α and what sense should be put in the notion of a power of a positive number with an irrational exponent.

If $a = 1$, then we set $1^\alpha = 1$.

Let $a > 1$. Consider two sets A and B of numbers: the set A consists of all numbers of the form a^r, where r is a rational number such that $r < \alpha$, the set B consists of all numbers of the form $a^{\tilde{r}}$, where $\tilde{r}$ is a rational number such that $\tilde{r} > a$. Since $r < \tilde{r}$, we have $a^r < a^{\tilde{r}}$ (this follows from the increase in the function $y = a^x$,

$x \in \mathbf{Q}$, $a > 1$). Hence, any element of the set A is less than any element of the set B, that is, the set A is situated to the left of the set B, and, consequently (see § 1.3, Item 11), there is a number c separating these sets: $a^r < c < a^{\tilde{r}}$ for any r and $\tilde{r}$. According to Property 8° from Item 1, the numbers $a^{\tilde{r}}$ and a^r can be chosen arbitrarily close to each other; then the number c separating the sets A and B is unique. It is taken as a^α.

Thus, we have arrived at the following definition: a^α, where $a > 1$, and α is an irrational number, is understood as the only number separating two sets: the set A consisting of numbers of the form a^r, where $r < \alpha$, r is a rational number, and the set B consisting of numbers of the form $a^{\tilde{r}}$, where $\tilde{r} > \alpha$, $\tilde{r}$ is a rational number.

This definition implies that for $a > 1$

$$r_1 < \alpha < r_2 \Rightarrow a^{r_1} < a^\alpha < a^{r_2}. \tag{1}$$

A power with an irrational exponent for $0 < a < 1$ is defined in a similar way, only in this case the designations for the sets A and B are interchanged since the function $y = a^x$, $x \in \mathbf{Q}$ is decreasing. Hence it follows that for $0 < a < 1$

$$r_1 < \alpha < r_2 \Rightarrow a^{r_2} < a^\alpha < a^{r_1}. \tag{2}$$

It is possible to prove that for such a definition of the notion of a power with an irrational exponent for any real numbers x_1, x_2 and for any positive numbers a, b the following equalities hold:

1°. $a^{x_1} \cdot a^{x_2} = a^{x_1+x_2}$.

2°. $a^{x_1} \div a^{x_2} = a^{x_1-x_2}$.

3°. $(a^{x_1})^{x_2} = a^{x_1 x_2}$.

4°. $a^{x_1} \cdot b^{x_1} = (ab)^{x_1}$.

5°. $\dfrac{a^{x_1}}{b^{x_1}} = \left(\dfrac{a}{b}\right)^{x_1}$.

4. Exponential Function $y = a^x$ ($x \in \mathbf{R}$, $a > 1$). Its Properties and Graph. In Item 1, we considered the exponential function on the set $\mathbf{Q}$ of rational numbers. Now, with the notion of a power with an irrational exponent introduced, we can study the function $y = a^x$ defined on the set $\mathbf{R}$ of real numbers. Here, we shall consider the case when $a > 1$.

1°. *The domain of definition is the set of all real numbers*: $D(f) = \mathbf{R}$.

2°. *The function is increasing.*

□ Let $x_1 < x_2$. Let us prove that $a^{x_1} < a^{x_2}$. If x_1 and x_2 are rational numbers, then $a^{x_1} < a^{x_2}$ which has already been proved (see Property 7° from Item 1). If one of the numbers is rational and the other irrational, then the inequality $a^{x_1} < a^{x_2}$ follows

from inequality (1) obtained in Item 3. Let, finally, x_1 and x_2 be irrational numbers. Then, choosing between them a rational number r, that is, $x_1 < r < x_2$, we get $a^{x_1} < a^r$, $a^r < a^{x_2}$, whence $a^{x_1} < a^{x_2}$.

Thus, in any case $x_1 < x_2 \Rightarrow a^{x_1} < a^{x_2}$. ■

Corollary. *If* $a^{x_1} = a^{x_2}$, *then* $x_1 = x_2$.

□ Indeed, suppose that $x_1 < x_2$, then, according to Property 2^0, we get $a^{x_1} < a^{x_2}$, and from $x_1 > x_2$ it follows that $a^{x_1} > a^{x_2}$. This contradicts the hypothesis, that is, $x_1 = x_2$. ■

3^0. *If* $x \to +\infty$, *then* $a^x \to +\infty$.

□ Let us take an arbitrary number $M > 0$. We have to prove that there is a number $P > 0$ such that it follows from $x > P$ that $a^x > M$ (see § 8.2, Item 3).

Let us set $a = 1 + h$. Since $a > 1$, then $h > 0$. Using the Bernoulli inequality (see § 7.1, Item 2), we get

$$(1 + h)^n > 1 + nh, \quad n \in \mathbf{N}.$$

It is clear that for a sufficiently large $n = P$ the inequality $1 + Ph > M$ is satisfied, then $a^P = (1 + h)^P > M$. Since the exponential function $y = a^x$ is increasing, it follows from $x > P$ that $a^x > a^P$. Thus, $x > P \Rightarrow a^x > M$. ■

4^0. *If* $x \to -\infty$, *then* $a^x \to 0$.

□ We set $t = -x$. By Property 3^0, if $t \to +\infty$, then $a^t \to +\infty$. Further, we have $a^x = a^{-t} = 1/a^t$. If $x \to -\infty$, then $t \to +\infty$, $a^t \to +\infty$, $1/a^t \to 0$, that is, $a^x \to 0$. ■

We have proved that $\lim\limits_{x \to -\infty} a^x = 0$. This means that the straight line $y = 0$ (the x-axis) is a horizontal asymptote to the graph of the exponential function $y = a^x$ as $x \to -\infty$.

5^0. *The function is continuous on the set* **R** *of all real numbers.*

This property is given without proof.

6^0. *The range of the exponential function is the set of all positive numbers*: $E(f) = (0, +\infty)$.

This follows directly from Properties 3^0, 4^0, and 5^0 and also from the theorem on an intermediate value of a continuous function (see § 8.1, Item 6).

Using the obtained results, we construct the graph of the function $y = a^x$ for $a > 1$. Let, for instance, $a = 2$. We compile a table of values of the function $y = 2^x$:

x	-2	-1	0	1	2	3
y	$1/4$	$1/2$	1	2	4	8

Using the found check points, we construct the graph of the function $y = 2^x$ (Fig. 240).

5. Exponential Function $y = a^x$ ($x \in \mathbf{R}$, $0 < a < 1$). Its Properties and Graph. Let $b = 1/a$. Then $0 < a < 1 \Rightarrow b > 1$, and

from the properties of the function $y = b^x$, $b > 1$ it is easy to obtain the properties of the function $y = a^x$ we are interested in. Let us list these properties.

1^0. $D(f) = \mathbf{R}$.

2^0. *The function is decreasing.*

3^0. *If* $x \to +\infty$, *then* $a^x \to 0$.

This means that the straight line $y = 0$ (the x-axis) is a horizontal asymptote to the graph of the function $y = a^x$ as $x \to +\infty$.

4^0. *If* $x \to -\infty$, *then* $a^x \to +\infty$.

5^0. *The function is continuous on the set* $\mathbf{R}$ *of all real numbers.*

6^0. $E(f) = (0, +\infty)$.

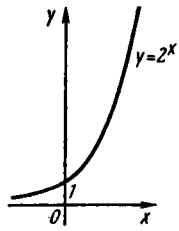


Fig. 240

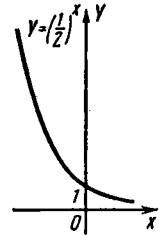


Fig. 241

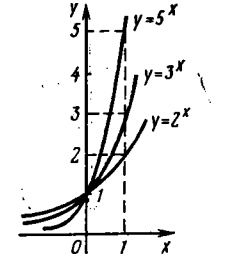


Fig. 242

Using the obtained results, we construct the graph of the function $y = a^x$ for $0 < a < 1$. Let, for instance, $a = 1/2$. We compile a table of values of the function $y = \left(\frac{1}{2}\right)^x$:

x	-2	-1	0	1	2	3
y	4	2	1	$1/2$	$1/4$	$1/8$

Using the found check points, we construct the graph of the function $y = \left(\frac{1}{2}\right)^x$ (Fig. 241).

Example 1. Which of the two numbers is greater: $a = 12^{-0.7}$ or $b = 12^{-0.8}$?

○ The function $x = 12^x$ is increasing, since $-0.7 > -0.8$, we have $12^{-0.7} > 12^{-0.8}$. ●

Example 2. Which of the intervals, $(0, 1)$ or $(1, +\infty)$, does the number $a = \left(\frac{1}{4}\right)^{-1/3}$ belong to?

○ The function $y = \left(\frac{1}{4}\right)^x$ is decreasing. Since $-1/3 < 0$, we have $\left(\frac{1}{4}\right)^{-1/3} > \left(\frac{1}{4}\right)^0$, that is, $\left(\frac{1}{4}\right)^{-1/3} > 1$. Consequently, $a \in (1, +\infty)$. ●

Example 3. Which of the intervals, $(-\infty, 0)$, $(0, 1)$ or $(1, +\infty)$, does the number a belong to if $\left(\frac{2}{7}\right)^a = \frac{3}{5}$?

○ We have $\frac{2}{7} < \frac{3}{5} < 1$, that is $\left(\frac{2}{7}\right)^1 < \left(\frac{2}{7}\right)^a < \left(\frac{2}{7}\right)^0$. Since the function $y = \left(\frac{2}{7}\right)^x$ is decreasing, the last inequality implies $1 > a > 0$. Thus, $a \in (0, 1)$. ●

6. Number e. Differentiating the Exponential Function $y = e^x$. The Notion of Differential Equation. When describing physical laws, it is convenient to use a specific number which is called the *number e*. This number can be defined in the following way. Let us graph the functions $y = 2^x$, $y = 3^x$, $y = 5^x$ in rectangular Cartesian coordinates (Fig. 242). We note that the greater the base, the lesser the angle formed by the graph of the function and the y-axis at the point $B(0, 1)$ (the angle between the curve $y = f(x)$ and the y-axis is understood as the angle between the tangent to the curve $y = f(x)$ at the point $x = 0$ and the y-axis). For example, the angle between the y-axis and the graph of $y = 2^x$ is approximately equal to 55°, and the angle between the y-axis and the graph of the function $y = 3^x$ is about 42°. It is natural to assume that between 2 and 3 there is a number such that the graph of the exponential function whose base is this number intersects the y-axis at an angle of 45°. This number is denoted by the letter e. It is irrational, the calculations show that $e = 2.7182818284590....$

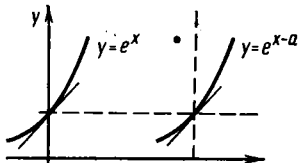


Fig. 243

By definition, the tangent to the graph of the function $y = e^x$ at the point $x = 0$ forms an angle of 45° with the coordinate axes. Consequently, for the function $y = e^x$ we have $f'(0) = \tan 45° = 1$. Let us now consider the function $g(x) = e^{x-a}$. The graph of this function is obtained from the graph of the function e^x by translating the latter by a units along the axis of abscissas (Fig. 243). When displaced, the tangent to the graph $y = e^x$ at the point $x = 0$ changes into the tangent line to the shifted curve at the point $x = a$. Therefore for the function $g(x) = e^{x-a}$ we have $g'(a) = \tan 45° = 1$. Since $f(x) = e^x = e^a e^{x-a} = e^a g(x)$, we have $(e^x)' = (e^a g(x))' = e^a g'(x)$, and, consequently,

$$f'(a) = (e^x)'|_{x=a} = e^a g'(a) = e^a.$$

Thus, we have shown that the derivative of the function $y = e^x$ at the point $x = a$ is equal to e^a. Since a is an arbitrary number, writing x instead of a, we get

$$(e^x)' = e^x. \tag{3}$$

Thus, the *derivative of the exponential function* $y = e^x$ *at any point is equal to the value of the function at this point.*

Example. Prove that for $x \geqslant 0$ the following inequality holds:

$$e^x \geqslant 1 + x.$$

○Consider the function $f(x) = e^x - 1 - x$ and prove that it increases on the open interval $(0, +\infty)$. For this purpose, we find the derivative

$$f'(x) = (e^x - 1 - x)' = e^x - 1.$$

For $x \geqslant 0$ we have $e^x \geqslant e^0$, that is, $e^x \geqslant 1$. Hence, $f'(x) \geqslant 0$ on the interval $(0, +\infty)$, the equality sign occurring only for $x = 0$. Thus, $f(x)$ is an increasing function, that is, from the inequality $x \geqslant 0$ there follows the inequality

$$e^x - 1 - x \geqslant e^0 - 1 - 0 \text{ or } e^x - 1 - x \geqslant 0,$$

whence we get $e^x \geqslant 1 + x$. ●

From formula (3) and the rule for differentiating a composite function it follows that $(e^{kx})' = ke^{kx}$. This equality shows that the function $y = e^{kx}$ is one of the solutions of the equation

$$y' = ky. \tag{4}$$

Equations containing functions and their derivatives are called *differential equations.* It is easy to verify that for any C the function $y = Ce^{kx}$ is also a solution of equation (4). Thus, this equation has infinitely many solutions depending on one parameter C. We can prove that, besides $y = Ce^{kx}$, the equation has no other solution.

The differential equation $y' = ky$ shows that the rate of variation of the quantity y is proportional to the value of this quantity. This variation is called *exponent growth.* According to this law, deposits grow at a savings bank (the larger the deposit, the larger the sum computed as interest due at the end of a certain period), the same law governs the growth of a population of animals having an unlimited supply of fodder (the more individuals, the greater the posterity), and so forth.

For $k < 0$ we are concerned with a decreasing quantity whose rate of decrease is proportional to its value. Such a quantity can be exemplified by the mass of a radioactive substance (the more the mass of the substance, the larger the number of atoms decaying during a given interval of time).

In conclusion, note that from the formula $(e^x)' = e^x$ we get the following integration formula:

$$\int e^x \, dx = e^x + C.$$

Exercises

1. Graph the following functions: (a) $y=4^x$, (b) $y=\left(\frac{1}{3}\right)^x$, (c) $y=2\cdot e^x$, (d) $y=2^x-5$, (e) $y=3^{x-2}$, (f) $y=\left(\frac{1}{2}\right)^{x+2}-1$, (g) $y=|e^x-1|$, (h) $y=e^{|x|}$.

2. What can be concluded about the sign of the number x if: (a) $3^x=0.9$, (b) $\left(\frac{1}{6}\right)^x=10$, (c) $10^x=4$, (d) $0.3^x=0.1$?

3. Which is greater α or β if $1.34^{\alpha}<1.34^{\beta}$?

4. What can be said about the number a if $a^{-2}>a^{-0.6}$?

5. Show graphically that the equation $2^x=\cos x$ has infinitely many roots.

6. Find the derivatives of the following functions:

(a) $y=\frac{e^x-1}{e^{x+1}}$, (b) $y=e^{\sin x}$, (c) $y=\tan e^x$.

7. Test the function $y=e^{x^2-4x+1}$ for monotonicity.

8. Compute the area of the figure bounded by the lines $y=e^x$, $y=1$, and $x=2$.

§ 9.2. Logarithmic Function

1. Logarithmic Function. Consider the exponential function $y=a^x$, where $a>0$ and $a\neq 1$. In the preceding section, we noted a number of properties of this function including

1^0. $D(f)=\mathbf{R}$.

2^0. $E(f)=(0, +\infty)$.

3^0. *$y=a^x$ is a monotonic function (increasing for $a>1$ and decreasing for $0<a<1$).*

4^0. *$y=a^x$ is a function continuous on* $\mathbf{R}$.

The first three properties ensure for the exponential function the existence and monotonicity of the inverse function $x=f^{-1}(y)$ defined on $(0, +\infty)$ and having the set $\mathbf{R}$ as its range (see the theorem from § 3.4). Since the function $y=a^x$ is continuous on $\mathbf{R}$, the inverse function is also continuous in its domain of definition (see the theorem from § 8.1).

This inverse function is called the *logarithmic function* and is denoted as follows: $x=\log_a y$ (read: "the logarithm of the number y to the base a"). Interchanging, as usually, x and y, we get $y=\log_a x$.

According to the foregoing, the logarithmic function $y=\log_a x$ possesses the following properties:

1^0. $D(f)=(0, +\infty)$, that is, the *domain of definition of the logarithmic function is represented by the set of all positive numbers.*

2^0. $E(f)=\mathbf{R}$, that is, the *range of the logarithmic function coincides with the set* $\mathbf{R}$ *of real numbers.*

3^0. *$y=\log_a x$ is a monotonic function: increasing if $a>1$ and decreasing if $0<a<1$.*

4^0. *$y=\log_a x$ is a function continuous on* $(0, +\infty)$.

From the monotonicity of the logarithmic function $y = \log_a x$ it follows, in particular, that if $\log_a x_1 = \log_a x_2$, then $x_1 = x_2$ (*if the logarithms of two positive numbers to the same base are equal, then the numbers themselves are also equal*).

The graph of the function $y = \log_a x$ can be obtained from the graph of the function $y = a^x$ using symmetry transformation of the latter with respect to the straight line $y = x$. Logarithmic functions constructed for the cases $a > 1$ and $0 < a < 1$ are shown in Fig. 244 and Fig. 245, respectively. Note that the straight line

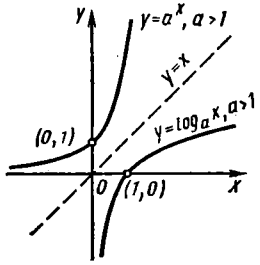


Fig. 244

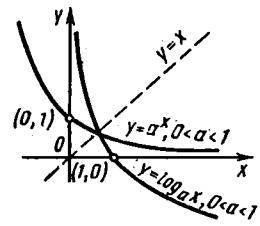


Fig. 245

$x = 0$ (the y-axis) is a vertical asymptote to the graph of the logarithmic function. In addition, we can note that the graph of the logarithmic function intersects the x-axis at one point (1, 0), that is, the equation $\log_a x = 0$ has the only solution $x = 1$.

2. Properties of Logarithms. We noted above that $y = \log_a x$ and $y = a^x$ are two mutually inverse functions. This means, in particular, that the equalities $y = \log_a x$ and $x = a^y$ express one and the same relation between x and y (as, for instance, in the case of the equalities $y = x^n$ and $x = \sqrt[n]{y}$). Thus, $y = \log_a x \Leftrightarrow x = a^y$.

Definition. The *logarithm* of a positive number x to the base a is defined as the exponent to which it is necessary to raise the number a to produce the number x (here and elsewhere in this chapter, it is assumed that $a > 0$ and $a \neq 1$):

$$a^{\log_a x} = x. \tag{1}$$

For example, $\log_3 81 = 4$ since $3^4 = 81$, $\log_{10} 0.001 = -3$ since $10^{-3} = 0.001$, $\log_{1/2} \sqrt{2} = -\frac{1}{2}$, since $\left(\frac{1}{2}\right)^{-1/2} = 2^{1/2} = \sqrt{2}$.

Two important equalities follow from the definition of logarithm:

$$\log_a 1 = 0, \quad \log_a a = 1.$$

The first equality follows from the fact that $a^0 = 1$, and the second from the fact that $a^1 = a$.

Note the basic properties of logarithms.

1°. *If* $x_1 > 0$ *and* $x_2 > 0$, *then*

$$\log_a (x_1 x_2) = \log_a x_1 + \log_a x_2$$

(the logarithm of the product of any two positive numbers (factors) is equal to the sum of the logarithms of the factors).

□ Let us set $y = \log_a (x_1 x_2)$, $y_1 = \log_a x_1$, $y_2 = \log_a x_2$. We have to prove that $y = y_1 + y_2$.

From the equality $y = \log_a (x_1 x_2)$ it follows that $a^y = x_1 x_2$. Similarly, from the equality $y_1 = \log_a x_1$ it follows that $a^{y_1} = x_1$, and from the equality $y_2 = \log_a x_2$ that $a^{y_2} = x_2$. We then have

$$a^y = x_1 x_2 = a^{y_1} \cdot a^{y_2} = a^{y_1 + y_2}.$$

But the equality $a^y = a^{y_1 + y_2}$ implies the equality $y = y_1 + y_2$ (see p. 384) which was required to be proved. ■

Property 1° is extended to the product of any number of positive factors.

Let us give a brief notation of the proof given above:

$$\left.\begin{array}{l|l} \log_a (x_1 x_2) = y, & a^y = x_1 x_2 \\ \log_a x_1 = y_1, & a^{y_1} = x_1, \\ \log_a x_2 = y_2. & a^{y_2} = x_2 \end{array}\right\} a^y = a^{y_1} \cdot a^{y_2} \Rightarrow a^y = a^{y_1 + y_2} \Rightarrow y = y_1 + y_2.$$

Prove:
$y = y_1 + y_2$.

2°. *If* $x_1 > 0$ *and* $x_2 > 0$, *then*

$$\log_a \frac{x_1}{x^2} = \log_a x_1 - \log_a x_2$$

(the logarithm of the quotient of two positive numbers is equal to the difference of the logarithms of the dividend and divisor).

□

$$\left.\begin{array}{l|l} \log_a \frac{x_1}{x_2} = y, & a^y = \frac{x_1}{x_2}, \\ \log_a x_1 = y_1, & a^{y_1} = x_1, \\ \log_a x_2 = y_2. & a^{y_2} = x_2 \end{array}\right\} a^y = \frac{a^{y_1}}{a^{y_2}} \Rightarrow a^y = a^{y_1 - y_2} \Rightarrow y = y_1 - y_2. \blacksquare$$

Prove:
$y = y_1 - y_2$.

3°. *If* $x > 0$, *then*

$$\log_a x^k = k \log_a x$$

(the logarithm of a power with a positive base is equal to the product of the exponent and the logarithm of the base).

□

$$\left.\begin{array}{l|l} \log_a x^k = y, & a^y = x^k, \\ \log_a x = z. & a^z = x \end{array}\right\} a^y = (a^z)^k \Rightarrow a^y = a^{zk} \Rightarrow y = kz. \blacksquare$$

Prove:
$y = kz$.

4^0. *If* $x>0$, *then*

$$\log_a x=\frac{\log_b x}{\log_b a}$$

(formula for change of base of logarithms, here, as usual, $b>0$ and $b\neq 1$).

□

$\log_a x=y,\ \log_b x=y_1,\ \log_b a=y_2.$

Prove:

$y=\frac{y_1}{y_2}.$

$a^y=x,\ b^{y_1}=x,\ b^{y_2}=a$

$a^y=x\Rightarrow(b^{y_2})^y=b^{y_1}\Rightarrow y_2y=y_1\Rightarrow y=\frac{y_1}{y_2}.$ ■

5^0. *If* $x>0$, *then*

$$\log_a x=\log_{a^k}x^k\quad(k\neq 0)$$

(*the value of the logarithm of a number remains unchanged if the base of the logarithm and the number are raised to the same power*).

□ Let us transform the right-hand member into the logarithm to base a. Using Property 4^0, we get

$$\log_{a^k}x^k=\frac{\log_a x^k}{\log_a a^k}.$$

According to Property 3^0, we have

$$\log_a x^k=k\log_a x,\quad \log_a a^k=k\log_a a=k.$$

Consequently,

$$\log_{a^k}x^k=\frac{k\log_a x}{k}=\log_a x.$$ ■

Example 1. Compute $\log_3 6$ if $\log_3 2=a$.

○ Using Property 1^0, we have

$$\log_3 6=\log_3(2\cdot 3)=\log_3 2+\log_3 3=a+1.$$ ●

Example 2. Compute $\log_2\sqrt[3]{\frac{3}{4}}$ if $\log_2 3=a$.

○ Using Properties 2^0 and 3^0 we find

$$\log_2\sqrt[3]{\frac{3}{4}}=\log_2\left(\frac{3}{4}\right)^{1/3}=\frac{1}{3}\log_2\frac{3}{4}=\frac{1}{3}(\log_2 3-\log_2 4)=\frac{1}{3}(a-2).$$ ●

Example 3. Compute $\log_5 6$ if $\log_2 3=a$, $\log_2 10=b$.

○ Let us change $\log_5 6$ to the base 2. Using Property 4^0 and then Properties 1^0 and 2^0, we get

$$\log_5 6=\frac{\log_2 6}{\log_2 5}=\frac{\log_2(2\cdot 3)}{\log_2(10/2)}=\frac{\log_2 2+\log_2 3}{\log_2 10-\log_2 2}=\frac{1+a}{b-1}.$$ ●

Example 4. Compute $\log_{\sqrt[3]{2}} \sqrt[6]{32}$.

○ According to Property 5°, the base of a logarithm and the number whose logarithm is taken may be raised to the same power. Consequently,

$$\log_{\sqrt[3]{2}} \sqrt[6]{32} = \log_{(\sqrt[3]{2})^3} (\sqrt[6]{32})^3 = \log_2 \sqrt{32} = \log_2 2^{5/2} = \frac{5}{2}. \quad \bullet$$

Example 5. Compute $49^{1-\frac{1}{4}\log_7 25}$.

○ We have

$$49^{1-\frac{1}{4}\log_7 25} = (7^2)^{1-\frac{1}{4}\log_7 25} = 7^{2-\frac{1}{2}\log_7 25}.$$

Let us transform the exponent of the obtained power:

$$2 - \frac{1}{2}\log_7 25 = 2 \cdot \log_7 7 - \log_7 25^{1/2} = \log_7 7^2 - \log_7 5 = \log_7 \frac{49}{5}.$$

Thus, $7^{2-\frac{1}{2}\log_7 25} = 7^{\log_7 \frac{49}{5}}$. From the definition of logarithm (see equality (1)) it follows that $7^{\log_7 \frac{49}{5}} = \frac{49}{5}$. $\bullet$

3. Taking Logarithms and Antilogarithms. If some expression A is made up from positive numbers using the operations of multiplication, division, and involution (raising a number to a certain power), then, using the properties of logarithms, $\log_a A$ can be expressed in terms of the logarithms of the numbers in expression A. Such transformation (operation) is called *taking logarithms*.

Example 1. Take the logarithm to base 5 of the expression $\frac{125\,a^3b^2}{\sqrt{c}}$, where a, b, and c are positive numbers.

○ Using Properties 1°-3° of logarithms, we get

$$\begin{aligned}\log_5 \frac{125a^3b^2}{\sqrt{c}} &= \log_5 (125a^3b^2) - \log_5 \sqrt{c} \\ &= \log_5 125 + \log_5 a^3 + \log_5 b^2 - \log_5 c^{1/2} \\ &= \log_5 5^3 + 3\log_5 a + 2\log_5 b - \frac{1}{2}\log_5 c \\ &= 3 + 3\log_5 a + 2\log_5 b - \frac{1}{2}\log_5 c. \quad \bullet\end{aligned}$$

In practice, we often have to solve an inverse problem, that is, to find the expression whose logarithm is represented in terms of logarithms of some numbers. Such transformation is referred to as *taking antilogarithms* (or *inverse logarithms*).

Example 2. Find x if

$$\log_3 x = 2\log_3 5 + \frac{1}{2}\log_3 8 - 3\log_3 10.$$

○ We have

$$2\log_3 5 + \frac{1}{2}\log_3 8 - 3\log_3 10 = \log_3 5^2 + \log_3 8^{1/2} - \log_3 10^3$$

$$= \log_3 \frac{25\cdot 2\sqrt{2}}{1000} = \log_3 \frac{\sqrt{2}}{20}.$$

From the equation $\log_3 x = \log_3 (\sqrt{2}/20)$ we find: $x = \sqrt{2}/20$. ●

4. Additional Remarks on Properties of Logarithms. In Item 2, we considered the properties of logarithms valid for positive values of the variables in the logarithm sign. However, when solving examples, one should take into consideration that the variables may be negative, and then the properties of logarithms must be used with care.

For example, if $x_1 < 0$ and $x_2 < 0$, then we cannot write

$$\log_a (x_1 x_2) = \log_a x_1 + \log_a x_2,$$

since the right-hand side of such "equality" has no sense (the logarithm of a negative number does not exist). We may reason here as follows: x_1 and x_2 are negative numbers, consequently, $x_1 x_2 > 0$. But then $x_1 x_2 = |x_1 x_2| = |x_1| \cdot |x_2|$. Hence

$$\log_a (x_1 x_2) = \log_a |x_1 x_2| = \log_a (|x_1| \cdot |x_2|).$$

Since $|x_1| > 0$ and $|x_2| > 0$, applying Property 1° from Item 2, we get

$$\log_a (|x_1| \cdot |x_2|) = \log_a |x_1| + \log_a |x_2|.$$

Thus, *if* $x_1 x_2 > 0$, *then*

$$\log_a (x_1 x_2) = \log_a |x_1| + \log_a |x_2|$$

and, analogously,

$$\log_a \frac{x_1}{x_2} = \log_a |x_1| - \log_a |x_2|$$

(it is easy to note that both of the last equalities also hold in the case when $x_1 > 0$, $x_2 > 0$).

Property 3° is also extended to the case of a negative value of the variable x when k is an even number. Namely, the following assertion is valid: *if k is an even number, then*

$$\log_a x^k = k \log_a |x|$$

for any $x \neq 0$.

5. Common (or Briggs) Logarithms. Logarithms to base 10 are called *common* (or *Briggs*) *logarithms*. They are frequently used in practice. This base is so convenient and so common that it is not

usually written as a subscript but is understood if no base is shown. The common logarithm of the number x is denoted by the symbol $\log x$ (but not $\log_{10} x$). In what follows, we shall justify why common logarithms are mainly used for computation.

Let us first of all introduce the notion of standard form of a positive number. Any positive number can be represented in the form $a_1 \cdot 10^n$, where $1 \leqslant a_1 < 10$, and $n \in \mathbf{Z}$.

Example 1. Let $a = 283$; then $a = 2.83 \cdot 10^2$; here $a_1 = 2.83$, $1 \leqslant a_1 < 10$, and $n = 2$.

Example 2. Let $a = 3.76$; then $a = 3.76 \cdot 10^0$, here $a_1 = 3.76$, $1 \leqslant a_1 < 10$, and $n = 0$.

Example 3. Let $a = 0.00075$; then $a = 7.5 \cdot 10^{-4}$; here $a_1 = 7.5$, $1 \leqslant a_1 < 10$, and $n = -4$.

If a positive number a is represented in the form $a_1 \cdot 10^n$, where $1 \leqslant a_1 < 10$, and $n \in \mathbf{Z}$, then the number a is said to be written in *standard form*, the exponent n being called the *exponent* of the number.

Let $\log a$ be the common logarithm of a positive number a. Writing this number in standard form and using the properties of logarithms, we find

$$\log a = \log (a_1 \cdot 10^n) = \log a_1 + \log 10^n = \log a_1 + n \log 10.$$

Since $\log 10 = \log_{10} 10 = 1$, we have

$$\log a = n + \log a_1; \tag{2}$$

since $1 \leqslant a_1 < 10$, we have $\log 1 \leqslant \log a_1 < \log 10$, that is, $0 \leqslant \log a_1 < 1$.

It follows from equality (2) that n is the greatest integer not exceeding the number $\log a$, in other words, n is the integral part of the number $\log a$, that is, $n = [\log a]$, while $\log a_1$ is the fractional part of the number $\log a$, that is, $\log a_1 = \{\log a\}$. The integral part of the number $\log a$, that is, the exponent of the number a, is called the *characteristic*, and the fractional part of the number $\log a$ is called the *mantissa*.

The following assertion holds true: *if a number $a > 0$ is multiplied by 10^k, where $k \in \mathbf{Z}$, then the mantissa of the logarithm remains unchanged, that is, the mantissas of the numbers* $\log a$ *and* $\log (a \cdot 10^k)$ *are equal.*

□ Indeed, we have

$$\log (a \cdot 10^k) = \log a + \log 10^k = \log (a_1 \cdot 10^n) + k \log 10$$
$$= \log a_1 + \log 10^n + k = n + k + \log a_1.$$

The mantissa of the number $\log (a \cdot 10^k)$ is $\log a_1$, that is, the same number which serves as mantissa for $\log a$. ■

This property demonstrates the convenience of using common logarithms in practice. The thing is that for the numbers from the interval

(1, 10) tables of common logarithms have been compiled. Thus, in *Tables of Common Logarithms* (base_{10}, $\log_{10} N$) by V. M. Bradis the values of the mantissas are given to a precision of 0.00005. The tables do not contain the characteristics of logarithms since the characteristic, which is equal to the exponent of a number, can be found without using the tables. Let us give a fragment of these tables and explain by way of examples how to use it for computations.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5

Example 4. Compute log 738.

○ At the intersection of row 73 and column 8, we find the mantissa of the number log 738, it is equal to 0.8681. Consequently,

$$\log 738 = \log 7.38 \cdot 10^2 = 2 + 0.8681 = 2.8681. \bullet$$

Example 5. Compute log 0.07453.

○ Using the table, we find the mantissa of the logarithm of the number 7453. At the intersection of row 74 and column 5 we find the number 8722. We then make a correction for the fourth digit of the number by adding it to the number 8722. This correction is located in the right-hand portion of the table at the intersection of the same row and the correction column marked 3. The correction is equal to 2, that is, we get $8722 + 2 = 8724$. Thus, the mantissa of the required logarithm is equal to 0.8724. Further, we find

$$\log 0.07453 = \log 7.453 \cdot 10^{-2} = -2 + 0.8724 = -1.1276. \bullet$$

Example 6. Compute the value of a if $a = \dfrac{0.737^3 \cdot 7489}{\sqrt[3]{70.52}}$.

○ Let us find $\log a$. Using the properties of logarithms, we get

$$\log a = 3 \log 0.737 + \log 7489 - \frac{1}{3} \log 70.52.$$

From the table of logarithms we find $\log 7.37 = 0.8675$; $\log 7.489 = 0.8744$; $\log 7.052 = 0.8483$. Hence we get

$$\log a = 3 \log 7.37 \cdot 10^{-1} + \log 7.489 \cdot 10^3 - \frac{1}{3} \log 7.052 \cdot 10$$

$$= 3 \cdot (-1 + 0.8675) + (3 + 0.8744) - \frac{1}{3} (1 + 0.8483) = 2.8608.$$

Thus, $\log a = 2.8608$. Using the table, we find that the mantissa 8608 corresponds to the number 7258, that is, $\log 7.258 = 0.8608$. Consequently,

$$\log a = 2.8608 = 2 + 0.8608 = \log 100 + \log 7.258 = \log 725.8,$$

that is, $a = 725.8$. ●

6. Natural (or Napierian) Logarithms. Differentiating a Logarithmic Function. Along with common logarithms, we often use logarithms to the base $e = 2.71828\ldots$ (see § 9.1, Item 6). These logarithms, called *natural*, are denoted by the symbol $\ln x$, that is, $\log_e x = \ln x$.

Great importance of natural logarithms in mathematics is explained first of all by simplicity of their differentiation. Let us derive a formula for finding the derivative of the function $y = \ln x$. This function is the inverse of the exponential function $y = e^x$, and we established above (p. 386) that $(e^x)' = e^x$.

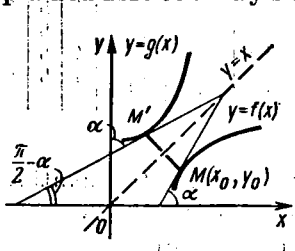


Fig. 246

Figure 246 represents the graphs of two mutually inverse functions $y = f(x)$ and $y = g(x)$. These graphs are symmetric with respect to the straight line $y = x$. Let us assume that at the point $M(x_0, y_0)$ taken on the graph of the function $y = f(x)$, there exists a nonvertical tangent line to the graph. This tangent line makes an angle α with the positive direction of the axis of abscissas. Then $f'(x_0) = \tan\alpha$.

Let us take on the graph of the inverse function $y = g(x)$ a point M' symmetric to the point M relative to the straight line $y = x$. The point M' has the coordinates (y_0, x_0), and at this point it is possible to draw a tangent to the graph of the function $y = g(x)$, this tangent being symmetric to the tangent drawn to the graph $y = f(x)$ at the point M with respect to the straight line $y = x$. The new tangent makes an angle α with the positive ray of the y-axis, and an angle $\frac{\pi}{2} - \alpha$ with the positive ray of the x-axis. Therefore

$$g'(y_0) = \tan\left(\frac{\pi}{2} - \alpha\right) = \cot\alpha = \frac{1}{\tan\alpha} = \frac{1}{f'(x_0)}.$$

Thus, we have obtained the following rule for computing the derivative of the inverse function: *the derivative of the inverse function is a quantity inverse to the derivative of the given function* (we do not give a rigorous proof of this assertion here).

This rule can be written briefly:

$$y'_x = \frac{1}{x'_y} \text{ if } x'_y \neq 0.$$

If $y = \ln x$, then

$$y'_x = (\ln x)' = \frac{1}{x'_y} = \frac{1}{(e^y)'} = \frac{1}{e^y}.$$

Since $y = \ln x$, we have $e^y = x$. Consequently

$$(\ln x)' = \frac{1}{x}.$$

By using the derived formula and the formula for change of base of logarithms (Property 4^0 from Item 2), it is easy to get the formula for computing the derivative of a logarithmic function to any base. Indeed, since $\log_a x = \frac{\ln x}{\ln a}$, we have

$$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a}(\ln x)' = \frac{1}{\ln a} \cdot \frac{1}{x}.$$

In particular. $(\log x)' = \frac{1}{\ln 10} \cdot \frac{1}{x}$; here $\frac{1}{\ln 10} = 0.434294$.

As we have found, the formula for the derivative of a logarithmic function takes the simplest form in the case when the number e serves as the base of the logarithm. Therefore, it is such logarithms that are frequently used, hence the term "natural logarithms".

Example 1. Find the derivative of the function $y = \ln^2 (3x + 1)$.

○ Here we use the rule for differentiating a composite function:

$$y' = (\ln^2 (3x-1))' = 2 \ln (3x-1) (\ln (3x-1))'$$

$$= 2 \ln (3x-1) \cdot \frac{1}{3x-1} \cdot (3x-1)'$$

$$= 2 \ln (3x-1) \cdot \frac{1}{3x-1} \cdot 3 = \frac{6 \ln (3x-1)}{3x-1}. \bullet$$

Example 2. Write the equation of the tangent line to the graph of the function $y = \log x$ at the point (10, 1).

○ The equation of the tangent line has the form

$$y = f(x_0) + f'(x_0) \cdot (x - x_0).$$

In this case $f(x) = \log x$, $x_0 = 10$, $f(x_0) = 1$, $f'(x) = (\log x)' = \frac{1}{\ln 10} \cdot \frac{1}{x}$, $f'(x_0) = \frac{1}{10 \ln 10}$. Consequently, the required equation can be written as follows:

$$y = 1 + \frac{1}{10 \ln 10}(x - 10). \bullet$$

7. Using the Logarithmic Function for Integration. According to the definition of indefinite integral, from the formula $(\ln x)' = \frac{1}{x}$

it follows that

$$\int \frac{dx}{x} = \ln x + C. \tag{3}$$

This integration formula is valid for $x > 0$.

If $x < 0$, then $-x > 0$, and we can find the derivative of the function $\ln(-x)$. According to the rule for differentiating a composite function, we find

$$(\ln(-x))' = \frac{1}{-x} \cdot (-x)' = \frac{1}{-x} \cdot (-1) = \frac{1}{x}.$$

Hence, if $x < 0$, then $\ln(-x) = \frac{1}{x}$ and therefore

$$\int \frac{dx}{x} = \ln(-x) + C. \tag{4}$$

Using formulas (3) and (4), we get

$$\int \frac{dx}{x} = \ln|x| + C, \text{ where } x \neq 0. \tag{5}$$

We know (see § 8.6, Item 3) that if $\int f(x)\,dx = F(x) + C$, then $\int f(ax+b)\,dx = \frac{1}{a} F(ax+b) + C$. This enables us to get the following generalization of formula (5):

$$\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax+b| + C \quad (a \neq 0).$$

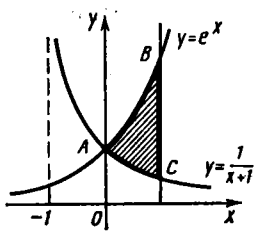


Fig. 247

Example 1. Compute $\int \frac{dx}{3x-5}$.

○ By formula (6), we get

$$\int \frac{dx}{3x-5} = \frac{1}{3} \ln|3x-5| + C. \bullet \tag{6}$$

Example 2. Find the area of the curvilinear triangle ABC bounded by the lines $y = e^x$, $y = \frac{1}{x+1}$ and $x = 1$ (Fig. 247).

○ The required area is computed by the formula

$$S = \int_0^1 \left(e^x - \frac{1}{x+1}\right) dx = \int_0^1 e^x\,dx - \int_0^1 \frac{dx}{x+1}$$

(see § 8.7, Item 6). We have

$$\int_0^1 e^x\,dx = e^x\Big|_0^1 = e^1 - e^0 = e - 1,$$

$$\int_0^1 \frac{dx}{x+1} = \ln(x+1)\Big|_0^1 = \ln 2 - \ln 1 = \ln 2.$$

Thus, $S = e - 1 - \ln 2$. ●

Example 3. Compute $\int \frac{x^3+x^2+2}{x^2-7x+12}\,dx$.

○ Consider the integrand $\frac{x^3+x^2+2}{x^2-7x+12}$. In § 2.2, Item 8, we spoke of representing an improper fraction in the form of a sum of a polynomial and a proper fraction and of expanding a proper fraction into a sum of partial fractions. This technique is just used in practice for integrating rational functions. Here we have

$$\frac{x^3+x^2+2}{x^2-7x+12} = x + 7 - \frac{32}{x-3} + \frac{70}{x-4}.$$

Consequently,

$$\int \frac{x^3+x^2+2}{x^2-7x+12}\,dx = \int x\,dx + \int 7\,dx - 32\int \frac{dx}{x-3} + 70\int \frac{dx}{x-4}$$

$$= \frac{x^2}{2} + 7x - 32\ln|x-3| + 70\ln|x-4| + C. \quad ●$$

8. Differentiating Exponential Function with Any Base and Power Function with Any Exponent. In the foregoing, we derived the formula for computing the derivative of the exponential function with base e:

$$(e^x)' = e^x.$$

Let us now consider an exponential function with an arbitrary base a, where $a > 0$ and $a \neq 1$. By the definition of logarithm, we have $a = e^{\ln a}$, consequently, $a^x = e^{x\ln a}$. Then

$$(a^x)' = (e^{x\ln a})' = e^{x\ln a}\cdot(x\ln a)' = (e^{\ln a})^x \cdot \ln a = a^x \ln a.$$

Thus,

$$(a^x)' = a^x \ln a.$$

For example, $(3^x)' = 3^x \ln 3$, $(10^x)' = 10^x \ln 10$.

Let us now consider the power function $y = x^\alpha$, where $x > 0$, $\alpha \in \mathbf{R}$. To compute its derivative, let us perform the following transformations:

$$x^\alpha = e^{\ln x^\alpha} = e^{\alpha \ln x}.$$

We then get

$$(x^\alpha)' = (e^{\alpha\ln x})' = e^{\alpha\ln x}(\alpha\ln x)' = (e^{\ln x})^\alpha \cdot \alpha \cdot \frac{1}{x} = x^\alpha \cdot \alpha \cdot \frac{1}{x} = \alpha x^{\alpha-1}.$$

Thus,

$$(x^\alpha)' = \alpha x^{\alpha-1}.$$

We have already used this formula in the case when $\alpha \in \mathbf{Q}$ (see § 8.4, Item 3).

Exercises

1. Graph the following functions: (a) $y = \log_2 x$, (b) $y = \log_{1/3} x$, (c) $y = \ln(x-3)$, (d) $y = \log x + 2$, (e) $y = |\log_3 x|$, (f) $y = 2\log_{1/2}|x|$.
2. Find the inverse of the function $y = 2^{x-3} - 5$. Graph both functions.
3. Solve graphically the equation $\log_2 x = -x^2 + 4x + 2$.
4. Which of the numbers a and b is greater: (a) $a = \log_2 2.7$, $b = \log_2 2.9$, (b) $a = \log_{1/3} 0.45$, $b = \log_{1/3} 0.44$, (c) $a = \log_{\pi/3} 22.77$, $b = \log_{\pi/3} 22.75$, (d) $a = \ln(1+\sqrt{2})$, $b = \ln(2\sqrt{3}-1)$?
5. Find the domain of definition of each of the given functions: (a) $f(x) = \ln(2x-4)$, (b) $f(x) = \log(x^2-5x+6)$, (c) $f(x) = \log_{1/2}\frac{3x-2}{2x+3}$, (d) $f(x) = \log_3(|x-3|-5)$.
6. It is known that $\log 2 = a$. Compute: (a) $\log 5$, (b) $\log \sqrt[3]{25}$, (c) $\log 40$, (d) $\log_2 5$, (e) $\log_5 500$, (f) $\log_{\sqrt{2}} \sqrt[3]{0.04}$.
7. Find the numerical values of the following expressions: (a) $\log_{1/3} \sqrt[5]{3}$, (b) $9^{-\log_3 4}$, (c) $\log_4 \log_{16} 256 + \log_4 \sqrt{2}$, (d) $36^{\log_6 5} + 10^{1-\log 2} - 3^{\log_9 36}$.
8. Take the logarithm to the base e of the expression $\frac{2e^2 a \sqrt[5]{b}}{c^{11}}$, where a, b, c are positive numbers.
9. Find x if $\log x = 2\log 5 - 3\log 2 - 0.5\log 625 + 0.25\log 256$.
10. Write the given number a in scientific notation and find the characteristic of $\log a$: (a) $a = 2.73$, (b) $a = 374.5$, (c) $a = 0.15$, (d) $a = 22.385$, (e) $a = 0.000013$.
11. Compute: (a) $f'(e)$ if $f(x) = \ln^2 x$, (b) $f'(\pi/3)$ if $f(x) = \ln \sin x$, (c) $f'(1)$ if $f(x) = \log(2x+1)$.
12. Find the points of extremum of the function $y = \ln(x^2 - 6x + 10)$.
13. Investigate the function $y = \ln(2x - x^2)$ and sketch its graph.
14. Compute:

(a) $\int_1^e \frac{dx}{x}$, (b) $\int_1^5 \frac{dx}{2x+1}$, (c) $\int \frac{3x-5}{x^2-7x+10}\,dx$,

(d) $\int \frac{x^3-5x^2+3-1}{x-4}\,dx$.

§ 9.3. Exponential and Logarithmic Equations

1. Exponential Equations. *Exponential equations* are defined as equations of the form $a^{f(x)} = a^{g(x)}$, where $a > 0$ and $a \neq 1$ and equations reduced to the equation of the indicated form. The solution of exponential equations is based on the following theorem.

Theorem 9.3.1. *The exponential equation*

$$a^{f(x)} = a^{g(x)} \tag{1}$$

is equivalent to the equation

$$f(x) = g(x). \tag{2}$$

□ Let x_0 be root of equation (1), that is, $a^{f(x_0)} = a^{g(x_0)}$. Then, according to the property of the exponential function (see p. 383), $f(x_0) = g(x_0)$, that is, x_0 is a root of equation (2).

Conversely, if x_1 is root of equation (2), that is, $f(x_1) = g(x_1)$, then $a^{f(x_1)} = a^{g(x_1)}$ and this means that x_1 is a root of equation (1).

Thus, equations (1) and (2) are equivalent. ■

Example 1. Solve the equation $2^{x^2-2x} = 2^{3x-6}$.

○ The given equation is equivalent to the equation $x^2 - 2x = 3x - 6$, whose roots $x_1 = 2$ and $x_2 = 3$ are also roots of the original equation.

Example 2. Solve the equation $\dfrac{(0.2)^{x-0.5}}{\sqrt{5}} = 5 \cdot 0.04^{x-1}$.

○ Let us reduce all the powers to the base 1/5:

$$\left(\frac{1}{5}\right)^{x-0.5} \cdot \left(\frac{1}{5}\right)^{0.5} = \left(\frac{1}{5}\right)^{-1} \cdot \left(\left(\frac{1}{5}\right)^2\right)^{x-1}.$$

Further, we have $\left(\frac{1}{5}\right)^x = \left(\frac{1}{5}\right)^{2x-3}$. The last equation is equivalent to the equation $x = 2x - 3$, whence $x = 3$. ●

Example 3. Solve the equation $3^{x^2-4} = 5^{2x}$.

○ Since $5 = 3^{\log_3 5}$, the given equation can be transformed to the form $3^{x^2-4} = 3^{\log_3 5 \cdot 2x}$. This equation is equivalent to the following:

$$x^2 - 4 = 2x \log_3 5, \quad \text{that is,} \quad x^2 - 2 \log_3 5 \cdot x - 4 = 0.$$

Let us solve the obtained quadratic equation: $x_{1,2} = \log_3 5 \pm \sqrt{\log_3^2 5 + 4}$. The two values found are roots of the original equation. ●

Example 4. Solve the equation $4^x + 2^{x+1} - 24 = 0$.

○ Since $4^x = (2^x)^2$ and $2^{x+1} = 2 \cdot 2^x$, the given equation can be rewritten as follows:

$$(2^x)^2 + 2 \cdot 2^x - 24 = 0.$$

Setting $2^x = z$, we get a quadratic equation $z^2 + 2z - 24 = 0$ with roots $z_1 = 4$ and $z_2 = -6$. Therefore the problem is reduced to solving the collection of equations: $2^x = 4$; $2^x = -6$. From the first equation of this collection we find $x = 2$. The second equation has no solution since $2^x > 0$ for any values of x. Thus, we get the answer: $x = 2$. ●

Example 5. Solve the equation $\left(\frac{3}{5}\right)^x + \frac{7}{5} = 2^x$.

○ This equation is not solved by any method considered in the previous examples. Let us try to find a solution of the equation by inspection. In this case it is an easy job: $x_1 = 1$. It is self-evident that the equation cannot be regarded as solved thus far since it can also have some other roots. Let us prove that there are no other roots.

Let $x>1$. Then, due to the decrease in the function $y=\left(\frac{3}{5}\right)^x$ we have $\left(\frac{3}{5}\right)^x+\frac{7}{5}<2$, and due to the increase in the function $y=2^x$ we have $2^x>2$. Hence, the equation has no roots greater than 1.

Let $x<1$. Then, due to the decrease in the function $y=\left(\frac{3}{5}\right)^x$ we have $\left(\frac{3}{5}\right)^x+\frac{7}{5}>2$, and due to the increase in the function $y=2^x$ we have $2^x<2$. Hence, the equation has no roots less than 1. Thus, $x=1$ is the only root of the equation. ●

In general, if one side of an equation contains a decreasing function and the other an increasing function, and the equation has a root, then it is unique (see Fig. 248).

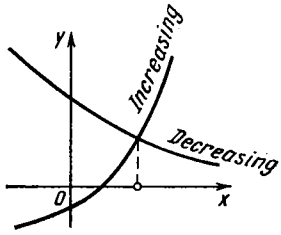


Fig. 248

2. Logarithmic Equations. *Logarithmic equations* are defined to be those of the form

$$\log_a f(x) = \log_a g(x), \qquad (3)$$

where $a>0$ and $a\neq 1$, or those which reduce to this form.

If, by analogy with exponential equations of the form $a^{f(x)}=a^{g(x)}$, we pass from equation (3) to the equation $f(x)=g(x)$, then it is, generally speaking, impossible to assert that the last equation is equivalent to equation (3). The thing is that the equation $f(x)=g(x)$ may have as its root such a number x_0 for which $f(x_0)\leqslant 0$ and $g(x_0)\leqslant 0$ and only a positive number may be found under the logarithm sign. This means that, when passing from equation (3) to the equation $f(x)=g(x)$, we have to find the roots of the last equation and choose those which satisfy the system of inequalities

$$\begin{cases} f(x)>0, \\ g(x)>0. \end{cases}$$

This system specifies the domain of definition of equation (3). Note, however, that we may confine ourselves to one of these inequalities, since the other follows from it and from the equation $f(x)=g(x)$.

As a result, we arrive at the following theorem.

Theorem 9.3.2. *The equation* $\log_a f(x)=\log_a g(x)$ *is equivalent to a mixed system*

$$\begin{cases} f(x)>0, \\ g(x)>0, \\ f(x)=g(x). \end{cases}$$

This system is, in turn, equivalent to each of the following systems:

$$\begin{cases} f(x) > 0, \\ f(x) = g(x) \end{cases} \quad \text{or} \quad \begin{cases} g(x) > 0, \\ f(x) = g(x). \end{cases}$$

This remark turns out to be convenient in practice.

When solving logarithmic equations, various properties of logarithms are used (see § 9.2). Consider, for example, the equation

$$\log_a f(x) + \log_a g(x) = \log_a h(x). \tag{4}$$

It is transformed to the form

$$\log_a (f(x)\, g(x)) = \log_a h(x). \tag{5}$$

However, equations (4) and (5) may turn out to be nonequivalent. Indeed, the domain of definition of the expression $\log_a f(x) + \log_a g(x)$ is given by the system of inequalities $\begin{cases} f(x) > 0, \\ g(x) > 0 \end{cases}$ whereas the domain of definition of the expression $\log_a (f(x)\, g(x))$ by the inequality $f(x)\, g(x) > 0$ which is, in turn, equivalent to the collection of systems of inequalities

$$\begin{cases} f(x) > 0, \\ g(x) > 0; \end{cases} \quad \begin{cases} f(x) < 0, \\ g(x) < 0. \end{cases}$$

Thus, by going over from equation (4) to equation (5), one can extend the domain of definition at the expense of the solutions of the last system of inequalities, and, hence, extraneous roots may occur. Therefore, on solving equation (5), one should either choose the roots belonging to the domain of definition of the original equation (4), that is, those which satisfy the system of inequalities

$$\begin{cases} f(x) > 0, \\ g(x) > 0, \\ h(x) > 0, \end{cases}$$

or verify the found roots by substituting them into the original equation (4).

Let us now consider equations of the form

$$\log_{a(x)} f(x) = \log_{a(x)} g(x). \tag{6}$$

Their solution is based on the following theorem (given without proof).

Theorem 9.3.3. *Equation* (6) *is equivalent to the "mixed" system*

$$\begin{cases} f(x) = g(x), \\ f(x) > 0, \\ g(x) > 0, \\ a(x) > 0, \\ a(x) \neq 1. \end{cases}$$

In other words, the roots of equation (6) are those and only those roots of the equation $f(x) = g(x)$ which at the same time satisfy the following conditions:

$$f(x) > 0, \quad g(x) > 0, \quad a(x) > 0, \quad a(x) \neq 1$$

(these conditions specify the domain of definition of equation (6)). Note that one of the inequalities $f(x) > 0$ or $g(x) > 0$ may be omitted.

Example 1. Solve the equation $\ln(x^2 - 3x - 5) = \ln(7 - 2x)$.

○ The given equation is equivalent to the following mixed system:

$$\begin{cases} 7 - 2x > 0, \\ x^2 - 3x - 5 = 7 - 2x. \end{cases}$$

Solving the equation of the system, we get $x_1 = 4$, $x_2 = -3$. Of these two values, only $x = -3$ satisfies the inequality of the system. Therefore $x = -3$ is the root of the given equation. ●

Example 2. Solve the equation $\log(x + 4) + \log(2x + 3) = \log(1 - 2x)$.

○ Let us transform the equation to the form

$$\log((x + 4)(2x + 3)) = \log(1 - 2x)$$
$$\text{or} \quad (x + 4)(2x + 3) = 1 - 2x.$$

From the last equation we find: $x_1 = -1$ and $x_2 = -5.5$. The domain of definition of the given equation is given by the system of inequalities

$$\begin{cases} x + 4 > 0, \\ 2x + 3 > 0, \\ 1 - 2x > 0. \end{cases}$$

Substituting the found values $x_1 = -1$ and $x_2 = -5.5$ one by one into the system, we make sure that -1 satisfies this system, while -5.5 does not. Thus, the value $x = -1$ is the only root of the given equation. ●

Note that in the example considered the system specifying the domain of definition is rather simple. Therefore, solving this system, we could find the domain of definition of the original equation:

$-1.5 < x < 0.5$. Since $x_1 = -1$ belongs to the interval $(-1.5, 0.5)$, $x_1 = -1$ is a root of the original equation. Since $x_2 = -5.5$ does not belong to the interval $(-1.5, 0.5)$, $x_2 = -5.5$ is an extraneous root.

Example 3. Solve the equation $\log_{x+4}(x^2 - 1) = \log_{x+4}(5 - x)$.

○ By Theorem 9.3.3, the given equation is equivalent to the system

$$\begin{cases} x^2 - 1 = 5 - x, \\ x^2 - 1 > 0, \\ 5 - x > 0, \\ x + 4 > 0, \\ x + 4 \neq 1. \end{cases}$$

Solving the equation entering the system, we get $x_1 = 2$, $x_2 = -3$. Of the two values only $x = 2$ satisfies the rest of the conditions of the system. Thus, we get the answer: $x = 2$. ●

Example 4. Solve the equation $\log^2 x + \log x + 1 = \dfrac{7}{\log(x/10)}$.

○ Since $\log(x/10) = \log x - 1$, the given equation can be rewritten as follows:

$$\log^2 x + \log x + 1 = \frac{7}{\log x - 1}.$$

Setting $\log x = z$, we get the equation

$$z^2 + z + 1 = \frac{7}{z - 1},$$

whence $z = 2$. Solving now the equation $\log x = 2$, we find $x = 100$ which is the only root of the given equation. ●

Example 5. Solve the equation $\ln^2 x^3 - \ln(x^{10}/e) = 0$.

○ We have

$$\ln^2 x^3 = (\ln x^3)^2 = (3 \ln x)^2 = 9 \ln^2 x,$$

$$\ln(x^{10}/e) = \ln x^{10} - \ln e = 10 \ln |x| - 1.$$

By the sense of the given equation, $x > 0$; hence, $|x| = x$, and therefore $\ln(x^{10}/e) = 10 \ln x - 1$.

Taking into consideration the performed transformations, we rewrite the given equation in the form

$$9 \ln^2 x - (10 \ln x - 1) = 0.$$

Setting $\ln x = y$, we get

$$9y^2 - 10y + 1 = 0,$$

whence $y_1 = 1$, $y_2 = 1/9$. From the equation $\ln x = 1$ we get $x_1 = e$, and from the equation $\ln x = 1/9$ we find $x_2 = e^{1/9}$. ●

Example 6. Solve the equation $\log (20 - x) = \log^3 x$.

○ We fail to solve this equation using the methods considered in the previous examples. Let us find some root of the equation by inspection. In this case, we get $x_1 = 10$.

However, the equation cannot be assumed to have been solved since it may have some other solutions. Let us prove that there are no other roots. It is self-evident that the roots of the equation should be sought in the domain of its definition, that is, in the interval (0, 20). We notice that on this interval the function $y = \log (20 - x)$ decreases, while the function $y = \log^3 x$ increases. In such a case if an equation has a root, then only one (see page 402). Thus, $x = 10$ is the only root of the given equation. ●

3. Exponential Equations Involving Logs. In this subsection, we shall consider equations which may be regarded both as exponential and logarithmic.

Example 1. Solve the equation

$$x^{1-\log x} = 0.01. \tag{7}$$

○ The domain of definition of the given equation is $x > 0$. In this domain the expressions on both sides of equation (7) take on only positive values; therefore, taking the logarithms of both of its sides to base 10, we get the equation

$$\log x^{1-\log x} = \log 0.01$$

which is equivalent to equation (7). Further, we have

$$(1 - \log x) \log x = -2.$$

Setting $u = \log x$, we get the equation $(1 - u)\, u = -2$, whence $u_1 = -1$, $u_2 = 2$. It remains to solve the collection of equations $\log x = -1$; $\log x = 2$. From this collection we get $x_1 = 0.1$, $x_2 = 100$ which are roots of equation (7). ●

Here, we have used the **method of taking logarithms** which consists in the passage from the equation $f(x) = g(x)$ to the equation $\log_a f(x) = \log_a g(x)$.

Example 2. Solve the equation

$$\log_x (3x^{\log_5 x} + 4) = 2 \log_5 x. \tag{8}$$

○ Using the definition of logarithm, we transform equation (8) to the form

$$x^{2 \log_5 x} = 3x^{\log_5 x} + 4.$$

Setting $u = x^{\log_5 x}$, we get the equation $u^2 - 3u - 4 = 0$, whose roots are $u_1 = -1$, $u_2 = 4$.

Now, the problem is reduced to solving the following collection of equations: $x^{\log_5 x} = -1$, $x^{\log_5 x} = 4$. Since $x^{\log_5 x} > 0$, and

$-1 < 0$, the first equation of the collection has no solution. Taking logarithms of both sides of the second equation to base 5, we get

$$\log_5^2 x = \log_5 4, \quad \text{that is,} \quad \log_5 x = \pm\sqrt{\log_5 4},$$

whence we find $x_{1,2} = 5 \pm x^{\sqrt{\log_5 4}}$ which are roots of equation (8). ●

Example 3. Solve the equation

$$\log_5\left(5^{\frac{1}{x}} + 125\right) = \log_5 6 + 1 + \frac{1}{2x}. \tag{9}$$

○ First we shall consider the given equation as a logarithmic one. Since $1 + \frac{1}{2x} = \log_5 5^{1+\frac{1}{2x}}$, we write equation (9) in the form

$$\log_5\left(5^{\frac{1}{x}} + 125\right) = \log_5 6 + \log_5 5^{1+\frac{1}{2x}}.$$

Further, we have

$$\log_5\left(5^{\frac{1}{x}} + 125\right) = \log_5\left(6 \cdot 5 \cdot 5^{\frac{1}{2x}}\right), \quad 5^{\frac{1}{x}} + 125 = 30 \cdot 5^{\frac{1}{2x}}.$$

We have obtained an exponential equation which can be solved by the method of introducing a new variable. Setting $u = 5^{1/(2x)}$, we get the equation $u^2 - 30u + 125 = 0$, whose roots are $u_1 = 5$ and $u_2 = 25$.

Now the problem is reduced to solving the collection of two equations: $5^{1/(2x)} = 5$; $5^{1/(2x)} = 25$. From the first equation we get $1/(2x) = 1$, whence $x_1 = 1/2$; from the second equation we get $1/(2x) = 2$, whence $x_2 = 1/4$. Thus, equation (9) has two roots: $x_1 = 1/2$ and $x_2 = 1/4$. ●

4. Systems of Exponential and Logarithmic Equations. The solution of systems of exponential and logarithmic equations involves no new techniques. We use here the standard methods for solving exponential and logarithmic equations and systems of equations which are familiar to the student. Here, we confine ourselves to considering only one example.

Example. Solve the system of equations

$$\begin{cases} 2^{(x-y)/2} + 2^{(y-x)/2} = 2.5, \\ \log(2x - y) + 1 = \log(y + 2x) + \log 6. \end{cases}$$

○ Let us consider the first equation of the given system and set $2^{(x-y)/2} = z$. Then $2^{(y-x)/2} = 2^{-(x-y)/2} = \frac{1}{2^{(x-y)/2}} = \frac{1}{z}$.

Hence, the equation can be written in the form $z + \frac{1}{z} = 2.5$. From the last equation we get $z_1 = 2$, $z_2 = 1/2$, and the problem is reduced to solving the collection of two exponential equations:

$$2^{(x-y)/2} = 2, \quad 2^{(x-y)/2} = 1/2.$$

From the first equation we find $(x - y)/2 = 1$, whence $x - y = 2$, from the second equation we get $(x - y)/2 = -1$, whence $x - y = -2$. Thus, the first equation of the original system is equivalent to the collection of two equations: $x - y = 2$; $x - y = -2$.

Let us now consider the second equation of the given system. We get in succession:

$$\log(2x - y) + 1 = \log(y + 2x) + \log 6,$$
$$\log(2x - y) + \log 10 = \log(6(y + 2x)),$$
$$10(2x - y) = 6(y + 2x), \quad 8x = 16y, \quad x = 2y.$$

As a result we arrive at the following collection of systems.

$$\begin{cases} x - y = 2, \\ x = 2y, \end{cases} \quad \begin{cases} x - y = -2, \\ x = 2y. \end{cases}$$

We find $x_1 = 4$, $y_1 = 2$ from the first system and $x_2 = -4$, $y_2 = -2$ from the second system.

The pairs found should be verified either by substitution into the original system or using the system of inequalities $\begin{cases} 2x - y > 0, \\ y + 2x > 0. \end{cases}$ (These inequalities are naturally related with the second equation of the original system.) We notice that the pair (4, 2) satisfies this system of inequalities, while the pair (−4, −2) does not. Thus, we get the answer: (4, 2). ●

Exercises

In Exercises 1 to 14, solve the given exponential equations:

1. $\sqrt{2^x} = 1024$.
2. $2^x \cdot 3^x = 216$.
3. $17^{3x} = 1$.
4. $3^{x-4} = 5^{x-4}$.
5. $3^{x-1} = 2^{x^2-4x+3}$.
6. $\left(\frac{2}{3}\right)^x \cdot \left(\frac{9}{8}\right)^x = \frac{27}{64}$.
7. $2^{x^2} \cdot 5^{x^2} = 0.001 \cdot (10^{3-x})^2$.
8. $10^x - 5^{x-1} \cdot 2^{x-2} = 950$.
9. $4^{x-1} + 4^x + 4^{x+1} = 84$.
10. $12 \cdot 3^{1/2x} - 3^{1/x} = 27$.
11. $2^x - 3 \cdot 2^{0.5(x-3)} = 26$.
12. $4^{\sqrt{x+1}} + 2^{\sqrt{x+1}+2} = 12$.
13. $9^x + 6^x = 2^{2x+1}$.
14. $5^x + 7^x = 12$.

In Exercises 15 to 34, solve the given logarithmic equations:

15. $\log_x (1/8) = -1.5$.
16. $\log_{1/16}(x/2) = -0.5$.
17. $\log_{\sqrt{2}/2}(1/32) = x$.
18. $\log_{\frac{1}{3}} x^4 = 256$.
19. $5 + \log_2 x = \log_2 7 - \log_2 x^2$.

20. $\ln \frac{2}{x-1} = \ln(4-x)$. 21. $\log(4.5-x) = \log 4.5 - \log x$.

22. $\log \sqrt{5x-4} + \log \sqrt{x+1} = 2 + \log 0.18$.

23. $\log_{1/5}(2x+5) = \log_{1/5}(16-x^2) + \ln e$.

24. $\frac{\ln(35-x^3)}{\ln(5-x)} = 3$. 25. $9^{\log_3(1-2x)} = 5x^2 - 5$.

26. $\log_{2x} x = \log_{x/2} x$. 27. $\log_4(x+12) \cdot \log_x 2 = 1$.

28. $\log_{x-2}(2x-9) = \log_{x-2}(23-6x)$.

29. $\frac{\log_2 x}{\log_2(x/2)} - 2\log_2 \sqrt{x} + \log_2 2x = 3$.

30. $2\log_x 27 - 3\log_{27} x = 1$. 31. $\log_{x^{3/2}} 2 + \log_4 x = 1$.

32. $\log_x 5 \sqrt{5} - \frac{5}{4} = \log_x^2 \sqrt{5}$. 33. $x^{\log x} = 100x$.

34. $(1.25)^{1-\log_2^2 x} = (0.64)^{2+\log_{\sqrt{2}} x}$.

In Exercises 35 and 36, solve given systems of equations:

35. $\begin{cases} 3^y \cdot 9^x = 81, \\ \log(y+x)^2 - \log x = 2\log 3. \end{cases}$

36. $\begin{cases} 8(\sqrt{2})^{x-y} = 0.5^{y-3}, \\ \log_3(x-2y) + \log_3(3x+2y) = 3. \end{cases}$

§ 9.4. Exponential and Logarithmic Inequalities

1. Exponential Inequalities. Any exponential inequality is reduced in the final analysis to an inequality of the form

$$a^{f(x)} > a^{g(x)}, \tag{1}$$

where, as usual, $a > 0$ and $a \neq 1$. When changing from inequality (1) to the inequality relating the exponents $f(x)$ and $g(x)$, one should take into account that the exponential function $y = a^x$ is increasing for $a > 1$ and decreasing for $0 < a < 1$. Hence, if $a > 1$, then, comparing the exponents $f(x)$ and $g(x)$, we have to retain the sign of inequality (1), that is, to write $f(x) > g(x)$. And in the case when $0 < a < 1$, comparing the exponents $f(x)$ and $g(x)$, we have to use the inequality sign of opposite sense, that is, to change from inequality (1) to the inequality $f(x) < g(x)$.

Thus, the following two theorems hold true.

Theorem 9.4.1. *If $a > 1$, then inequality (1) is equivalent to the inequality $f(x) > g(x)$.*

Theorem 9.4.2. *If $0 < a < 1$, then inequality (1) is equivalent to the inequality $f(x) < g(x)$.*

Example 1. Solve the inequality $(0.04)^{5x-x^2-8} < 625$.

○ Since $625 = (0.04)^{-2}$, we rewrite the given inequality in the form $(0.04)^{5x-x^2-8} < (0.04)^{-2}$. Since $0 < 0.04 < 1$, then, by The-

orem 9.4.2, this inequality is equivalent to the inequality

$$5x - x^2 - 8 > -2.$$

Solving the last inequality, we find $2 < x < 3$. Thus, we get the answer: (2, 3). ●

Example 2. Solve the inequality

$$\frac{1}{(0.5)^x - 1} - \frac{1}{1 - (0.5)^{x+1}} \geqslant 0.$$

○ Let us set $(0.5)^x = y$; then the given inequality takes the form

$$\frac{1}{y-1} - \frac{1}{1-0.5y} \geqslant 0,$$

whence after transformations we get

$$\frac{y - 4/3}{(y-1)(y-2)} \geqslant 0.$$

Using the method of intervals (see Fig. 249), we find $1 < y \leqslant 4/3$; $y > 2$. Thus, the problem is reduced to solving the following collection:

$$1 < (0.5)^x \leqslant 4/3;\ (0.5)^x > 2 \text{ or } (0.5)^0 < (0.5)^x \leqslant (0.5)^{\log_{0.5}(4/3)},$$
$$(0.5)^x > (0.5)^{-1}.$$

Taking into consideration that $0 < 0.5 < 1$, we go over from the collection obtained to the collection $0 > x \geqslant \log_{0.5}(4/3)$; $x < -1$. Thus, we get the answer $[\log_{0.5}(4/3), 0) \cup (-\infty, -1)$. ●

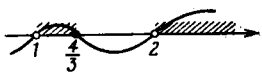


Fig. 249

2. Logarithmic Inequalities. Any logarithmic inequality is eventually reduced to an inequality of the form

$$\log_a f(x) > \log_a g(x), \qquad (2)$$

where, as usual, $a > 0$ and $a \neq 1$. Passing over from inequality (2) to the inequality relating $f(x)$ and $g(x)$, we should take into account that the logarithmic function increases for $a > 1$ and decreases for $0 < a < 1$. Hence, if $a > 1$, then we should pass from inequality (2) to the inequality of the same sense $f(x) > g(x)$, and if $0 < a < 1$ from inequality (2) to the inequality of opposite sense $f(x) < g(x)$. It is self-evident that in both cases the inequalities $f(x) > 0$ and $g(x) > 0$ must also be satisfied.

Thus, the following theorems hold.

Theorem 9.4.3. *If $a > 1$, then inequality (2) is equivalent to the system of inequalities*

$$\begin{cases} f(x) > 0, \\ g(x) > 0, \\ f(x) > g(x). \end{cases} \qquad (3)$$

Theorem 9.4.4. *If* $0 < a < 1$, *then inequality* (2) *is equivalent to the system of inequalities*

$$\begin{cases} f(x) > 0, \\ g(x) > 0, \\ f(x) < g(x). \end{cases} \tag{4}$$

Remarks. 1. In system (3) the inequality $f(x) > 0$ is a consequence of two other inequalities and therefore it may be omitted; similarly, we may omit the inequality $g(x) > 0$ in system (4).

2. Both theorems are extended to the case of any other inequality sign different from the sign $>$ in inequality (2). Note that in system (3) the sign of the last inequality coincides with the sign of the original logarithmic inequality while in system (4) it is opposite to it.

Example 1. Solve the inequality $\log_{1/2}(2x^2 - 4x - 14) \leqslant -1$.

○ Since $-1 = \log_{1/2} 2$, the given inequality can be rewritten as follows:

$$\log_{1/2}(2x^2 - 4x - 14) \leqslant \log_{1/2} 2.$$

Here the base of logarithms $a = 1/2$, that is, $0 < a < 1$, and, consequently, by Theorem 9.4.4, the given inequality is equivalent to the following system of inequalities:

$$\begin{cases} 2x^2 - 4x - 14 > 0, \\ 2x^2 - 4x - 14 \geqslant 2. \end{cases}$$

It is clear that the system obtained is equivalent to the inequality $2x^2 - 4x - 14 \geqslant 2$; solving this inequality, we find $x \leqslant -2$; $x \geqslant 4$. ●

Example 2. Solve the inequality

$$\ln(x + 27) - \ln(16 - 2x) > \ln x. \tag{5}$$

○ We have

$$\ln(x + 27) > \ln x + \ln(16 - 2x),$$

that is,

$$\ln(x + 27) > \ln(x(16 - 2x)). \tag{6}$$

Since the base of the natural logarithm satisfies the inequality $e > 1$, omitting the logarithm signs in inequality (6), we have to retain the inequality sign. However, we must be sure that the inequalities $x + 27 > 0$, $16 - 2x > 0$, and $x > 0$ specifying the domain of definition of inequality (5) are satisfied.

Thus, we get the following system of inequalities:

$$\begin{cases} x + 27 > 0, \\ 16 - 2x > 0, \\ x > 0, \\ x + 27 > x(16 - 2x). \end{cases}$$

Solving this system, we get

$$\begin{cases} x > -27, \\ x < 8, \\ x > 0, \\ 2x^2 - 15x + 27 > 0, \end{cases} \quad \text{that is,} \quad \begin{cases} 0 < x < 8, \\ 2(x-3)(x-4.5) > 0. \end{cases}$$

Consequently, $0 < x < 3$; $4.5 < x < 8$ (Fig. 250). ●

Example 3. Solve the inequality $\log_{x-2}(2x-3) > \log_{x-2}(24-6x)$.

Fig. 250

○ Here we have to consider two cases: (1) $x - 2 > 1$ and (2) $0 < x - 2 < 1$. In the first case, applicable to the given inequality is Theorem 9.4.3, and in the second, Theorem 9.4.4. Thus, the problem is reduced to solving the following collection of two systems of inequalities:

$$\begin{cases} x-2 > 1, \\ 2x-3 > 0, \\ 24-6x > 0, \\ 2x-3 > 24-6x; \end{cases} \quad \begin{cases} 0 < x-2 < 1, \\ 2x-3 > 0, \\ 24-6x > 0, \\ 2x-3 < 24-6x. \end{cases}$$

It follows from the first system that $27/8 < x < 4$, and from the second system that $2 < x < 3$. ●

Example 4. Solve the inequality $\log_2^2 (x-1)^2 - \log_{0.5}(x-1) > 5$.

○ On the basis of the properties of logarithms, we have $\log_2 (x-1)^2 = 2\log_2 |x-1| = 2\log_2 (x-1)$ (here $|x-1| = x-1$, since the given inequality implies that $x - 1 > 0$); $\log_{0.5}(x-1) = \log_{0.5^{-1}}(x-1)^{-1} = -\log_2 (x-1)$. Then the given inequality can be rewritten as follows:

$$4\log_2^2 (x-1) + \log_2 (x-1) > 5.$$

Setting $\log_2 (x-1) = y$, we get $4y^2 + y - 5 > 0$, whence $y < -5/4$; $y > 1$.

Now the problem is reduced to solving the collection of logarithmic inequalities $\log_2 (x-1) < -5/4$, $\log_2 (x-1) > 1$ or

$$\log_2 (x-1) < \log_2 2^{-5/4}, \quad \log_2 (x-1) > \log_2 2.$$

From the first inequality of this collection we get $0 < x - 1 < 2^{-5/4}$ and, consequently, $1 < x < 1 + 2^{-5/4}$. From the second inequality of the collection we have $x - 1 > 2$, that is, $x > 3$. Thus, we get the answer: $(1, 1 + 2^{-5/4}) \cup (3, +\infty)$. ●

Example 5. Solve the inequality $x^{\log x} > 10$.

○ Taking advantage of the fact that if $f(x) > 0$, $g(x) > 0$,

$f(x) > g(x)$ and $a > 1$, then $\log_a f(x) > \log_a g(x)$. Taking logarithms of both sides of the given inequality to base 10, we get

$$\log (x^{\log x}) > \log 10 \quad \text{or} \quad \log x \log x > 1,$$

that is, $\log^2 x > 1$, whence $\log x < -1$ and $\log x > 1$. We find $0 < x < 0.1$ from the first inequality of this collection, $x > 10$ from the second one.

3. Graphical Solution of Exponential and Logarithmic Inequalities. In § 3.9, Item 2, we considered examples of graphical solution of inequalities. In what follows, we shall dwell on graphical solution of exponential and logarithmic inequalities.

Example 1. Solve the inequality $2^x \geqslant 11 - x$.

○ We construct the graphs of the functions $y = 2^x$ and $y = 11 - x$ (Fig. 251). They intersect at the point with abscissa $x = 3$.

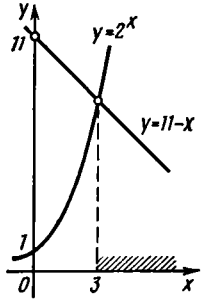


Fig. 251

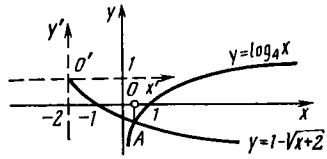


Fig. 252

It is obvious from the figure that the set of solutions of the inequality $2^x \geqslant 11 - x$ is contained in the interval $[3, +\infty)$. ●

Example 2. Solve the inequality $\log_4 (16x) + \sqrt{x+2} < 3$.

○ Noticing that $\log_4 (16x) = 2 + \log_4 x$, we transform the inequality to the form $\log_4 x < 1 - \sqrt{x+2}$. We then graph the functions $y = \log_4 x$ and $y = 1 - \sqrt{x+2}$ (to construct the graph of the function $y = 1 - \sqrt{x+2}$ it is convenient to translate the coordinate system xOy, choosing the point O' $(-2, 1)$ as the origin of the new coordinate system, and construct the graph of the function $y' = -\sqrt{x'}$ in the new system; this is just the graph of the function $y = 1 - \sqrt{x+2}$). In Fig. 252, the graph of the function $y = \log_4 x$ is located below the graph of the function $y = 1 - \sqrt{x+2}$ on the interval $0 < x < x_A$, where $x_A \approx 0.3$. Thus, we get the answer: $(0, x_A)$, $x_A \approx 0.3$. ●

Exercises

In Exercises 1 to 9, solve the given exponential inequalities:

1. $27^x \cdot 3^{1-x} < 1/3$. 2. $2^x \cdot 5^x > 0.1 \cdot (10^{x-1})^5$. 3. $2^{x^2-6x-2.5} > 16\sqrt{2}$.
4. $-2 < 5^x < 3$. 5. $2 < 6^x < 7$. 6. $3.3^{x^2+6x} < 1$.
7. $(1/3)^{-|x+2|} \geqslant 81$. 8. $4^x + 3 \cdot 2^x - 4 < 0$. 9. $2^{2+x} - 2^{2-x} > 15$.

In Exercises 10 to 23, solve the given logarithmic inequalities:

10. $\dfrac{\log_{1/5} 7 - \log_{1/5} 8}{x^2-3} > 0$. 11. $\log_{1/2}(x-2) > -1$. 12. $\log_2 x^2 < 1$.
13. $\log_2^2 x < 1$. 14. $\log_{16\cdot 3} \dfrac{2x}{2x-3} > 0$. 15. $\log_3(-x) > -\sqrt{3}$.
16. $(5x-2)\log_{1/3} x < 0$. 17. $\log_{0.1}(x^2+1) < \log_{0.1}(2x-5)$.
18. $\log_8(x^2-4x+3) > 1$.
19. $\log_{1/2} \dfrac{2x-1}{x+1} \geqslant 1$. 20. $\log x \leqslant \log 18 - \log(x-7)$.
21. $\log_{1/3} x > \log_x 3 - 2.5$. 22. $\log_{x-5}(2x-2.5) > 0$. 23. $\log_x \dfrac{x+1}{x-1} > 1$.

24. Solve graphically the given inequalities: (a) $x^2 < 2^x$, (b) $\log_3 x \leqslant 4 - x$.

In Exercises 25 to 34, solve the given systems of inequalities:

25. $\begin{cases} \ln(x^2-3x-3) \geqslant 0, \\ \log(x^2-15) < 1 \end{cases}$ 26. $\begin{cases} \dfrac{x^3}{(3x+1)^2} \leqslant 0, \\ (0,\ (6))^{x^2} > 1.5^{2x}. \end{cases}$
27. $\begin{cases} \log^2 x + \log 0.01\, x > 0, \\ 1/x < 1000. \end{cases}$ 28. $\begin{cases} 9^x - 3^{x+1} \leqslant 54, \\ \sqrt{3x-5} < 2. \end{cases}$
29. $\begin{cases} \log_{1/2}(x-3) > -1, \\ \sin x > 0. \end{cases}$ 30. $\begin{cases} 0.2^{3x^2-10x+3} \geqslant 1, \\ \cos x > 1/2. \end{cases}$
31. $\begin{cases} \log_{0.2}(x+2) - \log_5 x \leqslant \log_{0.2} 15, \\ |2x+1| < |x-3|. \end{cases}$ 32. $\begin{cases} \sqrt{\ln x} < \sqrt{2}, \\ \dfrac{3^x - 81}{81 - 3^{x+1}} > 0. \end{cases}$
33. $\begin{cases} \log|x-2| < 2, \\ (1/3)^{x^2} < 27^x. \end{cases}$ 34. $\begin{cases} \ln(x-3) < 1, \\ 4^x + 16 > 17 \cdot 2^x, \\ \sin x > -\sqrt{3}/2. \end{cases}$

35. Find the integral solutions of the following collection of systems of inequalities:

$$\begin{cases} \sin\left(\dfrac{x}{2}\right) > 0, \\ \ln x < 2; \end{cases} \quad \begin{cases} 2^{|x-1|} > 4, \\ x^2 + 3 < x. \end{cases}$$

Chapter 10

ELEMENTS OF COMBINATORICS

§ 10.1. Permutations, Arrangements, and Combinations

1. Permutations. A finite set M is said to be *ordered* if each of its elements has its own ordinal number. To denote an ordered set, we shall use parentheses enclosing in them the elements of a set in the chosen order. For example, given a three-element set $\{a, b, c\}$, we can form six different ordered sets: (a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), and (c, b, a).

A *permutation* is an ordered arrangement or sequence of all or part of a finite set of things. The number of all possible permutations of a finite set of n elements is denoted by P_n.

It is clear that the number of permutations of a set containing one element is equal to unity: $P_1 = 1$, and the number of permutations of a two-element set is equal to two: $P_2 = 2$. As we noted above, the number of permutations of a three-element set is equal to six: $P_3 = 6$. Given a four-element, set, $\{a, b, c, d\}$, we can compute the number of its permutations in the following way. We take a three-element set $\{a, b, c\}$ and order it in one of the six possible ways, for instance, (a, b, c). Making use of the ordered set (a, b, c), we can form four permutations of the given set $\{a, b, c, d\}$. To this end, we have to retain the chosen order of elements a, b, c, putting the fourth element d in succession in the first, second, third, and fourth place. In doing so, we get four permutations of the set $\{a, b, c, d\}$: (d, a, b, c), (a, d, b, c), (a, b, d, c), and (a, b, c, d). Since each of the six permutations of the three-element set $\{a, b, c\}$ yields four permutations of the four-element set $\{a, b, c, d\}$, we get a total of 24 permutations of the four-element set, that is, $P_4 = 24$.

We have proved that $P_4 = P_3 \cdot 4$. Reasoning in a similar way, we get a more general formula

$$P_{n+1} = P_n (n + 1). \tag{1}$$

Thus, the sequence $P_1, P_2, P_3, \ldots, P_n, \ldots$ can be represented by the recurrence formulas method: $P_1 = 1$, $P_{n+1} = P_n (n + 1)$. Now, it is easy to pass to the analytic way of representing the sequence (P_n), that is, to get the formula of the nth term of the sequ-

ence. We find

$$P_1 = 1,$$
$$P_2 = P_1 \cdot 2 = 1 \cdot 2,$$
$$P_3 = P_2 \cdot 3 = 1 \cdot 2 \cdot 3,$$
$$P_4 = P_3 \cdot 4 = 1 \cdot 2 \cdot 3 \cdot 4.$$

This suggests a hypothesis that the following formula holds true:

$$P_n = 1 \cdot 2 \cdot 3 \ldots (n-1)\, n.$$

Let us prove this hypothesis $A(n)$ applying the method of mathematical induction.

□ We have seen that $A(1)$ is true. Let us prove that $A(k) \Rightarrow A(k+1)$. Suppose that this assertion is true for $n = k$, that is, that $P_k = 1 \cdot 2 \cdot 3 \ldots k$. It follows from formula (1) that $P_{k+1} = P_k (k+1)$. Consequently

$$P_{k+1} = 1 \cdot 2 \cdot 3 \ldots k\,(k+1).$$

Thus, the assertion is true for $n = k+1$, that is, $A(k) \Rightarrow A(k+1)$.

On the basis of the principle of mathematical induction, we conclude that the equality being proved is valid for any $n \in \mathbf{N}$. ■

The product $1 \cdot 2 \cdot 3 \ldots n$ is usually denoted by the symbol $n!$ (read: "n-factorial"); it is also supposed that $0! = 1$, $1! = 1$. Therefore the formula proved above for finding the number of permutations of an n-element set can be written as follows:

$$P_n = n!, \tag{2}$$

and equality (1) obtained above takes the form

$$(n+1)! = n! \cdot (n+1). \tag{3}$$

Example 1. How many six-digit numbers, multiple of 5, can be formed from the digits 0, 1, 2, 3, 4, and 5 provided that there are no equal digits in a number?

○ A number multiple of 5 must end either in 0 or in 5. If the last digit of a number is 0, then the other five digits can be arranged in any order. Thus, the number of six-digit numbers ending in zero is equal to the number of permutations of five elements 1, 2, 3, 4, and 5, that is, P_5. If the last digit is 5, then the other five digits can be again arranged in P_5 ways, invalid arrangements being those beginning with the digit 0 (a number cannot begin with the digit 0). Therefore we have to subtract the number of permutations beginning with 0 from P_5 permutations. There are as many such permutations as is the number of ways in which the digits 1, 2, 3, and 4 can be arranged in the second, third, fourth, and fifth places of a six-digit number, that is, P_4. Hence, the total number of numbers that are, multiples of 5 and end in 5, is $P_5 - P_4$.

Summarizing, we get that there are $P_5 + (P_5 - P_4)$ six-digit numbers consisting of different numbers 0, 1, 2, 3, 4, and 5, and multiple of five. Computing, we get:

$$P_5 + (P_5 - P_4) = 2P_5 - P_4 = 2\,(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5) - 1 \cdot 2 \cdot 3 \cdot 4 = 216. \bullet$$

Example 2. Reduce the fraction $\frac{n!}{(n-m)!}$ $(m < n)$.

○ We find

$$\frac{n!}{(n-m)!} = \frac{1 \cdot 2 \cdot 3 \ \ldots \ (n-m)\,(n-m+1) \ \ldots \ n}{1 \cdot 2 \cdot 3 \ \ldots \ (n-m)}$$

$$= (n-m+1)\,(n-m+2) \ \ldots \ (n-1)\,n.$$

Thus,

$$\frac{n!}{(n-m)!} = n\,(n-1)\,(n-2) \ \ldots \ (n-m+1). \bullet \qquad (4)$$

2. Arrangements. Suppose that we are given a finite set M consisting of n elements. Its any ordered m-element subset $(m \leqslant n)$ is called an *arrangement* of n elements taken m at a time.

Let us find out the total number of arrangements of n elements taken m at a time; the number of such arrangements is denoted by A_m^n.

It is obvious that $A_0^n = 1$, since there is only one subset containing no elements (an empty subset). It is also clear that $A_1^n = 1$, since we can single n one-element subsets out of n-element set, each one-element subset being regarded an ordered one.

Let us compute A_2^n. We take some element of the set M and attach to it in turn, as a "second" one, each of the remaining $n - 1$ elements of the set M. As a result, we get $n - 1$ arrangements. Since a similar reasoning is applicable to any of n elements of the set M, we get a total of $n\,(n - 1)$ arrangements. Thus, $A_2^n = n\,(n - 1)$, that is, $A_2^n = A_1^n\,(n - 1)$.

Knowing A_2^n, it is easy to compute A_3^n. To this effect, it suffices to take an ordered two-element subset of the set M and annex to it in turn, as a "third" one, each of the remaining $(n - 2)$ elements of the set M. As a result, we get $n - 2$ arrangements. Since the number of ordered two-element subsets is equal to A_2^n, there is a total of $A_2^n\,(n - 2)$ arrangements of n elements taken 3 at a time. Thus, $A_3^n = A_2^n\,(n - 2)$.

A similar reasoning enables us to get the recurrence relation $A_{m+1}^n = A_m^n(n - m)$. Using this relation, we get in turn:

$$A_1^n = n.$$

$$A_2^n = A_1^n \cdot (n-1) = n\,(n-1).$$

$$A_3^n = A_2^n \cdot (n-2) = n\,(n-1)\,(n-2)$$

and, in general,

$$A_m^n = n(n-1)(n-2)(n-3)\ \ldots\ (n-m+1). \tag{5}$$

Making use of equality (4) proved in the preceding subsection, we get

$$A_m^n = \frac{n!}{(n-m)!}. \tag{6}$$

Note that the last formula is valid not only when $1 \leqslant m \leqslant n-1$, but also for $m = 0$ and $m = n$.

Example 1. A group consists of 25 students. They have to elect the monitor, Komsomol organizer, and the editor of wall newspaper. In how many ways can this be done?

○ The problem is reduced to finding the number of ordered three-element subsets in the given 25-element set. Consequently, this can be done in A_3^{25} ways:

$$A_3^{25} = 25 \cdot 24 \cdot 23 = 13\,800. \ ●$$

Example 2. Solve the equation $\dfrac{A_3^x + 3A_2^x}{P_{x+1}} = \dfrac{1}{2}$.

○ According to the sense of the equation, the variable x can assume only natural values not less than three. Making use of formulas (2) and (6), we get

$$\frac{A_3^x + 3A_2^x}{P_{x+1}} = \left(\frac{x!}{(x-3)!} + \frac{3x!}{(x-2)!}\right) \div (x+1)!$$

$$= \frac{x!\,(x-2) + 3x!}{(x-2)!\,(x+1)!} = \frac{x!\,(x+1)}{(x-2)!\,(x+1)!}$$

$$= \frac{(x+1)!}{(x-2)!\,(x+1)!} = \frac{1}{(x-2)!}.$$

Thus, the given equation is transformed to the form $1/(x-2)! = 1/2$, whence $(x-2)! = 2$, that is, $x - 2 = 2$, or $x = 4$. ●

3. Combinations. Let there be given a finite set M consisting of n elements. Any its m-element subset ($m \leqslant n$) is called a *combination* of n elements taken m at a time.

We first find out what is the total number of combinations of n elements taken m at a time; the number of such combinations is denoted by C_m^n.

We take an arbitrary m-element subset of the set M. Carrying out all possible permutations of the elements of the subset, we get P_m ordered m-element subsets of the set M, that is, P_m arrangements of n elements taken m at a time. To get all the arrangements of n elements taken m at a time, we have to perform P_m permutations in each m-element nonordered subset, that is, in each combination of n elements taken m at a time. Thus, $A_m^n = C_m^n P_m$, whence

$$C_m^n = \frac{A_m^n}{P_m}. \tag{7}$$

Since $A_m^n = \frac{n!}{(n-m)!}$, and $P_m = m!$, we get the following formula for determining the number of combinations of n elements taken m at a time:

$$C_m^n = \frac{n!}{(n-m)!\,m!}. \tag{8}$$

Example 1. A meeting of 80 persons elect a chairman, a secretary, and three members of an editorial board. In how many ways can this be done?

○ The number of ways for electing a chairman and a secretary from a body of 80 persons is equal to the number of ordered two-element subsets contained in the 80-element set, that is, to A_2^{80}. Suppose that the chairman and the secretary have been elected. Then the remaining 78 persons have to elect three members of the editorial board. This can be done in a number of ways equal to the number of nonordered three-element subsets contained in the 78-element set, that is, in C_3^{78} ways. Hence, the chairman, secretary, and three members of the editorial board can be elected in $A_2^{80} \cdot C_3^{78}$ ways.

We compute:

$$A_2^{80} = 80 \cdot 79 = 6\,320, \quad C_3^{78} = \frac{A_3^{78}}{P_3} = \frac{78 \cdot 77 \cdot 76}{1 \cdot 2 \cdot 3} = 76\,076.$$

Consequently, $A_2^{80} \cdot C_3^{78} = 6320 \cdot 76076 = 480\,800\,320$. ●

Example 2. Twelve workers are to make up three teams, each consisting of four men. In how many ways can this be done?

○ We first form one of the teams. To this end, we have to choose four out of the twelve workers available. This can be done in a variety of ways equal to the number of four-element subsets contained in the twelve-element set, that is, in C_4^{12} ways. Of the remaining eight workers we have to pick up another four men to form a second team; then the remaining four men will automatically form a third team.

We can choose four workers out of eight in C_4^8 ways. Thus, three four-men teams can be formed in $C_4^{12} \cdot C_4^8$ ways. We compute:

$$C_4^{12} \cdot C_4^8 = \frac{12!}{8! \cdot 4!} \cdot \frac{8!}{4! \cdot 4!} = \frac{12!}{(4!)^3} = 34\,650. \bullet$$

4. Properties of the Number of Combinations. Pascal's Triangle. Consider two properties of the number of combinations.

1°. *The following equality holds*:

$$C_m^n = C_{n-m}^n. \tag{9}$$

□ Making use of equality (8) we get

$$C_m^n = \frac{n!}{(n-m)!\,m!}, \quad C_{n-m}^n = \frac{n!}{(n-(n-m))!\,(n-m)!} = \frac{n!}{m!\,(n-m)!}.$$

Hence, $C_m^n = C_{n-m}^n$. ■

For example $C_5^7 = C_2^7$, $C_{11}^{20} = C_6^{20}$.

2°. *The following equality holds*:

$$C_m^n + C_{m+1}^n = C_{m+1}^{n+1}. \tag{10}$$

□ We find

$$C_m^n + C_{m+1}^n = \frac{n!}{(n-m)!\,m!} + \frac{n!}{(n-m-1)!\,(m+1)!}$$
$$= \frac{n!\,(m+1)}{(n-m)!\,(m+1)!} + \frac{n!\,(n-m)}{(n-m)!\,(m+1)!}$$
$$= \frac{n!\,(m+1+n-m)}{(n-m)!\,(m+1)!} = \frac{(n+1)!}{(n-m)!\,(m+1)!}.$$

On the other hand, we get

$$C_{m+1}^{n+1} = \frac{(n+1)!}{((n+1)-(m+1))!\,(m+1)!} = \frac{(n+1)!}{(n-m)!\,(m+1)!}.$$

Thus, $C_m^n + C_{m+1}^n = C_{m+1}^{n+1}$. ■

Equality (10) is a recurrence relation for the number of combinations. Using this equality, we can compile a table for finding the number of combinations C_m^n for $n = 2, 3, 4, \ldots$. Note that $C_0^0 = 1$ (an empty set contains only one empty subset) and that for any $n \in \mathbf{N}$ we have $C_0^n = C_n^n = 1$ (an n-element set contains one empty and one n-element subset); in particular, $C_0^1 = C_1^1 = 1$. Further, we find

$$C_0^2 = 1,\ C_1^2 = C_0^1 + C_1^1 = 1 + 1 = 2,\ C_2^2 = 1,$$
$$C_0^3 = 1,\ C_1^3 = C_0^2 + C_1^2 = 1 + 2 = 3,\ C_2^3 = C_1^2 + C_2^2 = 2 + 1 = 3,\ C_3^3 = 1.$$

We arrange all the numbers found to form a triangular array according to the following scheme:

$$\begin{array}{cccccc} & & & C_0^0 & & \\ & & C_0^1 & & C_1^1 & \\ & C_0^2 & & C_1^2 & & C_2^2 \\ C_0^3 & & C_1^3 & & C_2^3 & & C_3^3 \end{array}$$

Then, we get

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

Now it is clear how to form the fourth row of this array containing the numbers C_0^4, C_1^4, C_2^4, C_3^4, C_4^4. The terminal numbers of the fourth row are unities, each of the remaining numbers being the sum of those two numbers immediately above it. The rest of the rows are formed in much the same way. As a result, we get the following tri-

angular table called **Pascal's triangle** (in honour of the French mathematician of the seventeenth century):

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

The nth row of the table consists of the numbers, $C_0^n, C_1^n, C_2^n, \ldots, C_n^n$.

5. **Probability of an Event.** Let us begin with a very simple example. A bag contains five balls—red, blue, green, yellow, and black. What is the probability of the event that a ball randomly taken out of the bag turns out to be red?

Intuition prompts us that the probability is equal to 1/5 or 20%. A problem arises how to determine the probability of an event in the general case.

By the *probability* of an event we understand the ratio of the number of occurrences favourable to a given event to the total number of possible occurrences.

Thus, in the above example, there is only one occurrence favourable for the given event, that is, when a red ball is taken out of the bag, and the total number of possible occurrences is equal to five, since one ball out of the five contained in the bag can be picked up in five different ways. Therefore the probability turned out to be equal to 1/5.

Let us now consider a more complicated problem.

Example. A "Sportloto" card contains 45 numbers, six of which participate in a drawing. What is the probability to guess right: (1) six numbers, (2) five numbers, (3) four numbers, (4) three numbers.

○ (1) The total number of possible occurrences is equal to the number of all possible ways of choosing six numbers out of 45, that is, C_6^{45}. It is possible to guess right six "lucky" numbers on a given card only in one way, the probability of this event being equal to

$$\frac{1}{C_6^{45}} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{45 \cdot 44 \cdot 43 \cdot 42 \cdot 41 \cdot 40} = \frac{1}{8\,145\,060} \approx 0.000012\%.$$

(2) In order to guess five winning numbers, one has to choose five out of the six "lucky" numbers and add one of the remaining 39 "unlucky" numbers. Five numbers out of six can be chosen in C_5^6 ways, and one number out of 39 in C_1^{39} ways. The total number of combinations with five "lucky" numbers amounts to $C_5^6 \cdot C_1^{39} = 6 \cdot 39 = 234$, the probability to guess right five numbers being equal to

$$\frac{C_5^6 \cdot C_1^{39}}{C_6^{45}} = \frac{234}{8\,145\,060} \approx 0.003\%.$$

(3) In order to guess exactly four winning numbers, one has to choose four out of six "lucky" numbers and add to them two of the remaining 39 "unlucky" numbers. This can be done in $C_4^6 \cdot C_2^{39}$ ways, the probability to guess right four numbers being equal to

$$\frac{C_4^6 \cdot C_2^{39}}{C_6^{45}} = \frac{11\,115}{8\,145\,060} \approx 0.14\%.$$

(4) The probability to guess right three numbers is equal to

$$\frac{C_3^6 \cdot C_3^{39}}{C_6^{45}} = \frac{182\,780}{8\,145\,060} \approx 2.24\%. \bullet$$

Exercises

1. Find the numerical values of the given expressions:

(a) $\frac{45!}{43!}$, (b) $\frac{15!}{7! \cdot 9!}$, (c) $\frac{P_8}{A_3^3 C_2^3}$.

2. Simplify: (a) $\frac{(n+2)!\,(n^2-9)}{(n+4)!}$, (b) $\frac{P_n \cdot C_3^{n+1}}{A_{n-2}^n}$.

3. Transform into a fraction $\frac{1}{(n-2)!} - \frac{n^3+n}{(n+1)!}$.

4. Solve the given equations: (a) $A_8^x = 18A_4^{x-2}$, (b) $A_2^{x-1} - C_1^x = 79$.

5. Prove the equality $C_8^{15} + 2C_9^{15} + C_{10}^{15} = C_{10}^{17}$.

6. How many different three-colour flags with three horizontal stripes can be obtained by making use of: (a) three colours, (b) four colours?

7. Different five-digit numbers, containing no equal digits, are made up out of the digits, 1, 2, 3, 4, and 5. Find out how many numbers (among the formed ones) are such that: (a) begin with the digit 3, (b) do not begin with the digit 5, (c) begin with the number 54, (d) do not begin with the number 54, (e) are even, (f) are divisible by 4.

8. In how many ways can four persons be elected to occupy four different posts out of nine applicants?

9. Six workers out of twenty must be chosen to carry out a certain job. In what ways can this be done?

10. A total of 210 games were played in a chess tournament, each participant having played one game with each of the other participants. How many persons took part in the tournament?

11. In what ways can we use 12 different sweets to form a set of an even number of sweets.

12. A set consists of six elements: A, B, C, D, E, and F. If this set is ordered in all possible ways, then in how many cases: will (a) element A occupy neither the first nor the last place, (b) elements D and E be placed side by side, (c) elements A and B not be placed side by side?

13. A bag contains eight balls: three red and five green. What is the probability of the event that two balls randomly taken out of the bag will turn out to be: (a) different colours, (b) the same colour?

14. A "sportloto" card contains 36 numbers, five numbers participating in a drawing. What is the probability to guess right: (a) five numbers, (b) four numbers, (c) three numbers?

§ 10.2. Binomial Formula

1. Raising the Binomial $x + 1$ to Natural Power. Let us derive a formula for representing the expression $(x + 1)^n$ as a sum of monomials. We find in turn

$$(x + 1)^1 = x + 1,$$
$$(x + 1)^2 = x^2 + 2x + 1,$$
$$(x + 1)^3 = x^3 + 3x^2 + 3x + 1,$$
$$(x + 1)^4 = (x + 1)^3 (x + 1) = x^4 + 4x^3 + 6x^2 + 4x + 1.$$

Note that in all the cases considered $(x + 1)^n$ represents a polynomial of degree n whose coefficients coincide with the elements of the corresponding row of Pascal's triangle (see subsection 4 of the preceding section). For example, in the expansion $(x + 1)^3 = x^3 + 3x^2 + 3x + 1$ the coefficients of the polynomial are respectively equal to 1, 3, 3, 1, and the third row of Pascal's triangle has the same form; in the expansion $(x + 1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$ the coefficients of the polynomial are respectively equal to 1, 4, 6, 4, 1, the fourth row of Pascal's triangle being exactly the same, and so on.

In general, the nth row of Pascal's triangle has the form C_0^n, C_1^n, C_2^n, ..., C_n^n. Therefore a hypothesis arises that

$$(x+1)^n = C_0^n x^n + C_1^n x^{n-1} + C_2^n x^{n-2} + \ldots + C_{n-1}^n x + C_n^n. \qquad (1)$$

Let us prove this hypothesis $A(n)$ by the method of mathematical induction.

□ We saw above that this hypothesis is true for $n = 1$ (and even for $n = 2, 3, 4$). Let us prove that $A(k) \Rightarrow A(k + 1)$, that is, that the equality

$$(x+1)^k = C_0^k x^k + C_1^k x^{k-1} + C_2^k x^{k-2} + \ldots + C_{m-1}^k x^{k-m+1} + C_m^k x^{k-m} + \ldots + C_{k-1}^k x + C_k^k$$

implies the equality

$$(x+1)^{k+1} = C_0^{k+1} x^{k+1} + C_1^{k+1} x^k + C_2^{k+1} x^{k-1} + \ldots + C_m^{k+1} x^{k-m+1} + \ldots + C_k^{k+1} x + C_{k+1}^{k+1}.$$

We have

$$\begin{aligned}(x+1)^{k+1} &= (x+1)^k (x+1) = (C_0^k x^k + C_1^k x^{k-1} + C_2^k x^{k-2} \\ &\quad + \ldots + C_{m-1}^k x^{k-m+1} + C_m^k x^{k-m} + \ldots + C_{k-1}^k x + C_k^k)(x+1) \\ &= C_0^k x^{k+1} + (C_0^k + C_1^k) x^k + (C_1^k + C_2^k) x^{k-1} \\ &\quad + \ldots + (C_{m-1}^k + C_m^k) x^{k-m+1} + \ldots + (C_{k-1}^k + C_k^k) x + C_k^k.\end{aligned}$$

Equality (10) obtained in the preceding section implies that

$$C_0^k + C_1^k = C_1^{k+1},\ C_1^k + C_2^k = C_2^{k+1},$$
$$\ldots,\ C_{m-1}^k + C_m^k = C_m^{k+1},\ \ldots,\ C_{k-1}^k + C_k^k = C_k^{k+1}.$$

In addition, $C_0^k = C_k^k = C_0^{k+1} = C_{k+1}^{k+1} = 1$. Thus, we get

$$(x+1)^{k+1} = C_0^{k+1}x^{k+1} + C_1^{k+1}x^k + C_2^{k+1}x^{k-1} + \ldots + C_m^{k+1}x^{k-m+1} + \ldots + C_k^{k+1}x + C_{k+1}^{k+1}.$$

On the basis of the principle of mathematical induction, formula (1) is valid for any $n \in \mathbf{N}$. ■

Setting $x = 1$ in equality (1), we get

$$C_0^n + C_1^n + C_2^n + \ldots + C_n^{n-1} + C_n^n = 2^n. \qquad (2)$$

This means that the *number of all subsets of an n-element set is equal to* 2^n.

Example. Find the expansion of $(x+1)^6$.

○ We have

$$(x+1)^6 = C_0^6x^6 + C_1^6x^5 + C_2^6x^4 + C_3^6x^3 + C_4^6x^2 + C_5^6x + C_6^6$$
$$= x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$

(in the process of computation, we took advantage of the fact that $C_4^6 = C_2^6$, and $C_5^6 = C_1^6$). ●

2. Natural Power of a Binomial. Let us derive a formula for raising a binomial to any natural power.

Consider the expression $(a+b)^n$ and perform the following transformations:

$$(a+b)^n = \left(b\left(\frac{a}{b}+1\right)\right)^n = b^n\left(\frac{a}{b}+1\right)^n.$$

Setting $a/b = x$ and using formula (1), we get

$$(a+b)^n = b^n\left(\frac{a}{b}+1\right)^n = b^n(x+1)^n = b^n(C_0^n x^n + C_1^n x^{n-1} + C_2^n x^{n-2} + \ldots + C_{n-1}^n x + C_n^n) = b^n\left(C_0^n\left(\frac{a}{b}\right)^n + C_1^n\left(\frac{a}{b}\right)^{n-1} + C_2^n\left(\frac{a}{b}\right)^{n-2} + \ldots + C_{n-1}^n\frac{a}{b} + C_n^n\right) = C_0^n a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}b^2 + \ldots + C_{n-1}^n ab^{n-1} + C_n^n b^n.$$

Thus,

$$(a+b)^n = C_0^n a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}b^2 + \ldots + C_{n-1}^n ab^{n-1} + C_n^n b^n. \qquad (3)$$

This formula is called the **binomial formula**, its right-hand side being termed the **binomial expansion.** The coefficients C_m^n in the binomial expansion are called the *binomial coefficients.*

Note that the expansion of the nth power of the binomial $a + b$ contains $n + 1$ terms, the exponents of a successively decreasing from n to 0, while those of b successively increasing from 0 to n. In a binomial expansion, the coefficients of the terms equidistant from the beginning and the end of the expansion are equal: this follows from the equality $C_m^n = C_{n-m}^n$. We also note that the sum of the binomial coefficients in the expansion of $(a + b)^n$ is equal to 2^n (see equality (2)).

Example 1. Find the binomial expansion of $(2x - 3)^5$.

○ Setting $a = 2x$, $b = -3$, we get

$$(2x-3)^5 = (a+b)^5 = C_0^5 a^5 + C_1^5 a^4 b + C_2^5 a^3 b^2 + C_3^5 a^2 b^3 + C_4^5 a b^4 + C_5^5 b^5 = (2x)^5 + 5\cdot(2x)^4\cdot(-3) + 10\cdot(2x)^3\cdot(-3)^2 + 10\cdot(2x)^2\cdot(-3)^3 + 5\cdot(2x)\cdot(-3)^4 + (-3)^5 = 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243. \bullet$$

Example 2. The fifth term of the expansion of $\left(\sqrt[3]{x} + \frac{1}{x}\right)^n$ is independent of x. Find n.

○ The fifth term of the expansion T_5 has the following form:

$$T_5 = C_4^n \left(\sqrt[3]{x}\right)^{n-4} \cdot \left(\frac{1}{x}\right)^4 = C_4^n \cdot x^{\frac{n-4}{3}-4}.$$

By the hypothesis, T_5 is independent of x; this means that the exponent of x is zero, that is, $(n - 4)/3 - 4 = 0$. From the last equation we find $n = 16$. ●

Exercises

1. Write the binomial expansion of:

(a) $(x+1)^7$, (b) $(x-2)^5$, (c) $(3x+2y)^4$, (d) $\left(x^2 - \frac{1}{x}\right)^6$.

2. Find: (a) the binomial coefficient of the middle term of the expansion of $(a + b)^{20}$, (b) the fourth term of the expansion of $(8x - 5y)^6$.

3. The sum of the binomial coefficients of the expansion of $\left(\frac{2x}{3} + \frac{3}{2nx^2}\right)^n$ is equal to 64. Write the term containing no variable x.

4. The sum of the third from the beginning and the third from the end binomial coefficients of the expansion of $(\sqrt[4]{3} + \sqrt[3]{4})^n$ is equal to 9900. How many rational terms are contained in this expansion?

Chapter 11

COMPLEX NUMBERS

§ 11.1. Algebraic Form of a Complex Number

1. Notion of a Complex Number. The process of extending the concept of number from natural to real numbers was prompted both by man's practice and the needs of mathematics itself.

The concept of number grew out of the counting of objects. Counting resulted in the numbers 1, 2, 3, and so on, which are now termed natural numbers. Then, the necessity of performing the operation of division led to the notion of positive fractional numbers; further, the necessity of carrying out the operation of subtraction to the notions of zero and negative numbers (in other words, the negative numbers (and zero) came into mathematics with the development of algebra); finally, the necessity of taking roots of positive numbers to the notion of irrational numbers. The above operations are feasible in the set of real numbers. However, there are still impracticable operations, for instance, taking a square root of a negative number. Hence, there is a necessity in further extension of the concept of number, in appearing new numbers different from real ones.

Geometrically, real numbers are represented by points on the number line: every real number corresponds to one point of the line (the "image" of a real number), and, conversely, every point of the number line corresponds to one real number. The number line is completely filled with images of real numbers (that is, without gaps); figuratively speaking, "there is no place for new numbers on it". There arises a supposition that we do not have to seek for the geometrical images of new numbers on the straight line but in the plane. However, every point M of the x, y-plane can be identified with the coordinates of this point. Therefore, it is only natural to introduce ordered pairs of real numbers as new numbers which are called complex numbers (ordered in the sense that the pairs (a, b) and (b, a) are different points, and, hence, different numbers).

Definition. A *complex number* is any ordered pair (a, b) of real numbers a and b.

Note that this is not a new idea in mathematics. In fact, a common fraction is also an ordered pair of natural numbers (m, n), only another form of notation is conventionally used: m/n.

The aforesaid implies that every complex number (a, b) corresponds to the only point of the coordinate plane, namely, a point

with coordinates (a, b). In this case, the coordinate plane is called the *complex plane*, the same as the coordinate line is identified with the number line. The converse is also true: every point of the complex plane corresponds to the only complex number, that is, to a pair of real numbers representing the coordinates of the point. Therefore, the following definition of equality of complex numbers is quite relevant.

Definition. Two complex numbers (a, b) and (c, d) are said to be *equal* if and only if $a = c$ and $b = d$.

It is clear that for the numbers introduced to be actual numbers but not empty symbols, it is necessary that the arithmetic operations on them be defined so that the properties, similar to Properties 1^0-9^0 of real numbers (see § 1.3, Item 2), are fulfilled. The idea to give such a definition which would retain some customary properties is also familiar to us: we used it in § 2.4 for introducing powers with a fractional, zero, and negative exponent.

2. Addition and Subtraction of Complex Numbers.

Definition. The *sum of two complex numbers* $z = (a, b)$ and $w = (c, d)$ is a complex number $(a + c, b + d)$; it will be denoted by $z + w$.

For instance, $(2, 7) + (3, -4) = (2 + 3, 7 - 4) = (5, 3)$; $(-1, 0) + (4, 7) = (-1, +4, 0 + 7) = (3, 7)$.

The pair $(0, 0)$ will be regarded as a *complex zero* and denoted by 0. The number $(-a, -b)$ will be regarded as a number *opposite* to the number $z = (a, b)$ and denoted by $-z$.

With the definition of the operation of addition introduced, the following properties are fulfilled:

1^0. $z + w = w + z$.

2^0. $z + (w + u) = (z + w) + u$.

3^0. $z + 0 = z$.

4^0. $z + (-z) = 0$.

For instance, let us prove Properties 1^0 and 3^0.

□ Let $z = (a, b)$, $w = (c, d)$. Then $z + w = (a + c, b + d)$, $w + z = (c + a, d + b)$. But for real numbers a, b, c, and d the equalities $a + c = c + a$ and $b + d = d + b$ hold true; hence, $(a + c, b + d) = (c + a, d + b)$, that is, $z + w = w + z$, and Property 1^0 has been proved.

We have: $z + 0 = (a, b) + (0, 0) = (a + 0, b + 0) = (a, b) = z$; thus $z + 0 = z$, and Property 3^0 has been proved. ■

The *difference* of the numbers z and w is defined, as usual, as a number u such that $z = w + u$; let us show that the difference always exists and is unique.

Let $z = (a, b)$, $w = (c, d)$, $u = (x, y)$. Then $(a, b) = (c, d) + (x, y)$, that is, $(a, b) = (c + x, d + y)$. This means that $a = c + x$, $b = d + y$, whence we find: $x = a - c$, $y = b - d$, that is, $u = (x, y) = (a - c, b - d)$.

Thus, we get the following *rule for subtracting complex numbers*: $(a, b) - (c, d) = (a - c, b - d)$.

For example, $(9, 10) - (8, 12) = (9 - 8, 10 - 12) = (1, -2)$.

3. Multiplication and Division of Complex Numbers. We have established above that

$$(a, b) + (c, d) = (a + c, b + d), \quad (a, b) - (c, d) = (a - c, b - d).$$

There arises a quite natural desire to define the multiplication of complex numbers in a similar way, that is, to assume that $(a, b) \cdot (c, d) = (ac, bd)$. However, it turns out that with such a "definition" some customary properties of the operation of multiplication will not be fulfilled. Thus, the product of two nonzero numbers may turn out to be equal to zero, for instance, $(5, 0) \cdot (0, 7) = (0, 0)$. The result of division can be ambiguous, for instance: $(3, 5) \cdot (2, 0) = (6, 0)$; hence, $(6, 0) \div (2, 0) = (3, 5)$; $(3, 7) \cdot (2, 0) = (6, 0)$; hence, $(6, 0) \div (2, 0) = (3, 7)$; in general, $(3, y) \cdot (2, 0) = (6, 0)$, hence, $(6, 0) \div (2, 0) = (3, y)$.

Consequently, it is impossible to define $(a, b) \cdot (c, d)$ as (ac, bd). We have to look for another definition such that all the basic properties of the operation of multiplication are fulfilled. The following is such a definition.

Definition. The *product of two complex numbers* $z = (a, b)$ and $w = (c, d)$ is a complex number $(ac - bd, ad + bc)$ which will be denoted as zw.

For example, if $z = (2, 5)$ and $w = (3, 1)$, then $zw = (2 \cdot 3 - 5 \cdot 1, 2 \cdot 1 + 5 \cdot 3) = (1, 17)$.

It is possible to prove (we are not going to do this here) that with such a definition the properties similar to Properties 5^0-9^0 of real numbers are fulfilled (see page 35).

Let $z = (a, b)$, $w = (c, d) \neq (0, 0)$. Let us show that there is one and only one complex number $u = (x, y)$ such that $z = uw$. This number u will be called, as usual, the *quotient* of z divided by w and denoted by $z \div w$ or z/w.

We have $uw = (x, y) \cdot (c, d) = (xc - yd, xd + yc)$, $z = (a, b)$. Since $z = uw$, the following equalities must be satisfied:

$$\begin{cases} xc - yd = a, \\ xd + yc = b. \end{cases}$$

From this linear system of two equations in two variables we find $x = \frac{ac + bd}{c^2 + d^2}$, $y = \frac{bc - ad}{c^2 + d^2}$. Thus,

$$u = (x, y) = \left(\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2} \right).$$

We have obtained the following *rule for dividing complex numbers*: if $(c, d) \neq (0, 0)$, then

$$\frac{(a, b)}{(c, d)} = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right).$$

For example,

$$\frac{(2, 3)}{(1, 4)} = \left(\frac{2\cdot 1+3\cdot 4}{1^2+4^2}, \frac{3\cdot 1-2\cdot 4}{1^2+4^2}\right) = \left(\frac{14}{17}, \frac{5}{17}\right).$$

4. Algebraic Form of a Complex Number. Using the above definitions of the addition and multiplication of complex numbers, we can easily obtain the following equalities:

$$(0, 1)\cdot(0, 1) = (-1, 0), \tag{1}$$

$$(a, b) = (a, 0) + (b, 0)\cdot(0, 1), \tag{2}$$

$$(a, 0) + (b, 0) = (a + b, 0), \tag{3}$$

$$(a, 0)\cdot(b, 0) = (ab, 0). \tag{4}$$

Let us agree to write a instead of $(a, 0)$. This is justified from the geometrical point of view since the number $(a, 0)$ corresponds to the same point on the number line as the real number a. This is justified from the algebraic point of view as well since equalities (3) and (4) then take the form $a + b = a + b$ and $ab = ab$, respectively.

Further, let us agree to denote the complex number $(0, 1)$ by the letter i and call it the *unit imaginary number* (or *imaginary unit*). Then equality (1) takes the form $i\cdot i = 1$, that is,

$$i^2 = -1 \tag{5}$$

and equality (2) the form

$$(a, b) = a + bi. \tag{6}$$

The notation $a + bi$ is called the *algebraic form of the complex number* $z = (a, b)$, the number a being the *real part* of the complex number z, and bi its *imaginary part*.

For example, $(2, -4) = 2 - 4i$; $(3, 2) = 3 + 2i$; $-7 + \sqrt{3}\cdot i = (-7, \sqrt{3})$.

If both parts of a complex number $a + bi$ are different from zero, then such a number is called *imaginary*, if $a = 0$ and $b \neq 0$, then it is *pure imaginary* and has the form bi; finally, if the imaginary part of a complex number $a + bi$ is equal to zero, then a real number a is obtained.

The algebraic form of the complex number facilitates considerably the performance of all the arithmetic operations which we shall see below.

1. Addition. We know that

$$(a, b) + (c, d) = (a + c, b + d). \tag{7}$$

Carrying out the addition of the same numbers in algebraic form and assuming $a + bi$ and $c + di$ to be usual binomials, we find

$$(a + bi) + (c + di) = (a + c) + (b + d)\, i. \tag{8}$$

Comparing equalities (7) and (8), we see that the obtained result is correct.

2. **Subtraction.** We know that

$$(a,\ b) - (c,\ d) = (a - c,\ b - d). \tag{9}$$

Now, carrying out the subtraction of the same numbers in algebraic form and assuming $a + bi$ and $c + di$ to be usual binomials, we find

$$(a + bi) - (c + di) = (a - c) + (b - d)\, i. \tag{10}$$

Comparing equalities (9) and (10), we make sure that the result is correct.

3. **Multiplication.** We know that

$$(a,\ b)\cdot(c,\ d) = (ac - bd,\ ad + bc). \tag{11}$$

Carrying out the multiplication of the same numbers in algebraic form and assuming $a + bi$ and $c + di$ to be usual binomials, we have

$$(a + bi)\,(c + di) = ac + bci + adi + bdi^2.$$

Using the fact that $i^2 = -1$ (see equality (5)), we get $bdi^2 = -bd$. We then find

$$(a + bi)\,(c + di) = (ac - bd) + (ad + bc)\, i. \tag{12}$$

Comparing equalities (11) and (12), we see that the result obtained is correct.

4. **Division.** We know that if $(c,\ d) \neq (0,\ 0)$, then

$$\frac{(a,\ b)}{(c,\ d)} = \left(\frac{ac+bd}{c^2+d^2}\ ,\ \frac{bc-ad}{c^2+d^2}\right). \tag{13}$$

Let us now carry out the division of the same numbers in algebraic form, assuming $a + bi$ and $c + di$ to be usual binomials and $\frac{a+bi}{c+di}$ an ordinary common fraction. Multiplying the numerator and denominator of this fraction by $c - di$ (supposing that the value of the fraction will remain unchanged), we find

$$\frac{a+bi}{c+di} = \frac{(a+bi)\,(c-di)}{(c+di)\,(c-di)} = \frac{ac+bci-adi-bdi^2}{c^2-d^2i^2}$$

$$= \frac{ac+bd+(bc-ad)\,i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\, i.$$

Thus,

$$\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\, i. \tag{14}$$

Comparing equalities (13) and (14), we see that the result is correct.

Summarizing, we arrive at the following important practical conclusion: *we may carry out all the arithmetic operations on complex numbers represented in algebraic form, as on ordinary binomials, taking only into account that* $i^2 = -1$. *In order to transform a common fraction of the form* $\frac{a+bi}{c+di}$ *into a complex number, we have to multiply the numerator and denominator of the fraction by the number* $c - di$; $c + di$ *and* $c - di$ are said to be *conjugate complex numbers*.

Example 1. Compute $(2 - i)^2$.

○ Applying formula $(a - b)^2 = a^2 - 2ab + b^2$, we get

$$(2 - i)^2 = 4 - 4i + i^2 = 4 - 4i - 1 = 3 - 4i. \bullet$$

Example 2. Compute $(1 + 2i)\, i - \frac{3+2i}{1-i}$.

○ (1) $(1 + 2i)\, i = i + 2i^2 = -2 + i$.

(2) $$\frac{3+2i}{1-i} = \frac{(3+2i)(1+i)}{(1-i)(1+i)} = \frac{3+2i+3i+2i^2}{1-i^2}$$
$$= \frac{3+5i-2}{1+1} = \frac{1+5i}{1} = \frac{1}{2} + \frac{5}{2}\, i.$$

(3) $$(-2+i) - \left(\frac{1}{2} + \frac{5}{2}\, i\right) = -\frac{5}{2} - \frac{3}{2}\, i. \bullet$$

Example 3. Find the real numbers x and y satisfying the equality $(2x - 3yi)(2x + 3yi) + 4xi = 97 + 8i$.

○ We have $(2x - 3yi)(2x + 3yi) = 4x^2 - 9y^2 i^2 = 4x^2 + 9y^2$. Then the given equality can be rewritten as follows

$$4x^2 + 9y^2 + 4xi = 97 + 8i.$$

Since two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$, we get the system of equations

$$\begin{cases} 4x^2 + 9y^2 = 97, \\ 4x = 8, \end{cases}$$

from which we find $x_1 = 2$, $y_1 = 3$, $x_2 = 2$, $y_2 = -3$. ●

Example 4. Find the complex numbers z satisfying the equality $z^2 = -5 + 12i$.

○ We shall seek for a complex number z in the form $x + yi$. We have

$$(x + yi)^2 = -5 + 12i; \quad x^2 + 2xyi + y^2 i^2 = -5 + 12i;$$
$$x^2 - y^2 + 2xyi = -5 + 12i.$$

The last equality implies that $\begin{cases} x^2 - y^2 = -5 \\ 2xy = 12 \end{cases}$. This system has two solutions: (2, 3) and $(-2, -3)$. Hence, $z_1 = 2 + 3i$, $z_2 = -2 - 3i$. ●

Example 5. Compute $(1 + 2i)^6$.

○ Using the binomial formula (see § 10.2), we get

$$(1+2i)^6 = 1^6 + C_1^6 \cdot 1^5 \cdot 2i + C_2^6 \cdot 1^4 \cdot (2i)^2 + C_3^6 \cdot 1^3 \cdot (2i)^3 + C_4^6 \cdot 1^2 \cdot (2i)^4 + C_5^6 \cdot 1 \cdot (2i)^5 + (2i)^6.$$

Since $C_1^6 = 6$, $C_2^6 = \frac{6 \cdot 5}{1 \cdot 2} = 15$, $C_3^6 = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20$, $C_4^6 = C_2^6 = 15$; $C_5^6 = C_1^6 = 6$, we have

$$(1+2i)^6 = 1 + 12i + 60i^2 + 160i^3 + 240i^4 + 192i^5 + 64i^6.$$

Further we have $i^2 = -1$, $i^3 = i^2 \cdot i = -i$, $i^4 = i^2 \cdot i^2 = (-1) \times (-1) = 1$, $i^5 = i^4 \cdot i = 1 \cdot i = i$, $i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1$. Hence,

$$(1+2i)^6 = 1 + 12i - 60 - 160i + 240 + 192i - 64 = 117 + 44i. \bullet$$

5. Determining Complex Roots of Equations. Let $a > 0$. Since $(\sqrt{ai})^2 = ai^2 = -a$, we have $\sqrt{-a} = \sqrt{a}i$. Thereby we get a possibility to take square roots of negative real numbers. This enables finding not only real, but also imaginary roots of equations.

Example 1. Solve the equation $x^2 - 4x + 13 = 0$.

○ We have $x_{1,2} = 2 \pm \sqrt{4-13} = 2 \pm \sqrt{9} = 2 \pm 3i$. Thus, $x_1 = 2+3i$, $x_2 = 2-3i$. ●

Example 2. Solve the equation $x^3 = 8$.

○ We have $x^3 - 8 = 0$; $(x-2)(x^2 + 2x + 4) = 0$. Hence, either $x - 2 = 0$, whence we find $x_1 = 2$, or $x^2 + 2x + 4 = 0$, whence we find $x_{2,3} = -1 \pm \sqrt{1-4} = -1 \pm \sqrt{3}i$. Thus, $x_1 = 2$, $x_2 = -1 + \sqrt{3}i$, $x_3 = -1 - \sqrt{3}i$. ●

Exercises

In Exercises 1 to 3, compute the given expressions:

1. (a) i^{100}, (b) i^{30}, (c) $-i^5 - 2i^4 + i + 2$.
2. (a) $(3-2i)^2 - (1+2i)^3 + (1-i)(2+3i)$, (b) $(1+i)^7$, (c) $(2-i)^5$.
3. (a) $\frac{6+7i}{2i}$, (b) $\frac{-2+i}{1-i}$, (c) $\frac{(7+2i)(1-i)}{-3+i}$.
4. For what real values of x is the complex number $2x - 5ix - (3 + ix^2)$: (a) real, (b) pure imaginary?
5. For what real values of x is the equality $z_1 = z_2$ fulfilled if $z_1 = x^2 - 6 + 3xi^7$, and $z_2 = x^2 i - 10i - x$?
6. For what real values of x are the complex numbers $z_1 = -4xi + 3 - 11x$ and $z_2 = x^2 i - 21i + 4x^2$ conjugate?
7. For what real value of x is the equality $x^2 + i^5 - i^6 - xi^{11} = 0$ valid?
8. Factor the given binomial into conjugate complex multipliers: (a) $a^2 + 4b^2$, (b) $3m^2 + 7n^2$.
9. Set up a quadratic equation with real coefficients knowing its roots x_1 and x_2 if: (a) $x_1 = 1+2i$, $x_2 = 1-2i$, (b) $x_1 = \frac{4-2i}{1-i}$, $x_2 = \frac{4+2i}{1+i}$.
10. Find z if (a) $\sqrt{z} = 2-3i$, (b) $\sqrt[3]{z} = 5+i$, (c) $z^2 = 3+4i$, (d) $z^2 + z = -9+3i$.

In Exercises 11 to 14, find the complex roots of the given equations.

11. (a) $x^2 - x + 1 = 0$, (b) $x^2 - 8x + 25 = 0$.
12. (a) $27x^3 = 8$, (b) $8x^3 + 1 = 0$.
13. (a) $16x^4 = 625$, (b) $x^4 + 81 = 0$.
14. $x^4 + 4x^2 + 3 = 0$.

§ 11.2. Trigonometric Form of a Complex Number

1. Modulus of a Complex Number. Let there be given a complex number $z = a + bi$. As is known (see § 11.1), geometrically, it is represented on the complex plane by a point M with coordinates a and b (Fig. 253). Let us join the point M to the origin O. The line segment OM is called the *radius vector* of the point M, and its length, that is, the *modulus of the complex number* $z = a + bi$. The modulus of the number z is denoted, as usual, by $|z|$. Thus, if $z = a+bi$, then $|z| = \sqrt{a^2 + b^2}$.

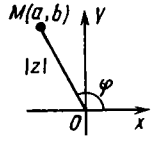


Fig. 253

For example, $|3-4i| = \sqrt{3^2+(-4)^2} = 5$.

If the complex number $z = a + bi$ is real, that is, if $b = 0$, then we get $|z| = |a| = \sqrt{a^2 + 0^2} = \sqrt{a^2}$. Hence, the notion of the modulus of a real number is a particular case of the more general notion of the modulus of a complex number.

The modulus of a complex number possesses the same properties as the modulus of a real number. Let us prove, for instance, that $|z_1 z_2| = |z_1| \cdot |z_2|$.

□ Let $z_1 = a+bi$, $z_2 = c+di$. Then $|z_1| = \sqrt{a^2+b^2}$, $|z_2| = \sqrt{c^2+d^2}$ and, consequently,

$$|z_1| \cdot |z_2| = \sqrt{a^2+b^2} \cdot \sqrt{c^2+d^2} = \sqrt{a^2c^2+b^2c^2+a^2d^2+b^2d^2}. \quad (1)$$

We now compute $|z_1 z_2|$. We have $z_1 z_2 = (a+bi)(c+di) = (ac-bd) + (ad+bc)i$. Hence,

$$|z_1 z_2| = \sqrt{(ac-bd)^2+(ad+bc)^2} = \sqrt{a^2c^2+b^2c^2+a^2d^2+b^2d^2}. \quad (2)$$

The equality of the right-hand members of expressions (1) and (2) implies the equality of their left-hand members, that is, $|z_1 z_2| = |z_1| \cdot |z_2|$. ■

2. Argument of a Complex Number. The angle φ that the radius vector of the point $M(a, b)$ makes with the positive horizontal axis is called the *argument* (or *amplitude*) *of the complex number* $z = a + bi$ (Fig. 253). It is clear that this angle is defined ambiguously: if φ is the argument of the number $z = a + bi$, then $\varphi + 2\pi k$ is also the argument of the number z for any integral value of k. For an angle to be defined uniquely, it must be chosen in the range from $-\pi$ to π. If the argument is chosen within the indicated limits, then it is denoted as follows: $\arg z$: $-\pi < \arg z \leqslant \pi$.

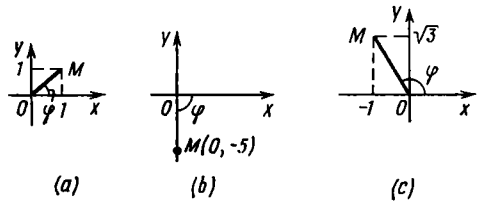
Example. Find $\arg z$ if (1) $z = 1 + i$, (2) $z = -5i$, (3) $z = -1 + \sqrt{3}i$.

○ (1) The number $z = 1 + i$ is represented by the point M (1, 1) in the complex plane. The radius vector of this point makes an angle $\pi/4$ with the positive direction of the x-axis (Fig. 254a). Hence, $\arg z = \pi/4$.

(2) The number $z = -5i$ is represented by the point M (0, −5). The radius vector of this point makes an angle $-\pi/2$ with the positive direction of the x-axis (Fig. 254b). Therefore $\arg z = -\pi/2$.

(3) The number $z = -1 + \sqrt{3}i$ is represented by the point M $(-1, \sqrt{3})$. The radius vector of this point makes an angle $2\pi/3$ with the positive direction of the x-axis (Fig. 254c). Consequently, $\arg z = 2\pi/3$. ●

3. Trigonometric Form of a Complex Number. Let a complex number $z = a + bi$ be represented by the point M (a, b) in the com-



(a) (b) (c)

Fig. 254

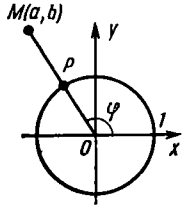


Fig. 255

plex plane, and let the radius vector of the point M intersect at the point P the unit circle with centre at the origin (Fig. 255). As is known, the point P has the coordinates $\cos\varphi$ and $\sin\varphi$. But $OP = 1$, and $OM = |z|$, and, hence, the point M has the coordinates $|z|\cos\varphi$, $|z|\sin\varphi$, that is,

$$a = |z|\cos\varphi, \quad b = |z|\sin\varphi. \tag{3}$$

Then, we get

$$\begin{aligned} z = a + bi &= |z|\cos\varphi + |z|\sin\varphi \cdot i \\ &= |z|(\cos\varphi + i\sin\varphi), \end{aligned}$$

that is,

$$z = |z|(\cos\varphi + i\sin\varphi). \tag{4}$$

Equality (4) is called the *trigonometric form of a complex number*. In order to find the trigonometric form of the complex number $z = a + bi$, we have to compute $|z| = \sqrt{a^2 + b^2}$ and find the argument z. Sometimes, the argument of the number z can be found geometrically as it was done in the example from Item 2, but more frequently the argument is found by using the formulas following

from the above equalities given by (3):

$$\cos\varphi = \frac{a}{|z|}, \quad \sin\varphi = \frac{b}{|z|}.$$

Example. Represent the complex number z in trigonometric form if: (1) $z = \sqrt{3} + i$, (2) $z = -8$, (3) $z = 3i$, (4) $z = 2 - 2i$.

○ (1) We have $a = \sqrt{3}$, $b = 1$; hence $|z| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$. Further, $\cos\varphi = a/|z| = \sqrt{3}/2$, $\sin\varphi = b/|z| = 1/2$; hence, $\varphi = \pi/6$. As a result, we get $z = \sqrt{3} + i = 2(\cos\pi/6) + i\sin(\pi/6))$.

(2) We have: $a = -8$, $b = 0$; hence, $|z| = \sqrt{(-8)^2 + 0^2} = 8$. Further, $\cos\varphi = -8/8 = -1$, $\sin\varphi = 0/8 = 0$; whence $\varphi = \pi$, $z = -8 = 8(\cos\pi + i\sin\pi)$.

(3) We have $a = 0$, $b = 3$; hence, $|z| = \sqrt{0^2 + 3^2} = 3$. Further, $\cos\varphi = 0$, $\sin\varphi = 1$; whence $\varphi = \pi/2$. Therefore $z = 3i = 3(\cos(\pi/2) + i\sin(\pi/2))$.

(4) We have $a = 2$, $b = -2$, hence $|z| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$. Further, $\cos\varphi = 2/(2\sqrt{2}) = \sqrt{2}/2$, $\sin\varphi = -2/(2\sqrt{2}) = -\sqrt{2}/2$, whence $\varphi = -\pi/4$. Thus, $z = 2 - 2i = 2\sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4))$. ●

4. De Moivre's Formula. The algebraic form of a complex number enables us to carry out rather easily such arithmetic operations on complex numbers as addition, subtraction, multiplication, and division, whereas raising a complex number to a natural power is more convenient in trigonometric form. For this purpose, *De Moivre's formula* is usually used:

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi. \tag{5}$$

□ Let us prove formula (5) using the method of mathematical induction. For $n = 1$ it is, obviously, true. Suppose that it is true for $n = k$, that is,

$$(\cos\varphi + i\sin\varphi)^k = \cos k\varphi + i\sin k\varphi. \tag{6}$$

Let us prove that formula (5) is then true for $n = k + 1$ as well, that is,

$$(\cos\varphi + i\sin\varphi)^{k+1} = \cos(k+1)\varphi + i\sin(k+1)\varphi. \tag{7}$$

Indeed, we have

$$(\cos\varphi + i\sin\varphi)^{k+1} = (\cos\varphi + i\sin\varphi)^k \cdot (\cos\varphi + i\sin\varphi).$$

Using equality (6), we get

$$(\cos\varphi + i\sin\varphi)^k (\cos\varphi + i\sin\varphi) = (\cos k\varphi + i\sin k\varphi) \times (\cos\varphi + i\sin\varphi) = (\cos k\varphi\cos\varphi - \sin k\varphi\sin\varphi) + i(\sin k\varphi\cos\varphi + \cos k\varphi\sin\varphi).$$

The expression in the first parentheses is the cosine of the sum of the arguments $k\varphi$ and φ, that is, $\cos(k+1)\varphi$; the expression in the second parentheses is the sine of the sum of the arguments $k\varphi$ and φ, that is, $\sin(k+1)\varphi$. Hence,

$$(\cos\varphi + i\sin\varphi)^{k+1} = \cos(k+1)\varphi + i\sin(k+1)\varphi,$$

and the validity of equality (7) has been proved.

According to the principle of mathematical induction, we conclude that formula (5) is true for any natural number n. ■

From De Moivre's formula it is not difficult to get **the rule for raising a complex number to a natural power:** if $z = |z|(\cos\varphi + i\sin\varphi)$, then

$$z^n = |z|^n(\cos n\varphi + i\sin n\varphi). \tag{8}$$

Example 1. Compute $(-1+i)^{13}$.

○ Let us represent the number $z = -1 + i$ in trigonometric form. We have $a = -1$, $b = 1$, $|z| = \sqrt{2}$, $\cos\varphi = -1/\sqrt{2}$, $\sin\varphi = 1/\sqrt{2}$, $\varphi = 3\pi/4$. Thus,

$$-1+i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).$$

Applying De Moivre's formula, we get

$$(-1+i)^{13} = (\sqrt{2})^{13}\cdot\left(\cos\left(13\cdot\frac{3\pi}{4}\right) + i\sin\left(13\cdot\frac{3\pi}{4}\right)\right).$$

It remains to carry out all necessary computations and simplifications. We have

$$\cos\left(13\cdot\frac{3\pi}{4}\right) = \cos\frac{39\pi}{4} = \cos\left(8\pi + \frac{7\pi}{4}\right) = \cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2},$$

$$\sin\left(13\cdot\frac{3\pi}{4}\right) = \sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}.$$

Further,

$$(\sqrt{2})^{13}\left(\frac{\sqrt{2}}{2} - i\cdot\frac{\sqrt{2}}{2}\right) = 64\sqrt{2}\left(\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}\right) = 64(1-i) = 64 - 64i.$$

Thus, $(-1+i)^{13} = 64 - 64i$. ●

In conclusion, we should like to note that complex numbers which were not recognized for a long time even in mathematics turned out to be useful when solving problems on propagation of sound and light, various problems of electrical and radio engineering, aircraft construction, motion of liquids and gases, and other problems of engineering and natural science. Complex numbers are also widely used in various fields of contemporary mathematics. By way of example, let us show how some trigonometric formulas can be derived using complex numbers.

Example 2. Express $\cos 5\varphi$ and $\sin 5\varphi$ in terms of $\cos\varphi$ and $\sin\varphi$.

○ Using De Moivre's formula, we find

$$(\cos\varphi + i\sin\varphi)^5 = \cos 5\varphi + i\sin 5\varphi. \tag{9}$$

On the other hand, we compute $(\cos\varphi + i\sin\varphi)^5$ by the binomial formula:

$$\begin{aligned}(\cos\varphi + i\sin\varphi)^5 = {} & \cos^5\varphi + C_1^5\cos^4\varphi\,(i\sin\varphi) \\ & + C_2^5\cos^3\varphi\,(i\sin\varphi)^2 + C_3^5\cos^2\varphi\,(i\sin\varphi)^3 \\ & + C_4^5\cos\varphi\,(i\sin\varphi)^4 + (i\sin\varphi)^5.\end{aligned}$$

We have $C_1^5 = C_4^5 = 5$, $C_2^5 = C_3^5 = \frac{5\cdot 4}{1\cdot 2} = 10$. Let us also make use of the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$. We then get

$$\begin{aligned}(\cos\varphi + i\sin\varphi)^5 = {} & \cos^5\varphi + 5\cos^4\varphi\sin\varphi\cdot i \\ & - 10\cos^3\varphi\sin^2\varphi - 10\cos^2\varphi\sin^3\varphi\cdot i \\ & + 5\cos\varphi\sin^4\varphi + \sin^5\varphi\cdot i.\end{aligned}$$

Thus,

$$\begin{aligned}(\cos\varphi + i\sin\varphi)^5 = {} & (\cos^5\varphi - 10\cos^3\varphi\sin^2\varphi \\ & + 5\cos\varphi\sin^4\varphi) + i\,(5\cos^4\varphi\sin\varphi \\ & - 10\cos^2\varphi\sin^3\varphi).\end{aligned} \tag{10}$$

Comparing equalities (9) and (10), we get

$$\cos 5\varphi = \cos^5\varphi - 10\cos^3\varphi\sin^2\varphi + 5\cos\varphi\sin^4\varphi,$$

$$\sin 5\varphi = 5\cos^4\varphi\sin\varphi - 10\cos^2\varphi\sin^3\varphi. \bullet$$

Exercises

1. Let z and $\bar{z}$ be conjugate complex numbers. Prove that $z\cdot\bar{z} = |z|^2$.

2. As is known, for any real number a the equality $a^2 = |a|^2$ is valid. Is a similar equality valid for any complex number z?

3. Represent the complex number z by a point in the complex plane and find $|z|$: (a) $z = 2i$, (b) $z = \sqrt{2} - \sqrt{2}i$, (c) $z = -3 + 4i$.

4. Construct in the complex plane the set of points z satisfying the given relationship: (a) $|z| = 1$, (b) $|z| \leqslant 2$, (c) $|z| > 3$, (d) $1 < |z| < 2$, (e) $|z - 1| = 2$, (f) $|z + i| = 3$.

5. Prove that in the complex plane the graph of the equation $|z - (a + bi)| = r$, where $r > 0$, is a circle of radius r centred at the point (a, b).

6. Construct in the complex plane the set of points z satisfying the following relationship: (a) $|z + 2i| < 2$, (b) $|z - 2| > 2$, (c) $1 < |z - 2i| < 2$, (d) $1 < |z - 1 - 2i| < 4$.

7. Represent the complex number z by a point in the complex plane and find $\arg z$: (a) $z = i$, (b) $z = -3$, (c) $z = 3 + 3i$, (d) $z = 2 - 2i$, (e) $z = -1 - \sqrt{3}i$, (f) $z = -3 + \sqrt{3}i$.

8. Construct in the complex plane the set of points z satisfying the given relationship: (a) $\arg z = \pi/6$, (b) $\arg z = -\pi/4$, (c) $\arg z = 0$, (d) $\arg z = \pi$, (e) $0 < \arg z < \pi/4$, (f) $-2\pi/3 \leqslant \arg z \leqslant 2\pi/3$.

9. Find in the complex plane a point z such that: (a) $|z| = 1$, $\arg z = \pi/4$, (b) $|z| = 2$, $\arg z = -\pi/2$, (c) $|z| = 3$, $\arg z = 5\pi/6$, (d) $|z| = 1$, $\arg z = -3\pi/4$.

10. Construct in the complex plane the set of points z meeting the following conditions:

(a) $\begin{cases} 1 < |z| < 2, \\ \pi/3 < \arg z < 2\pi/3. \end{cases}$ (b) $\begin{cases} |z| \leqslant 3, \\ -\pi/4 \leqslant \arg z \leqslant 3\pi/4. \end{cases}$

11. Represent the given number in trigonometric form: (a) $3+3i$, (b) $-2+2\sqrt{3}i$, (c) $5\sqrt{3}-5i$, (d) $-4i$, (e) 6.

12. Find the algebraic form of the complex number given in trigonometric form: (a) $2(\cos(\pi/3) + i\sin(\pi/3))$, (b) $3\sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4))$, (c) $5(\cos\pi + i\sin\pi)$, (d) $4(\cos(\pi/12) + i\sin(\pi/12))$.

13. Compute: (a) $(\cos(\pi/6) + i\sin(\pi/6))^6$, (b) $(\cos(2\pi/3) + i\sin(2\pi/3))^{12}$.

14. Using De Moivre's formula, compute: (a) $(1-i)^{16}$, (b) $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{17}$.

15. Using De Moivre's formula, prove that $\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi$, $\sin 3\varphi = 3\sin\varphi - 4\sin^3\varphi$.

ANSWERS TO EXERCISES

Chapter 1

§ 1.1

1. 500. **2.** 1. **3.** 100. **4.** 4. **5.** 56. **7.** (a) 14, (b) 36, (c) 44, (d) 35, (e) 45. **10.** (a) 36, (b) 360, (c) 1650, (d) 2520, (e) 14700. **13.** (a) $3n + 2$, (b) $4n + 1$, (c) $5n+3$. **14.** (a) $2n$, (b) $3n$, (c) $5n$. **15.** $2n-1$. **16.** (a) 1, (b) 3. **17.** (a) 0, 2, 4, 6, 8, (b) 2, 5, 8, (c) 0, 5, (d) 8, (e) 0, 4, 8, (f) 2, 8, (g) 0.

§ 1.2

1. (a) 2.4, (6), (b) 0.0048, (c) 3.(6), (d) -1, (142857), (e) 1.708 (3). **2.** (a) $\frac{1}{3}$, (b) $2\frac{2}{11}$, (c) $\frac{1}{33}$, (d) $5\frac{223}{990}$. **3.** -0.3. **4.** $1\frac{2}{15}$. **5.** -0.4. **6.** $3\frac{7}{8}$. **7.** $-10\frac{2}{3}$. **8.** 17.11. **9.** 4. **10.** 1.12. **11.** 0. **12.** 0. **13.** 0.214. **14.** $18\frac{103}{150}$. **15.** 2.24. **16.** $1\frac{7}{27}$. **17.** 0. **18.** 11. **19.** (a) $(-2.5)^{2n}$, (b) $(-1)^{2n+1}$. **20.** (a) 35%, (b) 2%, (c) 105%, (d) 224%. **21.** (a) 0.05, (b) 0.0108, (c) 1.2, (d) 0.002. **22.** (a) 6, (b) 224, (c) 1.53, (d) 3.6. **23.** (a) 50%, (b) 7%, (c) 2000%. **24.** 15%. **25.** 72 r 90 k. **26.** 50/13. **27.** 3. **28.** 1/6.

§ 1.3

1. (a) $a < b$, (b) $a > b$, (c) $a < b$, (d) $a < b$, (e) $a > b$, (f) $a < b$. **8.** (a) $a < 0$, (b) $a > 0$, (c) $a > 0$, (d) $a < 0$. **10.** (a) $x = -24$, (b) $x = 0.001$. **13.** 1.90532. **14.** 3.31 . . . **16.** (a) $x = y$ or $x = -y$, (b) $x = y$ or $x = -y$, (c) $x = y = 0$, (d) $x = y = 0$, (c) $x = -y$. **17.** (a) $-5/3$, (b) 1, if $x = 3$, -1, if $x = -3$.

§ 1.4

5. (a) $m \neq 0$, (b) no solution, (c) $m \neq 0$, (d) $a \neq 0$, $b \neq 0$. **6.** (a) 2, (b) 3. **7.** (a) 48, (b) -1, (c) 2, (d) 15. **8.** (a) 0.2, (b) 4, (c) 0.1, (d) 3. **10.** 120. **11.** (a) $-11x^2y + 1$, (b) $-7.7p^4$.

§ 1.5

1. 3. **2.** 8. **3.** 51. **4.** 0.6. **5.** 9. **6.** $5\frac{32}{33}$. **7.** No solution. **8.** $(-\infty, +\infty)$. **9.** (a) ± 2, (b) 0, (c) no solution. **10.** (a) 7, 3, (b) 4, (c) 1, -3, (d) 5, 2, (e) -2.5, (f) no solution. **11.** (a) 0.2, (b) 1, 2, -4, (c) no solution. **12.** (a) 700, (b) 2000, (c) 600. **13.** (a) Yes, (b) yes, (c) no, (d) no, (e) no. **15.** (a) No solution, (b) 2. **16.** (a) 2, (b) no solution. **17.** (a) 2, (b) -7, -3, -2, 2, 3. **18.** (a) $3/(3 - m)$ if $m \neq 3$, (b) 1, if $a \neq 1$, $(-\infty, +\infty)$ if $a = 1$, (c) no solution, (d) $3/m$ if $m \neq 0$, $m \neq 3$, no solution if $m = 0$ or $m = 3$. **19.** (a) The only solution for $k \neq 5$ is zero (for $k = 3$), (b) no solution for $k = 5$. **20.** (a) $a = \frac{S}{b}$, (b) $b = \frac{2Q}{h}$,

(c) $Q = \frac{3V}{h}$. **21.** In 8 days. **22.** 25. **23.** 128 205. **24.** 250 persons took entrance examinations, 30 of them failed at the examination in mathematics. **25.** 500 kg. **26.** 350 km.

§ 1.6

4. (a) (3, 5), (b) (2, −3), (c) (5/3, 3/4). **7.** (a) (1, −1), (b) (5, 2.5). **11.** 4/9. **12.** 76, 24. **13.** 73. **14.** 25. **15.** 55 m, 15 m. **16.** 4 r, 1 r 20 k. **17.** 5 km/h, 3 km/h. **18.** 75°, 75°, 30°. **19.** 11 m/s, 7 m/s.

§ 1.7

3. (a) $x > 4$, (b) $x < -60$. **4.** (a) $x < 43$, (b) no solution. **5.** (a) $x < 1/3$, (b) $(-\infty, +\infty)$. **6.** (a) $x \geqslant 1/2$, (b) $-7 < x < 7$. **7.** $-29/11 < x < 10/3$. **8.** $2.7 < x < 5.6$. **9.** No solution. **10.** $x < 5$. **11.** 1, 2. **12.** −3, −2, −1, 0, 1, 2, 3. **13.** 8, 7, 6, 5, 4, ... **14.** 57. **15.** (a) for $y > 5$, for $y > -4$, for any y, (b) for $y < 5$, for $y < -4$. **16.** (a) $x > 0.5$, (b) $x < 1.5$, (c) $x < -0.4$, (d) $x < -29.5$. **17.** $-5 \leqslant x < -2$. **18.** (a) no, (b) yes. **19.** (a) $x > -3.5$, (b) $x > \frac{1}{4}$, $x > \frac{7}{3}$. **20.** $x > 4.4$. **21.** (a) $x < 0$, $x > \frac{1}{2}$, (b) $x < -1$, $x > \frac{3}{4}$, (c) $x \leqslant -\frac{1}{2}$, $x > 2$, (d) $-4 \leqslant x < 5/3$. **22.** (a) $x < -2$, $x > 2$, (b) $x < 0$, $x > 3$. **23.** $x < 0$, $x > 18$. **24.** $1 < x < 4$. **25.** $x \leqslant -\frac{4}{3}$, $x \geqslant 2$. **26.** $x < -2$, $x > 4/3$. **27.** $x \geqslant 4.5$. **28.** $x \leqslant \frac{4}{3}$, $x \geqslant \frac{3}{2}$. **29.** $x < 3.5$, $x > 3.5$. **30.** $-1 < x < 5$. **31.** $x = 5$. **32.** No solution. **33.** $x > 2.5$. **34.** $-2 < x < 1$, $2 < x < 5$.

Chapter 2

§ 2.1

1. (a) $x^3 + x^2 - 2x + 3$, (b) $-3x^3 + 2x^2 - 19x + 1$, (c) $x^5 - 5x^4 + 5x^3 - 14x^2 + x + 2$, (d) $x^4 - 10x^3 + 29x^2 - 20x + 4$. **2.** (a) 13.75, (b) 260. **5.** 6.8. **6.** (a) $(ax)^2 - (ay)^2$, (b) $(a + b)^2 - 1^2$, (c) $(x^3)^2 - (x^2)^2$. **19.** (a) $(a - 2) \times (3x - 1)$, (b) $(c - 3)(a + 7b)$. **20.** (a) $9m^3n^3(7m + 3n - 5m^2n^4)$, (b) $(5bc - 2k)(bc^2 - k)$, (c) $(xy - z)(2x^2y + 3z^2)$, (d) $(12a - 19b)(2x - 1)$. **21.** (b) $(7a - 12b)(23a - 12b)$, (c) $4(4n - t)(7n + t)$. **22.** (a) $(2t - 5n - 6)(2t - 5n + 6)$, (b) $-(p^2 - 3k)^2$, (c) $-p^3q^6(4q - 9p)^2$. **23.** (a) $6(x - 2)^3$, (b) $15a(x + 1)^3$. **24.** (a) $4.5a^{2n-1}(2a^n - 1)$, (b) $4a^4b^2(a - 2b)(a^2 + 2ab + 4b^2)$, (c) $5p^{2n}q^n(1 + 3p^{3n}q^n)$. **25.** (a) $8(x^2 - 3y^2)(5x^2 + 2y^2)$, (b) $t^3(10 + 3t) \times (100 - 3t + 9t^2)$. **26.** (a) $7(1 - 2^2b)(1 + 2a^2b + 4a^4b^2)$, (b) $2a(4a - 1)^3$. **28.** 11 and 12. **31.** (a) 4/7, (b) 0.5, (c) 20.4. **32.** (a) 48, (b) 36. **33.** (a) 3.5, −2.5, (b) 5, −2. **35.** (1) 10 000, (b) 225, (c) 14. **36.** (a) Quotient: $2x - 1$, remainder: $18x^2 - 9x + 5$, (b) quotient: 2, remainder: $-4x^2 - 2x + 2$, (c) quotient: $x^2 + 4$, remainder: $16x^2 - 1$. **38.** (a) 0, 2, −2, (b) −10, −1, 1. **39.** (a) $5(a - 5)(a^2 + 2a - 5)$, (b) $3x^2(x - 3)(8x + 3)$, (c) $(a - 5)(a + 1)$, (d) $(x - 1)(x - 2)(x - 3)$. **40.** (a) 1, −3, (b) 3, 10.

§ 2.2

5. −0.9. **6.** (a) $\frac{9x^2}{11y}$, (b) $\frac{1-4x^2}{6x}$, (c) $\frac{4-x}{x+4}$, (d) $\frac{b}{2a^2}$. **7.** (a) $\frac{1}{2a(a-5)}$, (b) $-\frac{(x-1)^2}{x+1}$, (c) $\frac{x-y+z}{x-y-z}$, (d) $\frac{x-2}{x+1}$. **8.** $\frac{x-5}{4(x-2)}$. **9.** (a) $\frac{1}{9}$,

(b) $-\frac{14}{3}$. 10. (a) $-\frac{10}{9}$, (b) $\frac{3a^9}{2}$, (c) $-\frac{4a^8}{3x}$. 11. $\frac{15a}{4(a-1)}$.
12. $-\frac{3}{2(a+2)}$. 13. $\frac{a+1}{a-1}$. 14. $-\frac{(x-2)^{15}}{(2x-1)^5}$. 15. (a) $27a^2$, (b) 8. 16. $x-y$.
17. $\frac{54}{x-3}$. 18. (a) $\frac{13x^2+120}{336x^3}$, (b) $\frac{1}{y^{2n+1}}$. 19. (a) $\frac{1-33a}{2a(3a+1)}$,
(b) $\frac{9x+4}{12(3x-1)}$. 20. (a) $\frac{a+b}{c^2-d^2}$, (b) $\frac{2(2x^2+6x+5)}{4x^2-9}$. 21. 0. 22. $\frac{1}{3a+2}$.
23. $x+2$. 24. $\frac{2xy}{(x+y)^2(x-y)}$. 25. 1. 29. (a) $x=0$, (b) $x=11$, (c) $y=2$.
30. -10. 31. $\frac{12}{a}$. 32. $2(a-3)$. 33. -1. 34. -1. 35. $\frac{x}{x+y}$. 36. $-2x$.
37. $a-2$. 38. $a-5$. 39. (a) $\frac{4}{49}$, (b) $\frac{1}{2}$. 41. $a+b$. 42. (a) $\frac{2}{x-2}+\frac{3}{x+3}$, (b) $\frac{8}{x-5}-\frac{3}{x+2}$, (c) $\frac{3}{(x-1)^3}+\frac{2}{(x-1)^2}+\frac{1}{x-1}$. 43. (a) $x+7-\frac{32}{x-3}+\frac{70}{x-4}$, (b) $2x-1-\frac{9}{x}+\frac{9}{x-3}+\frac{9}{x+3}$.

§ 2.3

1. -1. 4. (a) $-2.4, 2.4$, (b) $-9, 3$, (c) $a \geqslant 0$, (d) an empty set. 5. (a) 450, (b) 864, (c) 144/161. 6. (a) $-3a\sqrt{b}$, (b) $5ab\sqrt{b}$, (c) $12ab\sqrt{ab}$, (d) $4a^2b\sqrt{2b}$, (e) $-5ab\sqrt[3]{3a}$, (f) $-2a^2b^3\sqrt{2ab}$. 7. (a) $\sqrt{3a^2}$, (b) $-\sqrt{3a^2}$, (c) $-\sqrt{3b^2}$, (d) $\sqrt{3b^2}$, (e) $\sqrt{2x}$, (f) $-\sqrt{-2x}$, (g) $\sqrt{3y^2}$ if $y \geqslant 0$, $-\sqrt{3y^2}$ if $y<0$, (h) $-\sqrt{7ac^2}$ if $c \geqslant 0$, $\sqrt{7ac^2}$ if $c<0$. 8. (a) the first, (b) the second. 10. (a) $8\sqrt{3}$, (b) $4.5\sqrt{13}$. 11. (a) 3/4, (b) $\sqrt{0.3}$. 12. (a) $2|a|\cdot\sqrt[6]{2a^2}$, (b) $\sqrt[10]{\sqrt{5}-2}$, 13. $2xy(4x^2\sqrt[3]{y}-5y\sqrt[3]{x^2y^2}+xy^2\sqrt[3]{x})$. 14. $\frac{\sqrt[6]{a^2b}}{ab^2}-6ab^3\times\sqrt{a}+\frac{b\sqrt[24]{a^3b}}{a^2}$. 15. (a) $-0.001\,a^2b\sqrt[5]{a^2b}$, (b) $\frac{x^4}{32y^4}\sqrt[4]{\frac{y}{x}}$, (c) $15\frac{1}{8}$.
16. (a) $\sqrt[4]{m}$, (b) $\sqrt[10]{a^9}$, (c) $\sqrt[3]{\frac{m}{n}}$. 17. (a) 29, (b) 35, (c) 74, (d) 103, (e) 147, (f) 356. 18. (a) $\sqrt{5}(2\sqrt{5}+1)$, (b) $\sqrt{3}(4\sqrt{3}-5)$, (c) $(2a-\sqrt{5})\times(2a+\sqrt{5})$, (d) $(b\sqrt{3}-4)(b\sqrt{3}+4)$. 19. (a) $-(\sqrt{10}+y)$, (b) $(\sqrt{x}+3)\times(\sqrt{x}-3)$, (c) $\frac{\sqrt{2}}{\sqrt{5}}$. 20. (a) $x>2$, (b) $x \geqslant 0$, $x \neq 4$, (c) $-2 \leqslant x \leqslant 15$, (d) an empty set, (e) all real numbers, (f) $x \geqslant 1$ or $x \leqslant -1$. 21. (a) $\sqrt{\sqrt{7}-2}$, (b) $\sqrt[1]{\pi-3}$, (c) -1. 22. (a) $\frac{1}{6}$, (b) $\frac{15}{29}$. 23. (a) 2, (b) $\frac{1}{8}(197-126\sqrt{2})$.

§ 2.4

1. (a) 8, (b) 5, (c) 0.2, (d) 10, (e) 10, (f) 0, (g) 10^5, (h) 4/9. 2. (a) $[0, +\infty)$, (b) $[1, +\infty)$, (c) $(-2, +\infty)$, (d) $(-\infty, 5) \cup (5, +\infty)$.

3. (a) 1.5, (b) -1.75, (c) 7200. 4. (a) $\frac{2b}{a^2}$, (b) $2a^2bx^3$, (c) $\frac{d^{3/2}}{ab^{3/4}c}$, (d) $\frac{d^{1/3}c^{1/3}}{a^{2/3}b^{1/3}}$. 5. (a) $\frac{16}{49}m^{10}n^{-1}x^{-1}$, (b) $2ab^{-1}$. 6. (a) $xy^{-\frac{1}{2}}$, (b) $\frac{100\,000}{243}\times mn^{5/3}$, (c) $125\,m^6n^{-9}$. 7. (a) $\frac{a^{2/3}}{b^{4/3}}$, (b) $\frac{a^{3/5}yx^{3/20}}{2}$. 8. (a) $\frac{4b^5c^2d}{5a^4}$, (b) $\frac{x^{48/5}}{2^{4/5}y^{32/5}}$. 9. (a) $a^{-2}-b^{-2}$, (b) $a^{-6}-b^{-6}$. 10. (a) abc, (b) $a^{2/7}b^{1/2}$. 11. (a) 9, (b) 1/8, (c) 0.5, (d) 0, (e) 64, (f) no roots, (g) $-1/3$, (h) 2. 12. (a) $(y^{1/5}-3)(y^{1/5}+3)$, (b) $(5-p^{2/7})(5+p^{2/7})$, (c) $(x^{1/6}-y^{5/2})(x^{1/6}+y^{5/2})$. 13. (a) $(p^{1/2}+1)(p-p^{1/2}+1)$, (b) $(5-b^{1/3})(25+5b^{1/3}+b^{2/3})$, (c) $(x^{1/12}-y^{1/6})\times(x^{1/6}+x^{1/12}y^{1/6}+y^{1/3})$. 14. (a) $y^{1/2}-1$, (b) $\frac{1}{a^{1/3}+b^{1/3}}$, (c) $n^{1/2}\left(n^{\frac{1}{2}}-1\right)$, (d) $(x^{1/2}+2)\,2x^{1/2}$. 15. (a) 1, (b) 0.1. 16. (a) $y=\frac{1}{x}$, (b) $x=y^2$, (c) $x^2y^3=72$, (d) $x^2-y^2=2$.

§ 2.5

1. $\frac{2}{2-a}$. 2. 0. 3. 1. 4. $-\frac{\sqrt{y}}{x}$. 5. 0. 6. 3. 7. $\left|\frac{x+y}{x-y}\right|$. 8. $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$. 9. 0. 10. $-\sqrt[3]{a^2b}$. 11. $x+1$. 12. $\frac{1}{a}$. 13. $\frac{1}{\sqrt{x}+1}$. 14. $\frac{\sqrt{a+1}}{a+3}$ if $a>1$, $-\frac{\sqrt{a+1}}{a+3}$ if $-1<a<1$. 15. $\frac{16a^4}{x^2}$. 16. $\frac{1}{a^{1/3}+1}$. 17. (a) $x-1$, (b) $\sqrt{x}$ if $x>2$, $-\sqrt{x}$ if $0<x<2$. 18. (a) $x+y$, (b) $a-b$. 19. $\frac{1}{\sqrt{x}}$. 20. $\frac{b+a}{b-a}$. 21. $\frac{x+3y}{x-y}$. 22. 10. 23. 39. 24. (a) $\frac{\sqrt[3]{ab^2}}{ab}$, (b) $\frac{2\sqrt{5}+\sqrt{7}}{13}$, (c) $\frac{\sqrt{2}+2-\sqrt{6}}{4}$, (d) $\frac{\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4}}{3}$, (e) $\sqrt[3]{3}-\sqrt[3]{2}$. 26. $\sqrt{11}+4-3\sqrt{7}$. 27. $\frac{13-\sqrt{5}}{2}$. 28. -10.

Chapter 3

§ 3.1

2. $f(0)=-27$, $f(a)=\frac{(2a-3)^3}{a^2+1}$, $f(-x)=\frac{(2x+3)^3}{x^2+1}$, $f(x+1)=\frac{(2x-1)^3}{x^2+2x+2}$. 4. 0. 5. $a=1$. 6. $f(-8)=1$, $f(-1.2)=1$, $f(0)=1$, $f(1)=1$, $f(4)=\sqrt[3]{4}$. 11. (a), (c) are even, (b), (d), (e) odd. 13. (a) $(-\infty, +\infty)$, (b) $x\neq 0$, (c) $x\neq 3$, (d) $x\neq\pm 1$, (e) $x\geqslant 0$, (f) $x\geqslant 4$. 16. No. 17. $x=3.5$. 18. (a) $3\sqrt{5}$, (b) 5.

§ 3.2

2. $y=3+x$. 3. $y=10+3x$. 6. (a) No, (b) yes, (c) yes. 7. $k=2$. 8. $b=-10$. 9. (a) [2, 6], (b) [1, 5], (c) [0.5, 1.5], (d) [2.5, 3.5]. 10. $y=x-7$. 11. (a) $k=$

-2, $b=3$, (b) $k=-2$, $b\neq 3$, (c) $k\neq -2$. **12.** $y=x-\frac{a\sqrt{2}}{2}$, $y=x+\frac{a\sqrt{2}}{2}$, $y=-x-\frac{a\sqrt{2}}{2}$, $y=-x+\frac{a\sqrt{2}}{2}$. **14.** $k=1$, $b=1$. **16.** (a) $(2, -3)$, (b) no solution, (c) $(3y+2, y)$, $y\in \mathbf{R}$. **17.** $a=3$. **18.** $a\neq 4$. **19.** $a=2.5$. **20.** (a) $a\neq \pm 3$, (b) $a=3$, (c) $a=-3$.

§ 3.3

2. (a) $a=3$, (b) $n=5$. **4.** (a) no, (b) yes, $n=8$.

§ 3.4

3. Yes. **4.** (a) $y=\frac{x+3}{2}$, (b) $y=-\sqrt[4]{x}$, (c) $y=\sqrt[3]{x}$. **5.** $y=\frac{x-3}{2}$. **6.** $y=\frac{3}{x}$. **7.** $y=\frac{2}{x}+1$. **8.** $y=\frac{1}{5-x}$. **9.** $y=2-\sqrt{x+1}$. **10.** No. **11.** $y=\sqrt[3]{x+1}$. **12.** $y=\begin{cases} 2x-2 & \text{for } x\geqslant 1, \\ 0.5(x-1) & \text{for } x<1. \end{cases}$ **13.** (0, 0), (1, 1). **16.** $y=\frac{x-b}{a}$. **17.** Nonmonotonic, reversible.

§ 3.5

4. (a) $[5, +\infty)$, (b) $(-\infty, +\infty)$, (c) $(-\infty, 2]$, (d) $(-\infty, 0]$.

§ 3.6

2. $a=25\,000$. **5.** $y=15/x$. **6.** $k=-162$. **8.** (a) 2, (b) $(0, 2)$, (c) $(-\infty, 0)\cup(2, +\infty)$, (2) $(-\infty, 0)\cup(4, +\infty)$. **13.** (a) $x<-4$, $x>2$, (b) $-4<x<2$. **14.** (a) $(-2, -9)$, (b) $(2.5, 6, 25)$, (c) $(-0.5, -6.25)$, (d) $(-0.5, -2.5)$. **16.** (a) yes, (b) yes, (c) no. **17.** (a) $c=20$, (b) $=12$. **18.** $a=-21$, $b=71$.

§ 3.7

3. (a) $(0.5, +\infty)$, (b) $(3, +\infty)$. **4.** (a) No, (b) yes. **5.** (a) Minus, (b) minus. **6.** (a) $(x-y)^2$, (b) a^2+ab. **7.** 3 and 1. **8.** Functions (a), (c), and (f) are even; functions (b), (d), and (e) are odd; function (g) is neither even nor odd. **9.** (a) $-\sqrt{3/2}$, (b) 0. **10.** (a) $a=b$, (b) $a>b$. **11.** (a) $a<0$, (b) $a<0$. **14.** (a) $-\sqrt{3}$, (b) $\sqrt{3}/3$, (c) $\sqrt{6}/2$. **15.** (a) $a>0$, (b) $a>0$. **17.** (a) $2\pi n<x<\pi+2\pi n$, (b) $2\pi n-\pi<x<2\pi n$, (c) $2\pi n+\pi/4<x<3\pi/4+2\pi n$, (d) $2\pi n+2\pi/3<x<7\pi/3+2\pi n$, (e) $2\pi n-\pi/6<x<7\pi/6+2\pi n$, (f) $2\pi n-2\pi/3<x<-\pi/3+2\pi n$. **18.** (a) $2\pi n-\pi/2<x<\pi/2+2\pi n$, (b) $2\pi n+\pi/2<x<3\pi/2+2\pi n$, (c) $2\pi n-\pi/3<x<\pi/3+2\pi n$, (d) $2\pi n+\pi/4<x<7\pi/4+2\pi n$, (e) $2\pi n-5\pi/6>x<5\pi/6+2\pi n$, (f) $2\pi n+2\pi/3<x<4\pi/3+2\pi n$. **19.** (a) $2\pi n<x<\pi/2+2\pi n$, (b) $2\pi n+\pi<x<4\pi/3+2\pi n$, (c) $2\pi n+3\pi/4<x<3\pi/2+2\pi n$, (d) $2\pi n+\pi/4<x<7\pi/6+2\pi n$, (e) $2\pi n-\pi/3<x<\pi/6+2\pi n$, $2\pi k+5\pi/6<x<4\pi/3+2\pi k$.

§ 3.8

1. (a) $7\pi/6$, (b) -1, (c) $-\sqrt{3}/2$, (d) $\pi/3$, (e) $\pi/4$, (f) $4\pi/5$. **2.** (a) $a<0$, (b) $a>0$, (c) $a>0$, (d) $a<0$. **3.** On (a) and (b). **4.** On (a). **5.** (a) $a>0$, (b) $a<0$, (c) $a>0$. **6.** (a) No roots, (b) $\sqrt{2}/2$, (c) 0.5 tan 1, (d) 1/2. **7.** (a) $[-1/2, 1/2]$, (b) $[-2/3, 4/3]$, (c) $x=0$, (d) $x\neq\pm 3$, (e) $(-\infty, +\infty)$. **9.** (a) $\arcsin(1/3)+2\pi n$, $\pi-\arcsin(1/3)+2\pi n$, (b) $-\arcsin(1/4)+2\pi n$, $\pi+\arcsin(1/4)+2\pi n$, (c) $\pm\arccos(2/3)+2\pi n$, (d) $\pm(\pi-\arccos(3/5))+2\pi n$. **10.** (a) $2\pi n+\arcsin(2/7)<x<\pi-\arcsin(2/7)+2\pi n$, (b) $2\pi n-\pi-\arcsin 0.3<x<\arcsin 0.3+2\pi n$, (c) $2\pi n-\pi+\arcsin 0.2<x<-\arcsin 0.2+2\pi n$,

(d) $2\pi n - \arcsin 0.4 < x < \pi + \arcsin 0.4 + 2\pi n$, (e) $2\pi n - \arccos 0.1 < x < \arccos 0.1 + 2\pi n$, (f) $2\pi n + \arccos (2/7) < x < 2\pi - \arccos (2/7) + 2\pi n$, (g) $2\pi n - \arccos (-0.7) < x < \arccos (-0.7) + 2\pi n$, (h) $2\pi n + \pi - \arccos 0.2 < x < \pi + \arccos 0.2 + 2\pi n$, (i) $\pi n - \pi/2 < x < \pi/4 + \pi n$, (j) $\pi n - \pi/6 < x < \pi/2 + \pi n$, (k) $\pi n < x < 3\pi/4 + \pi n$.

§ 3.9

1. (a) 1, (b) No solution, (c) 5, −1, (d) 0.5, (e) $(-\infty, 0]$, (f) 1, −4, (g) No solution, (h) −4.9, 2.9, (i) 1, −2, (j) 0.2, 1.8, (k) −0.7, (l) 0. 2. (a) $(-\infty, 2)$, (b) $(-\infty, 3.5)$. 3. (a) 3, −4, (b) $(-\infty, -4) \cup (0, 3)$, (c) $(-4, 0) \cup (3, +\infty)$. 6. (a) $(-\infty, -4) \cup (3, +\infty)$, (b) $(0, 4)$, (c) No solution. 7. (a) $2\pi n + \pi < x < 2\pi + 2\pi n$, (b) $2\pi n + \pi/6 < x < 5\pi/6 + 2\pi n$, (c) $2\pi n - 5\pi/4 < x < \pi/4 + 2\pi n$, (d) $2\pi n - \pi/6 < x < 7\pi/6 + 2\pi n$, (e) $2\pi n - \pi/2 < x < \pi/2 + 2\pi n$, (f) $2\pi n - \pi/4 < x < \pi/4 + 2\pi n$, (g) $2\pi n + 2\pi/3 \leqslant x \leqslant 4\pi/3 + 2\pi n$, (h) $\pi n < x < \pi/2 + \pi n$, (i) $\pi n + \pi/3 \leqslant x < \pi/2 + \pi n$, (j) $\pi n - \pi/2 < x < -\pi/4 + \pi n$, (k) $\pi n + \pi/2 \leqslant x < \pi + \pi n$, (l) $\pi n < x < 2\pi/3 + \pi n$, (m) $\pi n + \pi/4 < x < 3\pi/4 + \pi n$. 8. (a) $2\pi n + \pi/2 \leqslant x \leqslant 3\pi/2 + 2\pi n$, (b) $\pi n \leqslant x \leqslant \pi/4 + \pi n$. 9. (a) $(-4, 0)$, (b) No solution, (c) $(x, 1 - x)$, $x \in \mathbf{R}$) (d) $(5, 0)$, $(0, 5)$, (e) $(5.4, 2.7)$, $(-5.4, 2.7)$, (f) $(-4.3, 9)$, $(4.3, 9)$, (g) $(2, 4)$. 10. (a) 1, (b) 2, (c) 1, (d) 2, (e) 0

Chapter 4

§ 4.1

1. (a) −19/3, 19/3, (b) 26, −26. 2. (a) 2. 5, −2.5, (b) 0, 4, (c) No root. 3. (a) 6, −6, (b) 0.1, 1. 4. (a) −16, 4, (b) −12, −2, (c) 4, −3, (d) −1, 5, (e) No root. 5. 2. −9. 6. −1, −38/15. 7. 9, −4. 8. −1, 62/39. 9. 3, 1.4. 10. 9.2, 14. 11. (a) ± 2, ± 3, (b) $\pm\sqrt{3}/3$, ± 3, (c) $\pm\sqrt{3}/3$, $\pm\sqrt{2}$. 13. (a) $a = 1$, (b) $a = 13/14$, (c) $a = 2$, $a = -0.2$. 14. (a) $k = -8$, (b) $k = -5$, (c) $k = 15$, $x_2 = -1$. 15. (a) $x^2 - 6x - 16 = 0$, (b) $8x^2 - 10x + 3 = 0$, (c) $x^2 - (\sqrt{2} + \sqrt{3})\, x + \sqrt{6} = 0$, (d) $9x^2 - 18x - 11 = 0$, (e) $4x^2 - 4mx + m^2 - n^2 = 0$. 16. (a) $(x - 7)(x + 5)$, (b) $(a - 18)(a - 11)$, (c) $(m - 4)(m + 3)$, (d) $(5x - 18) \times (x + 7)$, (e) $(y + 1)(2y - 3)$. 17. (a) $\dfrac{x+13}{x+15}$, (b) $\dfrac{2(a+9)}{3(a-7)}$, (c) $\dfrac{b-7x}{b+x}$. 18. (a) 39, (b) −162. 19. $2x^2 - 9x + 9 = 0$. 20. $25x^2 - 21x + 4 = 0$. 21. $3x^2 - 8x + 2 = 0$. 22. 3, 5. 23. (a) $(-\infty, +\infty)$, (b) (-9.5). 25. $(-\infty, +\infty)$ if $a = 1$, no roots if $a = -1$, $(2a + 3)/(a + 1)$ if $a \neq \pm 1$. 26. $(-\infty, 1) \cup (1, +\infty)$ if $a = 1$, no roots if $a = 0$ or $a = -1$, $(a + 2)/(a + 1)$ if $a \neq 0, 1, -1$. 27. No roots if $a \neq 0$, $(-\infty, 0) \cup (0, +\infty)$ if $a = 0$. 28. $\dfrac{3a+1}{2(a+3)}$ if $a = 0.2$ or $a = 1, 3/8$ if $a = -3$, no roots if $0.2 < a < 1$, $\dfrac{3a+1 \pm \sqrt{5a^2-6a+1}}{2(a+3)}$ if $a < -3$ or $-3 < a < 0.2$ or $a > 1$. 29. $2a$, $-a/3$ for any a. 30. No root if $a = -1, 6$ if $a = 3$, $a + 3$, $a - 1$ if $a \neq 3$, $a \neq -1$. 31. $-2a$, $3a$ if $a \neq 0$, no roots if $a = 0$. 32. 12 km/h. 33. 80 km/h. 34. 2 m and 2.5 m. 35. 15 ha and 18 ha or 24 ha and 27 ha. 36. 5 hs 50 min, 4 hs 40 min. 37. 100 and 60 m. 38. 1936 cm^2. 39. 3 cm. 40. 7. 41. By 20%. 42. 48 km/h, 60 km/h. 43. 3 km/h. 44. 5 hs, 7 hs.

§ 4.2

1. −1, −2, 2. 2. −1, 3, 1/3. 3. −2/3, −1/2, 3. 4. −1/3, 0, −2, 2. 5. −1, 2, 3, 6. $-\sqrt{3}$, $-\sqrt{2}$, $\sqrt{2}$, $\sqrt{3}$. 7. −4, $-\sqrt{10}$, $\sqrt{10}$, 4. 8. 0, 3, 9. 1, −5, −1

$\pm\sqrt{6}$. **10.** $-3/2$, $(-3 \pm \sqrt{10})/2$. **11.** 1, $(-3 \pm \sqrt{5})/2$. **12.** 2, 1/2, 3, 1/3. **13.** 1, $(-3 \pm \sqrt{5})/2$. **14.** 0.5. **15.** 3/2, $(5 \pm \sqrt{17})/4$. **16.** -0.5. **17.** 2, 4, -1, $-1/2$.

§ 4.3

2. 1, -2, **3.** 4. **4.** -1, **5.** 3. **6.** -61, 30. **7.** 1. **8.** 2, -4. **9.** 1. **10.** 2. **11.** 15. **12.** 1. **13.** 4.

§ 4.4

1. $(-8, 6)$, $(6, -8)$. **2.** $(2, -5)$, $(-4, 3)$ $\left(1 + 2\sqrt{3}, \frac{3+8\sqrt{3}}{3}\right)$, $\left(1-2\sqrt{3}, \frac{3-8\sqrt{3}}{3}\right)$. **3.** $(-2, 0)$, $(-2, -1)$, $(1, 0)$, $(1, -1)$. **4.** $(4, 5)$, $(-4, -5)$. **5.** $(2, 1)$, $(-2, -1)$. **6.** $(9, 4)$. **7.** $(7, 1)$, $(56/9, 7/9)$. **8.** $(1/5, 1/3)$. **9.** $(3, 5)$. **10.** $(3\sqrt{2}, \sqrt{2})$, $(-3\sqrt{2}, -\sqrt{2})$, $(3\sqrt{2} - \sqrt{2})$, $(-3\sqrt{2}, \sqrt{2})$. **11.** $(2, -2)$. **12.** $(2, 3)$, $(3, 2)$. **13.** $(2, 0)$, $(0, 2)$. **14.** $(2, 3)$, $(3, 2)$, $(-3, -2)$, $(-2, -3)$. **15.** $(-3, 2, -1)$. **16.** $\left(\frac{7z+17}{5}, \frac{11(z+1)}{5}, z\right)$ $z \in \mathbf{R}$. **17.** $(-7z - 20, 4z + 10, z)$, $z \in \mathbf{R}$. **18.** No solution. **19.** $(1, -1, 0, 2)$. **20.** 80 cm, 18 cm. **21.** 40 km/h, 50 km/h. **22.** 400 ha, 30 centners. **23.** 0.15 m/s², 60 s. **24.** 5 km/h, 3 km/h. **25.** 13.5 hs, 18 hs, 27 hs.

Chapter 5

§ 5.1

6. 4. **7.** (a) $a > b$, (b) $a < b$, (c) $a < b$.

§ 5.2

1. $x < -5$, $x > 5$. **2.** $x \leqslant -6$, $x \geqslant 0$. **3.** $x < -1/3$, $x > -1/3$. **4.** 1.5. **5.** $(-\infty, +\infty)$. **6.** $0.5 < x < 1$. **7.** $x \leqslant -1$, $x \geqslant 5$. **8.** $-4 < x < -2$, $0 < x < 1$; $x > 3$. **9.** No solution. **10.** $x < -3$, $2 \leqslant x < 4$, $x \geqslant 5$. **11.** $-4/3 < x < -1/5$. **12.** $x < -1$, $0 < x < 3$. **13.** $x < -1$, $2/3 < x < 1$. **14.** $x < -4$, $-3 < x < -2.5$, $-2 < x < -1$, $x > 0$. **15.** (a) $[-9, 9]$, (b) $(0, 1/3]$. **16.** $v \geqslant 9$ km/h. **17** $v \geqslant 4$ km/h. **18.** $-1 < a < 3$. **19.** $-5 < a < -\sqrt{6}$, $-\sqrt{6} < a < -1$, 2.

§ 5.3

1. $x > 13/3$. **2.** 3, 4, **5.** 3. **3.** 2, 3, 4, 5. **4.** $-1 < a < 2$. **5.** $x < 16/7$. **6.** $1 < x < 2$, $x > 2$. **7.** $x < 1.5$, $x < 3.3$. **8.** $x < -0.4$, $x > 4$. **9.** $x < 0$, $x > 4/3$. **10.** $x < 7/4$, $x > 5/2$. **11.** $x < -5/3$, $x > 3$. **12.** $x < -5$, $x > -1$. **13.** $-7 \leqslant x \leqslant 7$. **14.** $x > 1/2$. **15.** (a) $(-\infty, -2) \cup [12, +\infty)$, (b) $[1/2, 5/2]$, (c) $[-2, -\sqrt{3}] \cup [\sqrt{3}, 2]$, (d) $[2, 3]$. **16.** 3, 4, **5.17.** $x < -4$, $x > 1$. **18.** $x \leqslant 0$, $x \geqslant 7$. **19.** $1/3 \leqslant x \leqslant 5/3$. **20.** $1/3 \leqslant x < 1/2$. **21.** $x \leqslant 0$, $x > 4.5$.

Chapter 6

§ 6.1

1. (a) $\sin\alpha - \cos\alpha$, (b) 1, (c) 1, (d) 0. **3.** (a) $\cos\alpha = -12/13$, $\tan\alpha = -5/12$, $\cot\alpha = -12/5$, (b) $\sin\alpha = -4/5$, $\tan\alpha = -4/3$, $\cot\alpha = -3/4$, (c) $\sin\alpha = -9/41$, $\cos\alpha = -40/41$, $\cot\alpha = 40/9$, (d) $\sin\alpha = 15/17$, $\cos\alpha = -8/17$, $\tan\alpha = -15/8$. **5.** (a) $\sqrt{3}/3$, (b) 1. **7.** (a) $\sqrt{2}/2$, (b) 1/2, (c) $-\sqrt{3}/3$, (d) $\sqrt{6}/3$, (e) $-(\sqrt{2} + \sqrt{6})/4$. **9.** (a) $-\cos^2\alpha$, (b) 0. **12.** 0.8, 0.6, 0.75. **14.** $\pi +$

$2\pi n < x < 2\pi + 2\pi n$. **15.** (a) $2 \sin 1^\circ \cos 8^\circ$, (b) $\sqrt{2} \sin \frac{\pi}{8}$, (c) $2 \cos^2 \frac{\alpha}{2}$, (d) $4 \sin\left(\frac{\pi}{6} + \frac{\alpha}{2}\right) \sin\left(\frac{\alpha}{2} - \frac{\pi}{6}\right)$, (e) $4 \cos \alpha \cdot \sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \cos\left(\frac{\pi}{4} - \frac{5\alpha}{2}\right)$. **16.** (a) $13 \sin(3t - \alpha)$, where $\alpha = \arcsin(12/13)$, (b) $25 \sin(5t + \alpha)$, where $\alpha = \arcsin(7/25)$. **17.** $[-5, 5]$. **23.** (a) $4 \cos 3\alpha \cos 4\alpha \sin 2\alpha$, (b) $8 \sin^4 \alpha$. **24.** $1/8$. **25.** (a) $2/\sqrt{5}$, (b) $-7/24$. **27.** (a) $2x\sqrt{1-x^2}$, (b) $\frac{x-y}{1+xy}$. **29.** (a) $3/5$, $3/7$, (b) $1/3$, $1/7$.

§ 6.2

1. (a) $\frac{1}{2} + (-1)^n \cdot \frac{\pi}{12} + \frac{\pi n}{2}$, (b) $-\frac{\pi}{4} \pm \arccos \frac{1}{3} + 2\pi n$, (c) $\frac{1}{3} \arctan 10^6 + \pi n$, (d) $\pm \sqrt{\pi + 2\pi n}$, $n = 0, 1, 2, 3, \ldots$, (e) No solution. **2.** $\pi + 2\pi n$. **3.** $\frac{\pi}{12} + \frac{\pi n}{3}$, $\frac{1}{3} \arctan 3 + \frac{\pi n}{3}$. **4.** $\frac{\pi}{6} + \frac{\pi n}{3}$. **5.** $(-1)^{n+1} \cdot \frac{\pi}{6} + \pi n$. **6.** $-\frac{1}{3} \arctan \frac{3}{2} + \frac{\pi n}{3}$. **7.** $-\arctan 5 + \pi n$, $\arctan 3 + \pi n$. **8.** $\pm 2 \arctan \frac{\sqrt{3}}{2} + 2\pi n$. **9.** $\frac{\pi}{2} + \pi n$, $\arctan \frac{3}{4} + \pi n$. **10.** $\frac{\pi}{2} + 2\pi n$, $-2 \arctan 2 + 2\pi n$. **11.** $2 \arctan 1.5 + 2\pi n$, $\pi + 2\pi n$. **12.** $\frac{\pi}{4} + \pi n$. **13.** $\arctan \frac{3 \pm \sqrt{6}}{3} + \pi$. **14.** $\frac{\pi}{2} + 2\pi n$, $\arctan \frac{1}{5} + \pi n$. **15.** $\pm \frac{\pi}{6} + \frac{\pi n}{2}$. **16.** $\frac{\pi}{7} + \frac{2\pi n}{7}$, $\frac{\pi}{3} + \frac{2\pi n}{3}$. **17.** $\frac{\pi}{2} + 2\pi n$, $\pi + 2\pi n$. **18.** $4 \arctan \frac{-3 \pm \sqrt{21}}{2} + 4\pi n$. **19.** $\pm \frac{\pi}{3} + \pi n$. **20.** $2\pi n$. **21.** $\frac{\pi n}{2}$. **22.** $\left(\pi n, \frac{\pi}{4} - \pi n\right)$, $\left(\frac{\pi}{4} + \pi n, -\pi n\right)$. **23.** $\left(\frac{\pi}{3} + \pi(n+k), \frac{\pi}{3} + \pi(k-n)\right)$, $\left(-\frac{\pi}{3} + \pi(n+k) - \frac{\pi}{3} + \pi(k-n)\right)$. **24.** $\frac{\pi k}{2} \pm \frac{1}{4} \arccos \frac{8a-5}{3}$ if $\frac{1}{4} \leqslant a \leqslant 1$, no solution if $a < \frac{1}{4}$ or $a > 1$, (b) πk or $\pi k \pm \frac{1}{2} \arccos \frac{1-a}{2a}$ if $a \geqslant \frac{1}{3}$ or $a \leqslant -1$, πk if $-1 < a < \frac{1}{3}$.

Chapter 7

§ 7.2

1. (a) $x_{10} = \frac{21}{13}$, $x_{25} = \frac{51}{28}$, $x_{n+1} = \frac{2n+3}{n+4}$, $x_{n+1} - x_n = \frac{5}{(n+3)(n+4)}$, (b) $b_3 = \frac{1}{9}$, $b_5 = \frac{5}{3^5}$, $b_{n+1} = \frac{n+1}{3^{n+1}}$, $\frac{b_{n+1}}{b_n} = \frac{n+1}{3n}$. **2.** $-10, -5, 0, 5, 10$, $x_n = 5(n-3)$. **3.** (a) $n \geqslant 103$, (b) $n \leqslant 43$. **4.** (a) n^2, (b) $(-1)^{n-1} \cdot 2$, (c) $2.5 \pm (-1)^{n-1} \cdot 2.5$, (d) $n^2 - 1$, (e) $\frac{1}{2^n}$, (f) $\frac{n+1}{n}$, (g) $\frac{1}{n(n+1)}$. **5.** $a_7 = b_7 = 308$. **6.** (a) No, (b) 1, (c) 0, (d) 0. **7.** (a) 0.6, (b) 0.15, (c) 0.03, (d) 0.003. **8.** No. **10.** Increasing.

§ 7.3

1. Yes, (a) $n > 10$, (b) $n > 25$. **2.** (c) $n > 97$. **4.** $n \geqslant 6$. **5.** (a) $-7/3$, (b) $3/2$, (c) 3, (d) 0. **6.** (a) Yes, (b) yes, (c) no, (d) no.

§ 7.4.

1. (a), (b), and (i) are arithmetic progressions, (c), (g), (i), and (k) are geometric progressions. **2.** (a) Geometric progression, (b) arithmetic progression. **3.** (a) $a_1 = 7$, $d = 4$, (b) $a_1 = 1$, $d = 2$, (c) $a_1 = 1$, $d = 0.6$, (d) $a_1 = 5/3$, $d = 7/3$, (e) $a_1 = 1.8$, $d = 1.6$. **5.** (a) $a_{11} = 10$, $S_{11} = 660$, (b) $d = 4$, $S_{26} = 1430$, (c) $a_1 = 1$, $a_{12} = 34$, (d) $a_1 = -38$, $S_{15} = -360$. **6.** (a) $n = 25$, $a_{25} = 3$ or $n = 12$, $a_{12} = -3.5$, (b) $d = 7$, $a_9 = 28$, (c) $n = 51$, $d = 0.1$, (d) $a_1 = -6$, $d = 0.75$. **7.** (a) $d = 3$, $a_1 = -7$, (b) $d = 0.3$, $a_1 = -0.7$, (c) $d = 1.5$, $a_1 = 0$, (d) $d = 3$, $a_1 = -7$. **8.** $-3/4$, $-1/4$, $1/4$, $3/4$. **9.** (a) 102.9 m, (b) 592.9 m, (c) 30 s. **10.** (a) 4905, (b) 494 550, (c) 249 500, (d) 98 550. **11.** 570. **12.** (a) $b_{10} = 19\,683$, $S_{10} = 29\,524$, (b) $b_1 = 256$, $S_8 = 510$, (c) $q = 3$, $S_7 = 2186$ or $q = -3$, $S_7 = 1094$, (d) $b_1 = 7$, $n = 5$, (e) $q = 0.5$, $n = 7$, (f) $n = 8$, $S_8 = 3280/6561$, (g) $b_1 = -1.5$, $b_4 = 40.5$. **14.** $n = 10$, $b_1 = 1$, $q = 2$. **15.** 3, -6, 12, -24. **16.** $b_1 = 5$, $b_5 = 405$. **17.** $n = 6$. **18.** $b_{10} = 117$, $S_{10} = (13/324)\,(3^{10} - 1)$. **19.** 2, 4, 8. **20.** $n = 14$. **21.** $n = 20$. **22.** 4, 10, 16 or 16, 10, 4. **23.** 3, 6, 12, 18. **24.** 27, 18. **25.** $-1/128$. **26.** 4, $-4/3$, $4/9$, $-4/27$, $4/81$.

Chapter 8

§ 8.1

1. (a) $|x-10| < 0.05$, (b) $|x-10| < 0.005$, (c) $|x-10| < 0.0005$, (d) $|x-10| < \varepsilon/20$. **2.** $|x-2| < 0.01$, where x is the side. **3.** $S(x) = \begin{cases} 9\pi \text{ for } 0 \leqslant x \leqslant 5, \\ 4\pi \text{ for } 5 < x \leqslant 10, \\ \pi \text{ for } 10 < x \leqslant 15. \end{cases}$ **5.** Continuous everywhere. **6.** $a = 1$. **7.** (a) A jump at the point $x = -1$, (b) a removable discontinuity at the point $x = 0$, (c) continuous everywhere, (d) continuous everywhere. **9.** $y_{\text{least}} = -91$, $y_{\text{great}} = -13.25$. **10.** $y_{\text{least}} = -16$, no y_{great}. **11.** (a) no y_{least}, $y_{\text{great}} = 7/3$, b) $y_{\text{least}} = 2$, $y_{\text{great}} = 5$, (c) $y_{\text{least}} = 5$, no y_{great}. **12.** $y_{\text{great}} = 6$.

§ 8.2

1. (a) -2.2, (b) 2. **2.** (a) 1/2, (b) 5/12, (c) 6. **3.** (a) 1/4, (b) $2\sqrt{2}$. **4.** (a) 1.4, (b) 2. **5.** (a) 3, (b) 0, (c) 0.6. **6.** (a) $y = 0$, $x = 0$, (b) $y = 1$, $x = 3$, (c) $y = 0$, $x = 2$. **7.** (a) $y = 2x + 1$, $x = 2$, (b) $y = x$, $x = 1$, $x = -1$. K,

§ 8.3

1. (a) $-3\Delta x$, (b) $2\left(\sqrt{x+\Delta x} - \sqrt{x}\right)$, (c) $2 \sin \frac{\Delta x}{2} \cos\left(x + \frac{\Delta x}{2}\right)$, (d) $\Delta x\,(2 - 2x - \Delta x)$. **2.** (a) $3(x+\Delta x) - 5$, $3\Delta x$, 3, (b) $(x+\Delta x)^2 - 3$, $2x\Delta x + (\Delta x)^2$, $2x + \Delta x$, (c) $\cos(x + \Delta x)$, $-2 \sin \frac{\Delta x}{2} \sin\left(x + \frac{\Delta x}{2}\right)$, $-\frac{2 \sin \frac{\Delta x}{2} \sin\left(x + \frac{\Delta x}{2}\right)}{\Delta x}$, (d) $a(x+\Delta x)^2 + b(x+\Delta x) + C$, $\Delta x\,(2ax + b + a\Delta x)$, $2ax + b + a\Delta x$. **3.** 0.1. **4.** (a) $y'(-1) = -2$, $y'(\sqrt{2}) = -2$, (b) $y'(0) = 0$, $y'\left(-\frac{2}{3}\right) = -\frac{4}{3}$. **5.** $y' = 2x + 7$. **6.** (a) ≈ 1.03, (b) ≈ 15.96, (c) ≈ 1.012. **7.** (a) $y = 0$, (b) $y = -2x - 1$, (c) $y = 3x - 2$, (d) $y = 12x + 16$, (e) $y = 2 - x$, (f) $y = -4x - 4$. **8.** (a) $v(t) = 4t + 13$ (m/s), (b) $a(t) = 4$ (m/s^2), (c) $F = 800$ N.

§ 8.4

1. (a) $4x-5$, (b) x^2-3x, (c) $-\frac{4}{x^2}-3x^2$. 2. (a) $2(x^2-7x-1)(2x-7)$, (b) $-\frac{18}{x^3}$. 3. (a) $-\frac{3}{x^4}$, (b) $4x^3$, (c) $6x^5$. 4. $15x^2+22x+2$. 5. (a) $\frac{1}{7}(6x^2+3)$, (b) $-\frac{7(6x^2+3)}{(2x^3+3x-1)^2}$. 6. (a) $-\frac{5}{(x-3)^2}$, (b) $\frac{x^2+4x-10}{4(x+2)^2}$. 7. (a) -3, (b) 80, (c) 4, (d) -1. 8. (a) $y=6-5x$, (b) $y=2-4x$, (c) $y=-4x-6$. 9. $y=x-\frac{1}{3}$ (for $x=-2$), $y=x-\frac{31}{12}$ (for $x=1$). 10. (a) 25 cm/s, (b) 16 m/s^2, (c) 32 N. 11. $t=4$ s. 12. $30x^4-\frac{12}{x^5}$. 13. $\frac{2}{3}x^{-1/3}\sin x+x^{2/3}\cos x$. 14. $\frac{\cos x+2x\sin x}{2\sqrt{x}\cos^2 x}$. 15. $\frac{3}{\cos^2 x}+\frac{1}{2\sin^2 x}$. 16. $\left(-\frac{6}{5}x^{-7/5}\right)\left(\frac{2}{x^5}+3\sqrt[3]{x}\right)+\left(\frac{3}{\sqrt[5]{x^2}}-1\right)\times\left(-\frac{10}{x^6}+\frac{1}{\sqrt[3]{x^2}}\right)$. 17. (a) $y=-x-\frac{\pi}{4}-\frac{1}{2}$, (b) $y=\frac{13}{12}x-\frac{20}{3}$. 18. $x=\pm\frac{\pi}{18}+\frac{2\pi n}{3}$. 19. (a) $\sin 2x$, (b) $-3\cos^2 x\sin x$. 20. (a) $\frac{1}{2\cos^2 x\sqrt{\tan x}}$, (b) $\frac{5}{\cot^6 x\sin^2 x}$. 21. (a) $\frac{5\cos 5x}{3\sqrt[3]{\sin^2 5x}}$, (b) $-6\cos^2(6x-5)\cdot\sin(6x-5)$. 22. (a) $\frac{3\cos(3x+5)+\cos(x+5)}{2}$, (b) $\frac{2x-1}{2\sqrt{x^2-x}}+\frac{3x^2\sin x-x^3\cos x}{\sin^2 x}$. 23. $\frac{5\sqrt{3}}{32}$.

§ 8.5

1. Decreases on $(-\infty, 1/3)$, increases on $(1/3, +\infty)$. 2. $x=7$ is a point of minimum, $x=-7$ is a point of maximum. 3. (a) $x=-1/3$ is a point of minimum, $x=1/3$ is a point of maximum, (b) $x=0$ is a point of maximum, $x=2$ is a point of minimum. 4. 1 and $-1/3$. 5. A right triangle. 6. 8 cm, $2\sqrt{3}$ cm. 7. $5\sqrt{3}$ cm. 8. $\sqrt[3]{V/(2\pi)}$. 9. 2 rad. 11. (a) $t=3$ s, (b) $v=243$ m/s, $a=0$, (c) 500 m.

§ 8.6

3. $2.5-\cos x$. 4. (a) $\frac{x^3}{3}+4x^{5/4}+C$, (b) $-\frac{1}{2}\cos(2x-3)-\tan x+C$. 5. $-\frac{4}{\sqrt{x}}+18\sqrt[6]{x}+5x+C$. 6. $-2\cos x-3\sin x+C$. 7. $x^2-\cot x+C$. 8. $\sin x+\cos x+\tan x+C$. 9. $-\frac{2}{9}\sqrt{(4-3x)^3}+C$. 10. $\frac{(2x-5)^{101}}{202}+C$. 11. $\frac{x}{2}-\frac{\sin 2x}{4}+C$. 12. $\frac{1}{4}\left(\sin 2x+\frac{1}{2}\sin 4x\right)$. 13. $\frac{1}{4}\left(\sin x-\frac{1}{3}\sin 3x\right)+C$.

§ 8.7

1. (a) 2, (b) 1, (c) $(1+7\sqrt{3})/3$. 2. (a) 32/3, (b) 4/3, (c) 3/4, (d) 1/5. 3. 60.9 m. 4. $s(t)=\frac{t^3}{3}-\frac{t^2}{2}+t-1$; $a(t)=2t-1$. 5. $1+2t+\frac{t^2}{2}-\sin t$. 6. $\pi/3$. 7. $41\frac{2}{3}$. 8. $16(3+2\sqrt{2})/3$.

Chapter 9

§ 9.1

2. (a) $x < 0$, (b) $x < 0$, (c) $x > 0$, (d) $x > 0$. 3. $\alpha < \beta$. 4. $0 < a < 1$. 6. (a) $1/e^{x+1}$, (b) $e^{\sin x} \cdot \cos x$, (c) $e^x/\cos^2 e^x$. 7. Decreases on $(-\infty, 2)$, increases on $(2, +\infty)$. 8. $e^2 - 3$.

§ 9.2

2. $y = 3 + \log_2 (x + 5)$. 4. (a) $a < b$, (b) $a < b$, (c) $a > b$, (d) $a < b$. 5. (a) $(2, +\infty)$, (b) $(-\infty, 2) \cup (3, +\infty)$, (c) $\left(-\infty, -\frac{3}{2}\right) \cup \left(\frac{2}{3}, +\infty\right)$, (d) $(-\infty, -2) \cup (8, +\infty)$. 6. (a) $1 - a$, (b) $2(1 - a)/3$, (c) $1 + 2a$, (d) $(1 - a)/a$, (e) $(3 - a)/(1 - a)$, (f) $(a - 1)/a$. 7. (a) $-1/5$, (b) $1/16$, (c) $3/4$, (d) 24. 8. $\ln 2 + 2 + \ln a + 0.2 \ln b - 11 \ln C$. 9. $x = 1/2$. 10. (a) $2.73 \cdot 10^0$, $n = 0$, (b) $3.745 \cdot 10^2$, $n = 2$, (c) $1.5 \cdot 10^{-1}$, $n = -1$, (d) $2.2385 \cdot 10^1$, $n = 1$, (e) $1.3 \cdot 10^{-5}$, $n = -5$. 11. (a) $2/e$, (b) $\sqrt{3}/3$, $(2/3) \log e$. 12. (3; 0) is a point of minimum. 14. (a) 1, (b) $\frac{1}{2} \ln \frac{11}{3}$, (c) $-\frac{1}{3} \ln |x - 2| + \frac{10}{3} \ln |x - 5| + C$, (d) $\frac{x^3}{3} - \frac{x^2}{2} - 3x - 13 \ln |x - 4| + C$.

§ 9.3

1. 20. 2. 3. 3. 0. 4. 4. 5. 1, $\log_2 24$. 6. 3. 7. 1, -3. 8. 3. 9. 2. 10. $\frac{1}{4}$, $\frac{1}{2}$. 11. 5. 12. 0. 13. 0. 14. 1. 15. 4. 16. 8. 17. 6.25. 18. $-\frac{1}{3}$, $\frac{1}{3}$, -3, 3. 19. $\frac{\sqrt[3]{14}}{4}$. 20. 2, 3. 21. $\frac{3}{2}$, 3. 22. 8. 23. -1. 24. 2, 3. 25. $-2 - \sqrt{10}$. 26. 1. 27. 4. 28. No root. 29. 4. 30. 9, $\frac{1}{27}$. 31. 2, $2\sqrt[3]{2}$. 32. $\sqrt[3]{5}$, 5. 33. 100, 0.1. 34. 1/2, 32. 35. (1, 2), (16, -28). 36. (3, -3).

§ 9.4

1. $x < -1$. 2. $x < 1.5$. 3. $x < -1$, $x > 7$. 4. $x < \log_5 3$. 5. $\log_6 2 < x < \log_6 7$. 6. $-6 < x < 0$. 7. $x \leqslant -6$; $x \geqslant 2$. 8. $x < 0$. 9. $x > 2$. 10. $x < -\sqrt{3}$, $x > \sqrt{3}$. 11. $2 < x < 4$. 12. $-\sqrt{2} < x < 0$, $0 < x < \sqrt{2}$. 13. $1/2 < x < 2$. 14. $x > 1.5$. 15. $x < -3^{-\sqrt{3}}$. 16. $0 < x < 0.4$, $x > 1$. 17. $x > 2.5$. 18. $x < -1$, $x > 5$. 19. $1/2 < x \leqslant 1$. 20. $7 < x \leqslant 9$. 21. $0 < x < 1$, $\sqrt{3} < x < 9$. 22. $x > 6$. 23. $1 < x < 3$. 25. $-5 < x < -1$, $4 < x < 5$. 26. $-2 < x < -1/3$, $-1/3 < x < 0$. 27. $0.001 < x < 0.01$, $x > 0$. 28. $\frac{5}{3} \leqslant x \leqslant 2$. 29. $3 < x < \pi$. 30. $\pi/3 < x \leqslant 3$. 31. $0 < x < 2/3$. 32. $3 < x < e^2$. 33. $-98 < x < -3$, $0 < x < 2$, $2 < x < 102$. 34. $4 < x < 4\pi/3$, $5\pi/3 < x < 3 + e$. 35. 1, 2, 3, 4, 5, 6.

Chapter 10

§ 10.1

1. (a) 1980, (b) 715, (c) 30/7. 2. (a) $\frac{n-3}{n+4}$, (b) $\frac{1}{3}(n + 1)\, n\, (n - 1)$. 3. $-\frac{2n}{(n+1)!}$. 4. (a) 9, 10, (b) 11. 6. (a) $P_3 = 6$, (b) $A_3^4 = 24$. 7. (a) $P_4 = 24$,

(b) $4P_4 = 96$, (c) $P_3 = 6$, (d) $P_5 - P_3 = 114$, (e) $2P_4 = 48$, (f) $4P_3 = 24$. 8. $A_4^9 = 3024$. 9. $C_6^{29} = 38\,760$. 10. 21. 11. 2047. 12. (a) 480, (b) 240, (c) 480. 13. (a) 15/28, (b) 13/28. 14. (a) 0.00027%, (b) 0.04%, (c) 1.23%.

§ 10.2

1. (a) $x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$, (b) $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$, (c) $81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$, (d) $x^{12} - 6x^9 + 15x^6 - 20x^3 + 15 - \frac{6}{x^3} + \frac{1}{x^6}$. 2. (a) $C_{10}^{20} = 20!/(10!)^2$, (b) $-1\,280\,000x^3y^3$. 3. 5/27. 4. 9.

Chapter 11

§ 11.1

1. (a) 1, (b) -1, (c) 0. 2. (a) $21 + 11$, (b) $8 - 8i$, (c) $-38 - 39i$. 3. (a) $\frac{7}{2} - 3i$, (b) $-\frac{3}{2} - \frac{1}{2}i$, (c) $-3.2 + 0.6i$. 4. (a) $x_1 = 0$, $x_2 = -5$, (b) $x = 1.5$. 5. 2. 6. -3. 7. -1. 8. (a) $(a - 2bi)(a + 2bi)$, (b) $(m\sqrt{3} + \sqrt{7}\,ni)(m\sqrt{3} - \sqrt{7}\,ni)$. 9. (a) $x^2 - 2x + 5 = 0$, (b) $x^2 - 6x + 10 = 0$. 10. (a) $-5 - 12i$, (b) $110 + 74i$, (c) $\pm(2 + i)$, (d) $z_1 = 3i$, $z_2 = -1 - 3i$. 11. (a) $\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}$. (b) $4 \pm 3i$. 12. (a) $\frac{2}{3}$, $-\frac{1}{3} + \frac{\sqrt{3}}{3}i$, (b) $-\frac{1}{2} \div \frac{1 \pm i\sqrt{3}}{4}$. 13. (a) $\frac{\pm 3\sqrt{2} + 3i}{2}$, (b) $\pm\frac{5}{2}$, $\pm\frac{5}{2}i$. 14. (a) $\pm i$, $\pm i\sqrt{3}$.

§ 11.2

2. No. 3. (a) $|z| = 2$, (b) $|z| = 2$, (c) $|z| = 5$. 11. (a) $3\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$, (b) $4(\cos(2\pi/3) + i\sin(2\pi/3))$, (c) $10(\cos(-\pi/6) + i\sin(-\pi/6))$, (d) $4(\cos(-\pi/2) + i\sin(-\pi/2))$, (e) $6(\cos 0 + i\sin 0)$. 12. (a) $1 + i\sqrt{3}$, (b) $3 - 3i$, (c) -5, (d) $\sqrt{6} + \sqrt{2} + (\sqrt{6} - \sqrt{2})\,i$. 13. (a) -1, (b) 1. 14. (a) 256, (b) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

MOST IMPORTANT FORMULAS

Fundamental Laws of Algebra

For any real numbers a, b, and c the following equalities are valid:
$a + b = b + a$ **(commutative law of addition).**
$a + (b + c) = (a + b) + c$ **(associative law of addition).**
$a + 0 = a$.
$a + (-a) = 0$.
$ab = ba$ **(commutative law of multiplication).**
$a(bc) = (ab)c$ **(associative law of multiplication).**
$a(b + c) = ab + ac$ **(distributive law of multiplication over addition).**
$a \cdot 1 = a$.
$a \cdot (1/a) = 1$, $a \neq 0$.

Numerical Inequalities

If $a > b$, then $b < a$.
If $a > b$ and $b > c$, then $a > c$.
If $a > b$, then $a + c > b + c$, where c is any real number.
If $a > b$ and $c > 0$, then $ac > bc$.
If $a > b$ and $c < 0$, then $ac < bc$.
If $a > b$ and $c > d$, then $a + c > b + d$.
If $a > 0$, $b > 0$, $c > 0$, and $d > 0$, where $a > b$ and $c > d$, then $ac > bd$.

Absolute Value of a Real Number

$$|a| = \begin{cases} a & \text{for } a \geqslant 0, \\ -a & \text{for } a < 0 \end{cases}$$ **(definition of the absolute value).**

$|a| \geqslant 0$.

$|a| = |-a|$.

$|ab| = |a| \cdot |b|$.

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}.$$

Distance Between Two Points

$\rho(x, a) = |x - a|$ **(distance between two points x and a on a straight line).**

$\rho(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ **(distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the plane).**

Identities for Short-cut Multiplication

$(a - b)(a + b) = a^2 - b^2$ **(difference of squares).**
$(a + b)^2 = a^2 + 2ab + b^2$ **(square of sum).**
$(a - b)^2 = a^2 - 2ab + b^2$ **(square of difference).**

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ **(cube of sum).**
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ **(cube of difference).**
$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ **(sum of cubes).**
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ **(difference of cubes).**

Arithmetic Root

If $a \geqslant 0$, then $\sqrt[n]{a} = x$ means: (1) $x \geqslant 0$, (2) $x^n = a$ **(definition of arithmetic root).**

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

$(\sqrt[n]{a})^k = \sqrt[n]{a^k}$.

$\sqrt[n]{\sqrt[k]{a}} = \sqrt[nk]{a}$.

$\sqrt[nm]{a^{km}} = \sqrt[n]{a^k}$.

$\sqrt{a^2} = |a|$.

Powers with a Rational Exponent

$a^n = \underbrace{a \cdot a \ \ldots \ a}_{n \text{ factors}}$, $a^1 = a$ **(definition of power with a natural exponent).**

$a^{p/q} = \sqrt[q]{a^p}$, where $a \geqslant 0$ **(definition of power with a positive fractional exponent).**

$a^0 = 1$, $a \neq 0$ **(definition of power with a zero exponent).**

$a^{-r} = 1/a^r$, where $a > 0$ **(definition of power with a negative exponent).**

$a^{r_1} \cdot a^{r_2} = a^{r_1 + r_2}$.

$a^{r_1} \div a^{r_2} = a^{r_1 - r_2}$.

$(a^{r_1})^{r_2} = a^{r_1 r_2}$.

$(ab)^{r_1} = a^{r_1} b^{r_1}$.

$\left(\frac{a}{b}\right)^{r_1} = \frac{a^{r_1}}{b^{r_1}}$.

Quadratic Equation $ax^2 + bx + c = 0$

$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-b \pm \sqrt{l^2 - 4ac}}{2a}$ **(formula of roots of quadratic equation).**

$x = \frac{-k \pm \sqrt{k^2 - ac}}{a}$, where $\frac{b}{2}$ **(formula of roots of quadratic equation if b is an even number).**

$x_1 + x_2 = -b/a$, $x_1 x_2 = c/a$, where x_1 and x_2 are roots of quadratic equation **(Viète's theorem).**

$ax^2 + bx + c = a(x - x_1)(x - x_2)$, where x_1 and x_2 are roots of the trinomial $ax^2 + bx + c$ **(formula of factoring a quadratic trinominal).**

Certain Values of Trigonometric Functions

α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
0	0	1	0	—
$\pi/6$	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1
$\pi/3$	$\sqrt{3}/2$	1/2	$\sqrt{3}$	$\sqrt{3}/3$
$\pi/2$	1	0	—	0
π	0	-1	0	—
$3\pi/2$	-1	0	—	0

Relationships Between Trigonometric Functions of the Same Argument

$\cos^2\alpha + \sin^2\alpha = 1.$
$1 + \tan^2\alpha = 1/\cos^2\alpha.$
$1 + \cot^2\alpha = 1/\sin^2\alpha.$

Addition Formulas

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$
$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$
$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$
$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.$$

$$\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}.$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.$$

$$\cot(\alpha-\beta)=\frac{1+\cot\alpha\cot\beta}{\cot\beta-\cot\alpha}.$$

Double-angle Functions

$\sin 2\alpha = 2\sin\alpha\cos\alpha.$
$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha.$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}.$$

$$\cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha}.$$

Half-angle Functions

$$1 + \cos\alpha = 2\cos^2(\alpha/2).$$
$$1 - \cos\alpha = 2\sin^2(\alpha/2).$$

Power Reduction Formulas

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}.$$

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}.$$

Formulas for Transforming the Sum of Trigonometric Functions into a Product

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}.$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$$

$$\tan\alpha + \tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}.$$

$$\tan\alpha - \tan\beta = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}.$$

$$\cot\alpha + \cot\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}.$$

$$\cot\alpha - \cot\beta = \frac{\sin(\beta-\alpha)}{\sin\alpha+\sin\beta}.$$

Formulas for Transforming the Product of Trigonometric Functions into a Sum

$$\sin\alpha\cos\beta = \frac{\sin(\alpha-\beta)+\sin(\alpha+\beta)}{2}.$$

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}.$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}.$$

Formulas for Solving Simple Trigonometric Equations

$\sin x = a$, $|a| \leqslant 1$, $x = (-1)^n \arcsin a + \pi n$.
$\cos x = a$, $|a| \leqslant 1$, $x = \pm \arccos a + 2\pi n$.
$\tan x = a$, $x = \arctan a + \pi n$.
$\cot x = a$, $x = \operatorname{arccot} a + \pi n$.

Arithmetic Progression

$a_{n+1} = a_n + d$ **(definition of arithmetic progression).**
$a_n = a_1 + d(n-1)$ **(formula of the nth term).**
$a_{n+1} = (a_n + a_{n+2})/2$ **(characteristic property).**

$S_n = \dfrac{(a_1 + a_n)\, n}{2}$, $S_n = \dfrac{2a_1 + d(n-1)}{2}\, n$ **(formula of the sum of the first n terms).**

Geometric Progression

$b_{n+1} = b_n q$, where $b_1 \neq 0$, $q \neq 0$ **(definition of geometric progression).**
$b_n = b_1 q^{n-1}$ **(formula of the nth term).**
$b_{n+1}^2 = b_n b_{n+2}$ **(characteristic property).**
$S_n = \dfrac{b_1 (q^n - 1)}{q - 1}$ **(formula of the sum of the first n terms).**

$S = \dfrac{b_1}{1 - q}$ **(formula of the sum of the terms of an infinitely decreasing geometric progression).**

Logarithms

$\log_a N = b$ if $a^b = N$, where $a > 0$, $a \neq 1$ **(definition of logarithm).**
$\log_a (x_1 x_2) = \log_a x_1 + \log_a x_2$, where $x_1 > 0$ and $x_2 > 0$.
$\log_a (x_1/x_2) = \log_a x_1 - \log_a x_2$, where $x_1 > 0$ and $x_2 > 0$.
$\log_a x^k = k \log_a x$, where $x > 0$.

$\log_a x = \dfrac{\log_b x}{\log_b a}$, where $x > 0$ **(formula for change of base).**
$\log_a x = \log_{a^k} x^k$, where $x > 0$, $k \neq 0$.

Derivatives

$y' = f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{\Delta f}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$ **(definition of the derivative).**
$(u + v)' = u' + v'$.
$(Cu') = Cu'$.
$(uv') = u'v + uv'$.
$\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$.
$y'_x = y_u \cdot u'_x$ **(definition of a composite function $y = y(u(x))$).**
$(x^r)' = rx^{r-1}$.

$(\sin x)' = \cos x.$

$(\cos x)' = -\sin x.$

$(\tan x)' = 1/\cos^2 x.$

$(\cot x)' = -1/\sin^2 x.$

$(e^x)' = e^x, \quad (a_x)' = a^x \ln a.$

$(\ln x)' = \frac{1}{x}, \quad (\log_a x)' = \frac{1}{\ln a} \cdot \frac{1}{x}.$

$f(x_0) = f'(x_0)(x - x_0)$ (equation of the tangent line to the graph of a function $y = f(x)$ at the point x_0).

Indefinite Integral

$$\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx.$$

$$\int kf(x)\, dx = k \int f(x)\, dx.$$

If $\int f(x)\, dx = F(x) + C$, then $\int f(ax + b)\, dx = \frac{1}{a} F(ax + b) + C$, $a \neq 0$.

Definite Integral

$S = \int_a^b (f_1(x) - f_2(x))\, dx$ (formula of the area of a curvilinear trapezoid bounded by two straight lines $x = a$ and $x = b$, where $a < b$, and the graphs of functions $y = f_1(x)$ and $y = f_2(x)$, where $f_2(x) < f_1(x)$ on $[a, b]$).

$\int_a^b f(x)\, dx = F(b) - F(a)$ (Newton-Leibniz formula).

Elements of Combinatorics

$P_n = 1 \cdot 2 \cdot 3 \ldots n = n!$ (formula for the number of permutations of n elements).

$A_m^n = \frac{n!}{(n-m)!}$ (formula for the number of arrangements of n elements taken m at a time).

$C_m^n = \frac{A_m^n}{P_m} = \frac{n!}{(n-m)!\, m!}$ (formula for the number of combinations of n elements taken m at a time).

$(a + b)^n = C_0^n a^n + C_1^n a^{n-1} b + C_2^n a^{n-2} b^2 + \ldots + C_{n-1}^n a b^{n-1} + C_n^n b^n$ (the binomial formula).

Complex Numbers

$z = a + bi$ (algebraic form of a complex number).

$|z| = \sqrt{a^2 + b^2}$ (modulus of a complex number).

$z = |z|(\cos \varphi + i \sin \varphi)$ (trigonometric form of a complex number).

$(\cos \varphi + i \sin \varphi)^n = \cos n\varphi + i \sin n\varphi$ (De Moivre's formula).

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TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

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